THEORY

OF

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THEORY

OF

DIFFERENTIAL EQUATIONS.

PART III.

ORDINARY LINEAR EQUATIONS.

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PREFACE.

The present volume, constituting Part III of this work, deals with the theory of ordinary linear differential equations. The whole range of that theory is too vast to be covered by a single volume; and it contains several distinct regions that have no organic relation with one another. Accordingly, I have limited the discussion to the single region specially occupied by applications of the theory of functions; in imposing this limitation, my wish has been to secure a uniform presentation of the subject.

As a natural consequence, much is omitted that would have been included, had my decision permitted the devotion of greater space to the subject. Thus the formal theory, in its various shapes, is not expounded, save as to a few topics that arise incidentally in the functional theory. The association with homogeneous forms is indicated only slightly. The discussion of combinations of the coefficients, which are invariantive under all transformations that leave the equation linear, of the associated equations that are covariantive under these transformations, and of the significance of these invariants

and covariants, is completely omitted. Nor is any application of the theory of groups, save in a single functional investigation, given here. The student, who wishes to consider these subjects, and others that have been passed by, will find them in Schlesinger's *Handbuch der Theorie der linearen Differentialgleichungen*, in treatises such as Picard's *Cours d'Analyse*, and in many of the memoirs quoted in the present volume.

In preparing the volume, I have derived assistance from the two works just mentioned, as well as from the uncompleted work by the late Dr Thomas Craig. But, as will be seen from the references in the text, my main assistance has been drawn from the numerous memoirs contributed to learned journals by various pioneers in the gradual development of the subject.

Within the limitations that have been imposed, it will be seen that much the greater part of the volume is assigned to the theory of equations which have uniform coefficients. When coefficients are not uniform, the difficulties in the discussion are grave: the principal characteristics of the integrals of such an equation have, as yet, received only slight elucidation. On this score, it will be sufficient to mention equations having algebraic coefficients: nearly all the characteristic results that have been obtained are of the nature of existence-theorems, and little progress in the difficult task of constructing explicit results has been made.

Moreover, I have dealt mainly with the general theory and have abstained from developing detailed properties of the functions defined by important particular equations. The latter have been used as illustrations; had they been developed in fuller detail than is

given, the investigations would soon have merged into discussions of the properties of special functions. Instances of such transition are provided in the functions, defined by the hypergeometric equation and by the modern form of Lamé's equation respectively.

A brief summary of the contents will indicate the actual range of the volume. In the first Chapter, the synectic integrals of a linear equation, and the conditions of their uniqueness, are investigated. The second Chapter discusses the general character of a complete system of integrals near a singularity of the equation. Chapters III, IV, and V are concerned with equations, which have their integrals of the type called regular; in particular, Chapter V contains those equations the integrals of which are algebraic functions of the variable. In Chapter VI, equations are considered which have only some of their integrals of the regular type; the influence of such integrals upon the reducibility of their equation is indicated. Chapter VII is occupied with the determination of integrals which, while not regular, are irregular of specified types called normal and subnormal; the functional significance of such integrals is established, in connection with Poincaré's development of Laplace's solution in the form of a definite integral. Chapter VIII is devoted to equations, the integrals of which do not belong to any of the preceding types; the method of converging infinite determinants is used to obtain the complete solution for any such equation. Chapter IX relates to those equations, the coefficients of which are uniform periodic functions of the variable: there are two

classes, according as the periodicity is simple or double. The final Chapter deals with equations having algebraic coefficients; it contains a brief general sketch of Poincaré's association of such equations with automorphic functions.

In the revision of the proof-sheets, I have received valuable assistance from three of my friends and former pupils, Mr. E. T. Whittaker, M.A., and Mr. E. W. Barnes, M.A., Fellows of Trinity College, Cambridge, and Mr. R. W. H. T. Hudson, M.A., Fellow of St John's College, Cambridge; I gratefully acknowledge the help which they have given me.

And I cannot omit the expression of my thanks to the Staff of the University Press, for the unfailing courtesy and readiness with which they have lightened my task during the printing of the volume.

A. R. FORSYTH.

TRINITY COLLEGE, CAMBRIDGE,
1 *March*, 1902.

CONTENTS.

CHAPTER I.

LINEAR EQUATIONS: EXISTENCE OF SYNECTIC INTEGRALS: FUNDAMENTAL SYSTEMS.

CHAPTER II.

GENERAL FORM AND PROPERTIES OF INTEGRALS NEAR A SINGULARITY.

CHAPTER III.

REGULAR INTEGRALS: EQUATION HAVING ALL ITS INTEGRALS REGULAR NEAR A SINGULARITY.

CHAPTER IV.

EQUATIONS HAVING THEIR INTEGRALS REGULAR IN THE VICINITY OF EVERY SINGULARITY (INCLUDING INFINITY).

CHAPTER V.

LINEAR EQUATIONS OF THE SECOND AND THE THIRD ORDERS POSSESSING ALGEBRAIC INTEGRALS.

CHAPTER VI.

EQUATIONS HAVING ONLY SOME OF THEIR INTEGRALS REGULAR NEAR A SINGULARITY.

CHAPTER VII.

NORMAL INTEGRALS: SUBNORMAL INTEGRALS.

CHAPTER VIII.

INFINITE DETERMINANTS, AND THEIR APPLICATION TO THE SOLUTION OF LINEAR EQUATIONS.

CHAPTER IX.

EQUATIONS WITH UNIFORM PERIODIC COEFFICIENTS.

CHAPTER X.

EQUATIONS HAVING ALGEBRAIC COEFFICIENTS.

CHAPTER I.

Linear Equations; Existence of Synectic Integrals: Fundamental Systems.

1. The course of the preceding investigations has made it manifest that the discussion of the properties of functions, which are defined by ordinary differential equations of a general type, rapidly increases in difficulty with successive increase in the order of the equations. Indeed, a stage is soon reached where the generality of form permits the deduction of no more than the simplest properties of the functions. Special forms of equations can be subjected to special treatment; but, when such special forms conserve any element of generality, complexity and difficulty arise for equations of any but the lowest orders. There is one exception to this broad statement; it is constituted by ordinary equations which are linear in form. They can be treated, if not in complete generality, yet with sufficient fulness to justify their separate discussion; and accordingly, the various important results relating to the theory of ordinary linear differential equations constitute the subject-matter of the present Part of this Treatise.

Some classes of linear equations have received substantial consideration in the construction of the customary practical methods used in finding solutions. One particular class is composed of those equations which have constants as the coefficients of the dependent variable and its derivatives. There are, further, equations associated with particular names, such as Legendre, Bessel, Lamé; there are special equations, such as those of the hypergeometric series and of the quarter-period in the Jacobian theory of elliptic functions. The formal solutions of such equations

can be regarded as known; but so long as the investigation is restricted to the practical construction of the respective series adopted for the solutions, no indication of the range, over which the deduced solution is valid, is thereby given. It is the aim of the general theory, as applied to such equations, to reconstruct the various methods of proceeding to a solution, and to shew why the isolated rules, that seem so sourceless in practice, actually prove effective. In prosecuting this aim, it will be necessary to revise for linear equations all the customarily accepted results, so as to indicate their foundation, their range of validity, and their significance.

For the most part, the equations considered will be kept as general as possible within the character assigned to them. But from time to time, equations will be discussed, the functions defined by which can be expressed in terms of functions already known; such instances, however, being used chiefly as illustrations. For all equations, it will be necessary to consider the same set of problems as present themselves for consideration in the discussion of unrestricted ordinary equations of the lowest orders: the existence of an integral, its uniqueness as determined by assigned conditions, its range of existence, its singularities (as regards position and nature), its behaviour in the vicinity of any singularity, and so on: together with the converse investigation of the limitations to be imposed upon the form of the equation in order to secure that functions of specified classes or types may be solutions. As is usual in discussions of this kind, the variables and the parameters will be assumed to be complex. It is true that, for many of the simpler applications to mechanics and physics, the variables and the parameters are purely real; but this is not the case with all such applications, and instances occur in which the characteristic equations possess imaginary or complex parameters or variables. Quite independently of this latter fact, however, it is desirable to use complex variables in order to exhibit the proper relation of functional variation.

2. Let z denote the independent variable, and w the dependent variable; z and w varying each in its own plane. The differential equation is considered *linear*, when it contains no term of order higher than the first in w and its derivatives: and a linear equation is called *homogeneous*, when it contains no term independent of w

and its derivatives. By a well-known formal result*, the solution of an equation that is not homogeneous can be deduced, merely by quadratures, from the solution of the equation rendered homogeneous by the omission of the term independent of w and its derivatives; and therefore it is sufficient, for the purposes of the general investigation, to discuss homogeneous linear equations. The coefficients may be uniform functions of z, either rational or transcendental; or they may be multiform functions of z, the simplest instance being that in which they are of a form $\phi(s, z)$, where ϕ is rational in s and z, and s is an algebraic function of z. Examples of each of these classes will be considered in turn. The coefficients will have singularities and (it may be) critical points; all of these are determinable for a given equation by inspection, being fixed points which are not affected by any constants that may arise in the integration. Such points will be found to include all the singularities and the critical points of the integrals of the equation; in consequence, they are frequently called *the singularities of the equation.* Accordingly, the differential equation, assumed to be of order m, can be taken in the form

$$\frac{d^m w}{dz^m} = p_1 \frac{d^{m-1} w}{dz^{m-1}} + p_2 \frac{d^{m-2} w}{dz^{m-2}} + \dots + p_m w,$$

where the coefficients $p_1, p_2, \dots, p_m$ are functions of z. In the earlier investigations, and until explicit statement to the contrary is made, it will be assumed that these functions of z are uniform within the domain considered; that their singularities are isolated points, so that any finite part of the plane contains only a limited number of them: and that all these singularities (if any) for finite values of z are poles of the coefficients, so that their only essential singularity (if any) must be at infinity. Let ζ denote any point in the plane which is ordinary for all the coefficients p; and let a domain of ζ be constructed by taking all the points z in the plane, such that

$$|z - \zeta| \leqslant |a - \zeta|,$$

where a is the nearest to ζ among all the singularities of all the coefficients. Then within this domain (but not on its boundary) we have

$$p_s = P_s(z - \zeta), \qquad (s = 1, 2, \dots, m),$$

* See my *Treatise on Differential Equations*, § 75.

where P_s denotes a regular function of $z-\zeta$, which generally is an infinite series of powers of $z-\zeta$ converging within the domain of ζ. An integral of the equation existing in this domain is uniquely settled by the following theorem:—

In the domain of an ordinary point ζ, the differential equation possesses an integral, which is a regular function of $z-\zeta$ and, with its first $m-1$ derivatives, acquires arbitrarily assigned values when $z=\zeta$; and this integral is the only regular function of $z-\zeta$ in the specified domain, which satisfies the equation and fulfils the assigned conditions.*

The integral thus obtained will be called† the *synectic* integral.

Synectic Integrals.

(Analytic)

3. The existence of an integral which is a holomorphic function of $z-\zeta$ within the domain will first be established.

Let r be the radius of the domain of ζ; let $M_1, \ldots, M_m$ denote quantities not less than the maximum values of $|p_1|, \ldots, |p_m|$ respectively, for points within the domain; and let dominant functions $\phi_1, \ldots, \phi_m$, defined by the expressions

$$\phi_s = \frac{M_s}{1-\dfrac{z-\zeta}{r}}, \qquad (s=1, \ldots, m),$$

be constructed. Then‡

$$\left|\frac{d^\alpha p_s}{dz^\alpha}\right|_{z=\zeta} \leqslant \left|\frac{d^\alpha \phi_s}{dz^\alpha}\right|_{z=\zeta},$$

for every positive integer α. The dominant functions ϕ are used to construct a dominant equation

$$\frac{d^m u}{dz^m} = \phi_1 \frac{d^{m-1}u}{dz^{m-1}} + \phi_2 \frac{d^{m-2}u}{dz^{m-2}} + \ldots + \phi_m u,$$

which is considered concurrently with the given equation.

* The conditions, as to the arbitrarily assigned values to be acquired at ζ by w and its derivatives, are called the *initial conditions*; the values are called the *initial values*.

† As it is a regular function of the variable, it would have been proper to call it the regular integral. This term has however been appropriated (see Chapter III, § 29) to describe another class of integrals of linear equations; as the use in this other connection is now widespread, confusion would result if the use were changed.

‡ See my *Theory of Functions*, 2nd edn., § 22: quoted hereafter as *T. F.*

Any function which is regular in the domain of ζ can be expressed as a converging series of powers of $z-\zeta$; and the coefficients, save as to numerical factors, are the values of the various derivatives of the function at ζ. Accordingly, if there is an integral w which is a regular function of $z-\zeta$, it can be formed when the values of all the derivatives of w at ζ are known. To $w, \dfrac{dw}{dz}, \ldots, \dfrac{d^{m-1}w}{dz^{m-1}}$, the arbitrary values specified in the initial conditions are assigned. All the succeeding derivatives of w can be deduced from the differential equation in the form

$$\frac{d^\alpha w}{dz^\alpha} = A_{\alpha 1}\frac{d^{m-1}w}{dz^{m-1}} + A_{\alpha 2}\frac{d^{m-2}w}{dz^{m-2}} + \ldots + A_{\alpha m}w,$$

(for $\alpha = m$, $m+1$, ... ad inf.), by processes of differentiation, addition, and multiplication: as the coefficient of the highest derivative of w in the equation (and in every equation deduced from it by differentiation) is unity, new critical points are not introduced by these processes, so that all the coefficients A are regular within the domain of ζ.

The successive derivatives of u are similarly expressible in the form

$$\frac{d^\alpha u}{dz^\alpha} = B_{\alpha 1}\frac{d^{m-1}u}{dz^{m-1}} + B_{\alpha 2}\frac{d^{m-2}u}{dz^{m-2}} + \ldots + B_{\alpha m}u,$$

(for $\alpha = m$, $m+1$, ... ad inf.), obtained in the same way as the equation for the derivatives of w. The coefficients B have the same form as the coefficients A, and can be deduced from them by changing the quantities p and their derivatives into the quantities ϕ and their derivatives respectively.

The values of the derivatives of w and u at ζ are required. When $z=\zeta$, all the terms in each quantity B are positive; on account of the relation between the derivatives of the quantities p and ϕ, it follows that

$$B_{\alpha s} \geqslant |A_{\alpha s}|, \qquad (s=1, \ldots, m),$$

when $z=\zeta$. Let the initial values of $|w|$, $\left|\dfrac{dw}{dz}\right|, \ldots, \left|\dfrac{d^{m-1}w}{dz^{m-1}}\right|$, when $z=\zeta$, be assigned as the values of $u, \dfrac{du}{dz}, \ldots, \dfrac{d^{m-1}u}{dz^{m-1}}$ when $z=\zeta$; then

$$\left|\frac{d^\alpha w}{dz^\alpha}\right| \leqslant \frac{d^\alpha u}{dz^\alpha},$$

when $z = \zeta$, for the values $m, m+1, \ldots$ of α. If the series

$$(u) + (z-\zeta)\left(\frac{du}{dz}\right) + \frac{(z-\zeta)^2}{2\,!}\left(\frac{d^2u}{dz^2}\right) + \ldots$$

converges, where $\left(\dfrac{d^a u}{dz^a}\right)$ denotes the value of $\dfrac{d^a u}{dz^a}$ when $z = \zeta$, the series

$$(w) + (z-\zeta)\left(\frac{dw}{dz}\right) + \frac{(z-\zeta)^2}{2\,!}\left(\frac{d^2w}{dz^2}\right) + \ldots,$$

where $\left(\dfrac{d^a w}{dz^a}\right)$ denotes the value of $\dfrac{d^a w}{dz^a}$ when $z = \zeta$, also converges; it then represents a regular function of $z - \zeta$ which, after the mode of formation of its coefficients, satisfies the differential equation.

We therefore proceed to consider the convergence of the series for u, obtained as a purely formal solution of the dominant equation. To obtain explicit expressions for the various coefficients in this series, let $z - \zeta = rx$, taking x as the new independent variable. Points within the domain of ζ are given by $|x| < 1$; and the dominant equation becomes

$$(1-x)\frac{d^m u}{dx^m} = \sum_{s=1}^{m} M_s r^s \frac{d^{m-s}u}{dx^{m-s}}.$$

When the series for u, taken in the form

$$u = \sum_{a=0}^{\infty} b_a x^a,$$

is substituted in the equation which then becomes an identity, a comparison of the coefficients of x^k on the two sides leads to the relation

$$(m+k)\,!\, b_{m+k}$$
$$= (m+k-1)\,!\,(k + M_1 r)\, b_{m+k-1} + \sum_{s=2}^{m} (m+k-s)\,!\, M_s r^s\, b_{m+k-s},$$

holding for all positive integer values of k.

This relation shews that all the coefficients b are expressible linearly and homogeneously in terms of $b_0, b_1, \ldots, b_{m-1}$: and that, as the first m of these coefficients have been made equal to the moduli of the m arbitrary quantities in the initial conditions for w and therefore are positive, all the coefficients b are positive. Hence

$$b_{m+k} > \frac{k + M_1 r}{k+m}\, b_{m+k-1}.$$

By the initial definition of M_1, it was taken to be not less than the maximum value of $|p_1|$ within the domain of ζ; it can therefore be chosen so as to secure that $M_1 r > m$. Assuming this choice made, we then have

$$b_{m+k} > b_{m+k-1},$$

so that the successive coefficients increase.

From the difference-equation satisfied by the coefficients b, it follows that

$$\frac{b_{m+k}}{b_{m+k-1}} = \frac{k+M_1 r}{k+m} + \sum_{s=2}^{m} \frac{(m+k-s)!}{(m+k)!} M_s r^s \frac{b_{m+k-s}}{b_{m+k-1}}.$$

So far as regards the $m-1$ terms in the summation, the ratio $b_{m+k-s} \div b_{m+k-1}$ is less than unity for each of them; $M_s r^s$ is finite for each of them; and $(m+k-s)! \div (m+k)!$ is zero for each of them, in the limit when k is made infinite. Hence we have

$$\operatorname*{Lim}_{k=\infty} \frac{b_{m+k}}{b_{m+k-1}} = 1,$$

and therefore

$$\begin{aligned}\operatorname*{Lim}_{l=\infty} \frac{b_{l+1}|x|^{l+1}}{b_l |x|^l} &= |x| \\ &< 1,\end{aligned}$$

for points within the domain of ζ, so that* the series

$$\sum_{a=0}^{\infty} b_a x^a$$

converges within the domain of ζ. The convergence is not established for the boundary, so that it can be affirmed only for points within the domain; it holds for all arbitrary positive values assigned to $b_0, b_1, \ldots, b_{m-1}$.

It therefore follows that, at all points within the domain of ζ, a regular function of $z-\zeta$ exists which satisfies the original differential equation for w, and, with its first $m-1$ derivatives, acquires at ζ arbitrarily assigned values.

4. Now that the existence of a synectic integral is established, the explicit expression of the integral in the form of a power-series in $z-\zeta$, this series being known to converge, can be obtained

* Chrystal's *Algebra*, vol. II, p. 121.

directly from the equation. As ζ is an ordinary point for each of the coefficients p, we have

$$p_s = P_s(z-\zeta), \qquad (s = 1, 2, \ldots, m),$$

where P_s denotes a regular function of $z-\zeta$. Let $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ be the arbitrary values assigned to $w, \frac{dw}{dz}, \ldots, \frac{d^{m-1}w}{dz^{m-1}}$, when $z = \zeta$; and take

$$w = \sum_{n=0}^{\infty} \frac{\alpha_n}{n!}(z-\zeta)^n,$$

which manifestly satisfies the initial conditions. In order that this may satisfy the equation, it must make the equation an identity when the expression is substituted therein. When the substitution is effected, and the coefficients of $(z-\zeta)^s$ on the two sides of the identity are equated, we have a relation of the form

$$\frac{\alpha_{m+s}}{s!} = A_{m+s},$$

where A_{m+s} is a linear homogeneous function of the coefficients α_κ, such that $\kappa < m+s$, and is also linear in the coefficients in the quantities $P_1(z-\zeta), \ldots, P_m(z-\zeta)$; and the relation is valid for $s = 0, 1, 2, \ldots$, ad inf. Using the relation for these values of s in succession, we find $\alpha_m, \alpha_{m+1}, \alpha_{m+2}, \ldots$ expressed (in each instance, after substitution of the values of the coefficients which belong to earlier values of s) as a linear homogeneous function of the quantities $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$: and in α_{m+s}, the expressions, of which the initial constants $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ are coefficients, are polynomials of degree $s+1$ in the coefficients of the functions $P_1(z-\zeta), \ldots, P_m(z-\zeta)$. The earlier investigation shews that the power-series for w converges; accordingly, the determination of the coefficients α in this manner leads to the formal expression of an integral w satisfying the equation.

5. Further, the integral thus obtained is the only regular function, which is a solution of the equation and satisfies the initial conditions associated with $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$. If it were possible to have any other regular function, which also is a solution and satisfies the same initial conditions, its expression would be of the form

$$w' = \sum_{\mu=0}^{m-1} \frac{\alpha_\mu}{\mu!}(z-\zeta)^\mu + \sum_{\mu=m}^{\infty} \frac{\alpha'_\mu}{\mu!}(z-\zeta)^\mu,$$

a regular function of $z-\zeta$. The coefficients would be determinable, as before, from a relation

$$\frac{\alpha'_{m+s}}{s\,!}=A'_{m+s},$$

where A'_{m+s} is the same function of $\alpha_0, \ldots, \alpha_{m-1}, \alpha'_m, \ldots, \alpha'_{m+s-1}$ as A_{m+s} is of $\alpha_0, \ldots, \alpha_{m-1}, \alpha_m, \ldots, \alpha_{m+s-1}$. Hence

$$\begin{aligned}\alpha'_m &= A'_m \;\;= A_m = \alpha_m;\\ \alpha'_{m+1} &= A'_{m+1} = A_{m+1}, \text{ after substitution for } \alpha'_m,\\ &\qquad\quad\;\; = \alpha_{m+1};\end{aligned}$$

and so on, in succession. The coefficients agree, and the two series are the same, so that $w=w'$; and therefore the initial conditions uniquely determine an integral of the equation, which is a regular function of $z-\zeta$ in the domain of the ordinary point ζ.

COROLLARY I. *If all the initial constants* $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ *are zero, then the synectic integral of the equation is identically zero.* For in the preceding discussion it has been proved that α_{m+s}, for all the values of s, is a linear homogeneous function of $\alpha_0, \ldots, \alpha_{m-1}$; hence, in the circumstances contemplated, $\alpha_{m+s}=0$ for all the values of s. Thus every coefficient in the series vanishes; accordingly, the integral is an identical zero.

COROLLARY II. *The initial constants* $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ *occur linearly in the expression of the synectic integral; and each of the* m *variable quantities, which have those constants for coefficients, is a synectic integral of the equation.* The first part is evident, because all the coefficients in w are linear and homogeneous in $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$. As regards the second part, the variable quantity multiplied by α_s is derivable from w by making $\alpha_s=1$, and all the other constants α equal to zero; these constitute a particular set of initial values which, according to the theorem, determine a synectic integral of the equation. Thus the synectic integral, determined by the initial values $\alpha_0, \ldots, \alpha_{m-1}$, is of the form

$$\alpha_0 u_1+\alpha_1 u_2+\ldots+\alpha_{m-1}u_m,$$

where each of the quantities $u_1, u_2, \ldots, u_m$ is a synectic integral of the equation.

Note 1. The series of powers of $z-\zeta$, which represents the synectic integral, has been proved to converge within the domain

of ζ, so that its radius of convergence is $|a - \zeta|$, where a is the singularity of the coefficients which is nearest to ζ. All these singularities lying in the finite part of the plane are determinable by mere inspection of the forms of the coefficients: another method must be adopted in order to take account of a possible singularity when $z = \infty$ because, even though $z = \infty$ may be an ordinary point of the coefficients, infinite values of the variable affect the character of w and its derivatives.

For this purpose, we may change the variable by the substitution

$$zx = 1,$$

and we then consider the relation of the x-origin to the transformed equation as a possible singularity. The transformation of the equation is immediately obtained by means of the formula

$$\frac{d^k w}{dz^k} = (-1)^k \sum_{\alpha=1}^{k} \frac{k!\,(k-1)!}{\alpha!\,(\alpha-1)!} \frac{x^{k+\alpha}}{(k-\alpha)!} \frac{d^\alpha w}{dx^\alpha};$$

inspection of the transformed equation then shews whether $x = 0$ is, or is not, a singularity. Or, without changing the independent variable, we may consider a series for w in descending powers of z: examples will occur hereafter.

It may happen that there is no singularity of the coefficients in the finite part of the plane, infinite values then providing the only singularity. In that case, we should not take the quantity r in the preceding investigation as equal to $|\infty - \zeta|$, that is, as infinite; it would suffice that r should be finite, though as large as we please.

It may happen that there is no singularity of the coefficients for either finite or infinite values of z; if the coefficients are uniform, they then can only be constants. The dominant equation is then effectively the same as the original equation; the investigation is still applicable, but it furnishes less information as to the result than a method which will be indicated later (§ 6).

Note 2. The preceding proof is based upon that which is given* by Fuchs in his initial, and now classical, memoir on the theory of linear differential equations.

* *Crelle*, t. LXVI (1866), pp. 122—125.

The theorem can also be established by regarding it as a particular case of Cauchy's theorem, which relates to the possession of unique synectic integrals by a system of simultaneous equations. If

$$w_\alpha = \frac{d^\alpha w}{dz^\alpha}, \qquad (\alpha = 0, 1, \ldots, m-1),$$

the homogeneous linear equation of order m can be replaced by the system

$$\frac{dw_s}{dz} = w_{s+1}, \text{ for } s = 0, 1, \ldots, m-2,$$

$$\frac{dw_{m-1}}{dz} = p_1 w_{m-1} + p_2 w_{m-2} + \ldots + p_m w_0.$$

These equations possess integrals, expressible as regular functions of $z - \zeta$, such that $w_0, w_1, \ldots, w_{m-1}$ assume arbitrarily assigned values when $z = \zeta$, and the integrals are unique when thus determined: which, in effect, is the theorem as to the synectic integral of the linear equation*.

Note 3. A different method for establishing the existence of the integrals, though it does not indicate fully the region of their convergence, can be based upon a suggestion made by Günther†. It consists in the adoption of another subsidiary equation

$$\frac{d^m v}{dz^m} = \psi_1 \frac{d^{m-1} v}{dz^{m-1}} + \psi_2 \frac{d^{m-2} v}{dz^{m-2}} + \ldots + \psi_m v,$$

where

$$\psi_\mu = \frac{M_\mu}{\left(1 - \dfrac{z-\zeta}{r}\right)^\mu},$$

for $\mu = 1, \ldots, m$. The advantage of this form of equation is that its integrals are explicitly given in the form

$$v = \left(1 - \frac{z-\zeta}{r}\right)^\sigma,$$

where σ is a root of the equation

$$\begin{aligned}\sigma(\sigma-1)\ldots(\sigma-m+1) = &- rM_1\sigma(\sigma-1)\ldots(\sigma-m+2) \\ &+ r^2M_2\sigma(\sigma-1)\ldots(\sigma-m+3) + \ldots \\ &+ (-1)^{m-1}r^{m-1}M_{m-1}\sigma + (-1)^m r^m M_m.\end{aligned}$$

* See Part II of this Treatise, §§ 4, 10—13.

† *Crelle*, t. CXVIII (1897), pp. 351—353; see also some remarks thereupon by Fuchs, *ib.*, pp. 354, 355.

If a root σ is multiple, the corresponding group of integrals is easily obtained*.

The construction of the actual proof on the foregoing lines is left as an exercise.

Ex. 1. Consider the equation

$$\frac{d^2w}{dz^2} - \frac{2z}{1-z^2}\frac{dw}{dz} + \frac{\kappa}{1-z^2}w = 0,$$

where κ is a constant.

The singularities in the finite part of the plane are $z=1$, $z=-1$. On transforming the equation by the substitution $zx=1$, so that it becomes

$$\frac{d^2w}{dx^2} - \frac{2x}{1-x^2}\frac{dw}{dx} - \frac{\kappa}{x^2(1-x^2)}w = 0,$$

we see that $x=0$ (and therefore $z=\infty$) is another singularity of the coefficients: so that the preceding investigation does not apply to the immediate vicinity of $x=0$.

It is clear that the z-origin is an ordinary point of the coefficients of the original equation: the domain of $z=0$ is a circle of radius unity. The equation therefore possesses a synectic integral, which is a series of powers of z converging within the circle; it is uniquely determined by the conditions that $w=a$, $\dfrac{dw}{dz}=\beta$, when $z=0$, where a and β are arbitrary constants. To obtain its expression, let

$$w = \sum_{n=0}^{\infty} b_n z^n$$

be substituted in

$$(1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + \kappa w = 0,$$

which then must be an identity. In order that the coefficient of z^n may vanish after substitution, we must have

$$(n+2)(n+1)\,b_{n+2} - (n^2+n-\kappa)\,b_n = 0,$$

so that

$$b_{n+2} = \frac{n^2+n-\kappa}{(n+2)(n+1)}\,b_n.$$

Now by the initial conditions, we have

$$b_0 = a, \quad b_1 = \beta;$$

hence

$$b_{2m} = \frac{(2m-1)(2m-2)-\kappa}{2m(2m-1)}\,b_{2m-2},$$

$$= \frac{a}{2m\,!}\prod_{s=1}^{s=m}\{(2s-1)(2s-2)-\kappa\};$$

* See my *Treatise on Differential Equations*, §§ 47, 48.

and, similarly,

$$b_{2m+1}=\frac{\beta}{(2m+1)\,!}\prod_{s=1}^{s=m}\{2s(2s-1)-\kappa\};$$

the expressed products being taken for integer values of s from 1 to m. The synectic integral satisfying the initial conditions is

$$a\sum_{m=0}\frac{z^{2m}}{2m\,!}\prod_{s=1}^{s=m}\{(2s-1)(2s-2)-\kappa\}+\beta\sum_{m=0}\frac{z^{2m+1}}{(2m+1)\,!}\prod_{s=1}^{s=m}\{2s(2s-1)-\kappa\};$$

both series, if infinite, converging for values of z such that $|z|<1$.

The best known instance of this equation is that which is usually associated with Legendre's name: κ then is $p(p+1)$, and p (in the simplest form) is a positive integer. If p be an even integer, all the coefficients b_{2m}, for $2m>p$, vanish, so that the quantity multiplying a is then a polynomial; the quantity multiplying β is an infinite series. If p be an odd integer, all the coefficients b_{2m+1}, for $2m+1>p$, vanish, so that the quantity multiplying β is then a polynomial; the quantity multiplying a is an infinite series. In all other cases, the quantities multiplying a and β are, each of them, infinite series; in every instance, the series converge when $|z|<1$.

Ex. 2. Obtain the synectic integral of the equation

$$\frac{d^2w}{dz^2}+\frac{1}{z}\frac{dw}{dz}+\left(a-\frac{b}{z^2}\right)w=0,$$

(which includes Bessel's equation as a special case), with the initial conditions that $w=a$, $\frac{dw}{dz}=\beta$ when $z=c$, where $|c|>0$.

Ex. 3. Determine the synectic integral of the equation of the hypergeometric series

$$z(1-z)\frac{d^2w}{dz^2}+\{\gamma-(a+\beta+1)z\}\frac{dw}{dz}-a\beta w=0,$$

the initial conditions being that $w=A$, $\frac{dw}{dz}=B$, when $z=\frac{1}{2}$.

Ex. 4. Determine the synectic integrals in the domain of $z=0$, possessed by the equation

$$\frac{d^2w}{dz^2}=zw,$$

with the initial conditions (i) that $w=1$, $\frac{dw}{dz}=0$, when $z=0$;

(ii) that $w=0$, $\frac{dw}{dz}=1$, when $z=0$.

Ex. 5. Prove that the synectic integral in the domain of $z=0$, possessed by the equation

$$\frac{d^2w}{dz^2}=we^{az},$$

with the initial conditions that $w=1$, $\dfrac{dw}{dz}=0$, when $z=0$, is

$$w=1+\frac{z^2}{2\,!}+\frac{az^3}{3\,!}+\frac{1+a^2}{4\,!}z^4+\frac{4a+a^3}{5\,!}z^5+\frac{1+11a^2+a^4}{6\,!}z^6+\ldots;$$

and if the term in w involving z^n be $\dfrac{c_n}{n\,!}z^n$, then

$$c_n=a^{n-2}+(2^{n-2}-n+1)\,a^{n-4}+\{\tfrac{1}{4}\,3^{n-2}-(n-4)\,2^{n-3}-\tfrac{1}{2}n-\tfrac{1}{4}\}\,a^{n-6}+\ldots.$$

Prove also that the primitive can be expressed in terms of Bessel's functions of order zero and argument $\dfrac{2i}{a}e^{\frac{1}{2}az}$.

Ex. 6. The equation with constant coefficients may be taken in the form

$$\frac{d^mw}{dz^m}=c_1\frac{d^{m-1}w}{dz^{m-1}}+c_2\frac{d^{m-2}w}{dz^{m-2}}+\ldots+c_mw;$$

it possesses a synectic integral in the form

$$w=\sum_{k=0}^{\infty}a_k\frac{z^k}{k\,!},$$

which converges everywhere in the finite part of the plane: and $a_0, \ldots, a_{m-1}$, are the arbitrarily assigned initial constants.

Substituting in the differential equation this value of w, and equating coefficients of $\dfrac{1}{n\,!}z^n$, we have

$$a_{m+n}=c_1a_{m+n-1}+c_2a_{m+n-2}+\ldots+c_ma_n.$$

The expression of the coefficients $a_m, a_{m+1}, \ldots$ in terms of $a_0, a_1, \ldots, a_{m-1}$ depends (by the solution of the foregoing difference-equation) upon the algebraical equation

$$\phi(\theta)=\theta^m-c_1\theta^{m-1}-c_2\theta^{m-2}-\ldots-c_m=0.$$

When the roots of $\phi(\theta)=0$ are different from one another, let them be denoted by $\alpha_1, \alpha_2, \ldots, \alpha_m$; and in connection with the m arbitrary constants $a_0, a_1, \ldots, a_{m-1}$, determine m new constants $A_1, A_2, \ldots, A_m$, by the relations

$$a_r=\sum_{\mu=1}^{m}\alpha_\mu{}^rA_\mu, \qquad (r=0, 1, \ldots, m-1).$$

The determination is unique: for on solving these m relations as m linear equations in $A_1, \ldots, A_\mu$, the determinant of the right-hand sides is

$$\begin{vmatrix} 1 & , 1 & , \ldots, & 1 \\ \alpha_1 & , \alpha_2 & , \ldots, & \alpha_m \\ \alpha_1{}^2 & , \alpha_2{}^2 & , \ldots, & \alpha_m{}^2 \\ & & \vdots & \\ \alpha_1{}^{m-1}, & \alpha_2{}^{m-1}, & \ldots, & \alpha_m{}^{m-1} \end{vmatrix},$$

which is equal to the product of the differences of the roots and is therefore not zero. Hence, as the constants $a_0, a_1, \ldots, a_{m-1}$ are arbitrary, the m new

constants $A_1, \ldots, A_m$, when used to replace the former set, can be regarded as m independent arbitrary constants. With these constants thus determined, we have

$$\sum_{\mu=1}^{m} a_\mu^{m+n} A_\mu = \sum_{\mu=1}^{m} (c_1 a_\mu^{m+n-1} + c_2 a_\mu^{m+n-2} + \ldots + c_m a_\mu^{n}) A_\mu$$

$$= c_1 \sum_{\mu=1}^{m} a_\mu^{m+n-1} A_\mu + c_2 \sum_{\mu=1}^{m} a_\mu^{m+n-2} A_\mu + \ldots + c_m \sum_{\mu=1}^{m} a_\mu^{r} A_\mu,$$

for all values of n. When $n=0$, we have

$$\sum_{\mu=1}^{m} a_\mu^{m} A_\mu = c_1 \alpha_{m-1} + c_2 \alpha_{m-2} + \ldots + c_m \alpha_0 = \alpha_m ;$$

when $n=1$, we have

$$\sum_{\mu=1}^{m} a_\mu^{m+1} A_\mu = c_1 \alpha_m + c_2 \alpha_{m-1} + \ldots + c_m \alpha_1 = \alpha_{m+1} ;$$

and so on, the general result being that

$$\sum_{\mu=1}^{m} a_\mu^{m+n} A_\mu = \alpha_{m+n},$$

for all values of n. Hence

$$w = \sum_{k=0}^{\infty} \alpha_k \frac{z^k}{k!}$$

$$= \sum_{k=0}^{\infty} (A_1 a_1^k + A_2 a_2^k + \ldots + A_m a_m^k) \frac{z^k}{k!}$$

$$= A_1 e^{a_1 z} + A_2 e^{a_2 z} + \ldots + A_m e^{a_m z},$$

the customary form of the solution, $A_1, \ldots, A_m$ being m independent arbitrary constants.

Ex. 7. Apply the preceding method to obtain a similar expression in finite terms, when the roots of the equation $\phi(\theta)=0$ are not all different from one another.

6. A different method of discussing the linear equation with constant coefficients has been given by Hermite.

Taking the equation, as before, in the form

$$\frac{d^m w}{dz^m} = c_1 \frac{d^{m-1} w}{dz^{m-1}} + c_2 \frac{d^{m-2} w}{dz^{m-2}} + \ldots + c_m w,$$

we associate with it the expression

$$\phi(\zeta) = \zeta^m - (c_1 \zeta^{m-1} + c_2 \zeta^{m-2} + \ldots + c_m).$$

Denoting by $f(\zeta)$ any polynomial in ζ, let

$$W = \frac{1}{2i\pi} \int e^{z\zeta} \frac{f(\zeta)}{\phi(\zeta)} d\zeta,$$

integration being taken round any simple contour in the ζ-plane.

In the first place, the degree of the polynomial $f(\zeta)$ may be taken to be less than m. If initially it is not so, then we have

$$\frac{f(\zeta)}{\phi(\zeta)} = g(\zeta) + \frac{f_1(\zeta)}{\phi(\zeta)},$$

on division, $g(\zeta)$ being a polynomial, and $f_1(\zeta)$ a polynomial of order less than that of ϕ, that is, less than m. Now

$$\int e^{z\zeta} g(\zeta)\, d\zeta = 0,$$

round any simple contour in the ζ-plane; in the remaining integral, the polynomial is of the form indicated. Accordingly, $f(\zeta)$ will be assumed to be of order less than m.

We have

$$\frac{d^r W}{dz^r} = \frac{1}{2i\pi}\int e^{z\zeta} \frac{f(\zeta)}{\phi(\zeta)} \zeta^r\, d\zeta, \qquad (r = 0, 1, 2, \ldots),$$

taken round the same contour; so that

$$\begin{aligned} &\frac{d^m W}{dz^m} - \left(c_1 \frac{d^{m-1} W}{dz^{m-1}} + c_2 \frac{d^{m-2} W}{dz^{m-2}} + \ldots + c_m W\right) \\ &\quad = \frac{1}{2i\pi}\int e^{z\zeta} \frac{f(\zeta)}{\phi(\zeta)} \{\zeta^m - (c_1\zeta^{m-1} + c_2\zeta^{m-2} + \ldots + c_m)\}\, d\zeta \\ &\quad = \frac{1}{2i\pi}\int e^{z\zeta} f(\zeta)\, d\zeta \\ &\quad = 0, \end{aligned}$$

because $f(\zeta)$ is a polynomial and the integral is taken round a simple contour in the ζ-plane. Thus W is a solution of the equation.

The only restriction upon $f(\zeta)$ is that, effectively, its degree must be less than m. It may therefore be taken as the most general polynomial of degree $m - 1$; in this form, it will contain m disposable coefficients which can be used to satisfy the initial conditions. Let these conditions require that, when $x = 0$, the variable w and its first $m - 1$ derivatives acquire values $k_0, k_1, \ldots, k_{m-1}$ respectively; then we determine $f(\zeta)$ as follows. Since

$$\left(\frac{d^r W}{dz^r}\right)_{z=0} = k_r = \frac{1}{2i\pi}\int \frac{f(\zeta)}{\phi(\zeta)} \zeta^r d\zeta,$$

we shall draw the simple contour in the ζ-plane so as to enclose the origin; and then the preceding relation shews that, when

$\dfrac{f(\zeta)}{\phi(\zeta)}$ is expanded in descending powers of ζ, the coefficient of ζ^{-r-1} is k_r; so that, as it holds for $r = 0, 1, \ldots, m-1$, we have

$$\frac{f(\zeta)}{\phi(\zeta)} = \frac{k_0}{\zeta} + \frac{k_1}{\zeta^2} + \ldots + \frac{k_{m-1}}{\zeta^m} + \ldots,$$

and therefore

$$f(\zeta) = \phi(\zeta)\left\{\frac{k_0}{\zeta} + \frac{k_1}{\zeta^2} + \ldots + \frac{k_{m-1}}{\zeta^m} + \ldots\right\}.$$

As $f(\zeta)$ is a polynomial in ζ, all terms involving negative powers of ζ must disappear, when multiplication is effected on the right-hand side; and therefore

$$f(\zeta) = \sum_{r=0}^{m-1} k_r \{\zeta^{m-r-1} - (c_1\zeta^{m-r-2} + \ldots + c_{m-r-1})\},$$

the coefficient of k_{m-1} being unity. If therefore w and its first s derivatives are all to acquire the value zero when $z = 0$, then the degree of the polynomial $f(\zeta)$ is $m - s - 2$.

In order to obtain the customary expression for W, let the contour be chosen so as to include all the zeros of $\phi(\zeta)$. Let α_1 be a zero, and let its multiplicity be n_1, so that

$$\phi(\zeta) = (\zeta - \alpha_1)^{n_1}\,\phi_1(\zeta),$$

where the roots of $\phi_1(\zeta)$ are the other roots of $\phi(\zeta)$. Let

$$\frac{f(\zeta)}{\phi(\zeta)} = \frac{A'_{11}}{\zeta - \alpha_1} + \frac{A'_{21}}{(\zeta - \alpha_1)^2} + \ldots + \frac{A'_{n_1 1}}{(\zeta - \alpha_1)^{n_1}} + \frac{f_1(\zeta)}{\phi_1(\zeta)},$$

$A'_{11}, A'_{21}, \ldots,$ being constants, and $f_1(\zeta)$ a polynomial of order $m - n_1 - 1$. So far as the first n_1 terms are concerned, their contribution to the value of the expression for W is given by taking a contour round α_1 only. We then have

$$\begin{aligned}\frac{1}{2\pi i}\int e^{z\zeta}\frac{A'_{r1}}{(\zeta - \alpha_1)^r}\,d\zeta &= \frac{A'_{r1}}{(r-1)!}\frac{d^{r-1}}{d\alpha_1^{r-1}}(e^{z\alpha_1}) \\ &= \frac{A'_{r1}}{(r-1)!}z^{r-1}e^{z\alpha_1} \\ &= A_{r1}z^{r-1}e^{z\alpha_1},\end{aligned}$$

on changing the constants; and therefore the part, arising through the root α_1 of multiplicity n_1, in the expression for the integral is

$$(A_{11} + A_{21}z + \ldots + A_{n_1 1}z^{n_1 - 1})\,e^{z\alpha_1},$$

involving a number of constants equal to the multiplicity of the root. This form holds for each root in turn; and therefore the number of constants is the sum of the multiplicities, that is, it is equal to m, the degree of $\phi(\zeta)$. But m is the number of arbitrary constants in $f(\zeta)$, when it is initially chosen: these can therefore be replaced by the constants A in the expression

$$\Sigma (A_1 + A_2 z + \ldots + A_n z^{n-1}) e^{\alpha z},$$

the summation extending over the roots α of $\phi(\zeta) = 0$, and n denoting the multiplicity of α. The simplest case, of course, occurs when all the roots of $\phi(\zeta) = 0$ are different from one another.

The method can be applied to the equation

$$\frac{d^m w}{dz^m} - \left(c_1 \frac{d^{m-1} w}{dz^{m-1}} + \ldots + c_m w\right) = F(z),$$

where $F(z)$ is any function of z. Consider

$$W = \int e^{z\zeta} \frac{f(z, \zeta)}{\phi(\zeta)} d\zeta,$$

where $\phi(\zeta)$ has the same significance as before, $f(z, \zeta)$ is a polynomial in ζ with (unknown) functions of z as coefficients of the powers of ζ, and integration extends round a simple contour that includes all the roots of $\phi(\zeta) = 0$. Then

$$\frac{dW}{dz} = \int e^{z\zeta} \zeta \frac{f(z, \zeta)}{\phi(\zeta)} d\zeta,$$

provided

$$\int \frac{e^{z\zeta}}{\phi(\zeta)} \frac{\partial}{\partial z} f(z, \zeta)\, d\zeta = 0;$$

also

$$\frac{d^2 W}{dz^2} = \int e^{z\zeta} \zeta^2 \frac{f(z, \zeta)}{\phi(\zeta)} d\zeta,$$

provided

$$\int \frac{e^{z\zeta}}{\phi(\zeta)} \zeta \frac{\partial}{\partial z} f(z, \zeta)\, d\zeta = 0;$$

and so on in succession, until we have

$$\frac{d^{m-1} W}{dz^{m-1}} = \int e^{z\zeta} \zeta^{m-1} \frac{f(z, \zeta)}{\phi(\zeta)} d\zeta,$$

provided

$$\int \frac{e^{z\zeta}}{\phi(\zeta)} \zeta^{m-2} \frac{\partial}{\partial z} f(z, \zeta)\, d\zeta = 0.$$

Then

$$\frac{d^m W}{dz^m} = \int e^{z\zeta} \zeta^m \frac{f(z, \zeta)}{\phi(\zeta)} d\zeta + \int \frac{e^{z\zeta}}{\phi(\zeta)} \zeta^{m-1} \frac{\partial}{\partial z} f(z, \zeta)\, d\zeta.$$

Hence, remembering that $f(z, \zeta)$ is a polynomial in ζ and that therefore

$$\int e^{z\zeta} f(z, \zeta)\, d\zeta = 0,$$

we have W as a solution of the given equation if, in addition to the other conditions, which are that

$$\int \frac{e^{z\zeta}}{\phi(\zeta)} \zeta^{m-r} \frac{\partial}{\partial z} f(z, \zeta)\, d\zeta = 0,$$

for $r = 2, 3, \ldots, m$, we have

$$\int \frac{e^{z\zeta}}{\phi(\zeta)} \zeta^{m-1} \frac{\partial}{\partial z} f(z, \zeta)\, d\zeta = F(z).$$

Now as the contour embraces all the roots of $\phi(\zeta)$, we have*

$$\int \frac{\zeta^{m-r}}{\phi(\zeta)}\, d\zeta = 0,$$

for $r = 2, \ldots, m$; so that, taking

$$\frac{\partial}{\partial z} f(z, \zeta) = \theta(z)\, e^{-z\zeta},$$

where $\theta(z)$ is a function of z at our disposal, we satisfy the $m-1$ formal conditions unconnected with $F(z)$; and then $\theta(z)$ must be such that

$$\int \frac{\zeta^{m-1}}{\phi(\zeta)} \theta(z)\, d\zeta = F(z).$$

But as

$$\phi(\zeta) = \zeta^m - (c_1 \zeta^{m-1} + \ldots + c_m),$$

we have†

$$\int \frac{\zeta^{m-1}}{\phi(\zeta)}\, d\zeta = 2\pi i;$$

and therefore

$$\theta(z) = \frac{1}{2\pi i} F(z).$$

Hence

$$\frac{\partial}{\partial z} f(z, \zeta) = \frac{1}{2\pi i} e^{-z\zeta} F(z),$$

so that

$$f(z, \zeta) = g(\zeta) + \frac{1}{2\pi i} \int^z e^{-u\zeta} F(u)\, du,$$

where $g(\zeta)$ is, so far as concerns this mode of determining $f(z, \zeta)$, any function of ζ, and integration with regard to u is along any path that ends in z. When $F(z)$ is zero, $f(z, \zeta)$ reduces to $g(\zeta)$; and then the solution of the differential equation shews that $g(\zeta)$ is a polynomial in ζ, of degree not higher than $m-1$. Accordingly, as $g(\zeta)$ is independent of z, we take it to be a polynomial of degree $m-1$ in ζ, with arbitrary constants for the coefficients; and then the integral of the equation has the form

$$W = \int e^{z\zeta} \frac{g(\zeta)}{\phi(\zeta)}\, d\zeta + \frac{1}{2\pi i} \int \frac{d\zeta}{\phi(\zeta)} \int^z F(u)\, e^{(z-u)\zeta} du,$$

* *T. F.*, § 24, III.

† *T. F.*, § 24, III, Cor.

where the ζ-integration extends round any simple contour including all the roots of $\phi(\zeta)=0$, and the u-integration extends from any arbitrary initial point along any path (the simpler the better) to z.

The single integral in the expression for W is clearly the complementary function, and the double integral is the particular integral, in the primitive of the differential equation. The expression can be developed into the customary form, in the same way as in the simpler case when $F(z)$ vanishes.

Hermite's investigation, based upon Cauchy's treatment by the calculus of residues as expounded in the *Exercices de Mathématiques*, is given in a memoir in Darboux's *Bull. des Sciences Math.*, 2^{me} Sér. t. III (1879), pp. 311—325: it is followed by a brief note (*l. c.*, pp. 325—328), due to Darboux. A memoir by Collet, *Ann. de l'Éc. Norm. Sup.*, 3^{me} Sér. t. IV (1887), pp. 129—144, may also be consulted.

The Process of Continuation applied to the Synectic Integral.

7. The synectic integral $P(z-\zeta)$ is known at all points in the domain of ζ, being uniquely determined by the assigned initial conditions at ζ. So long as the variable remains within this domain, the integral at z does not depend upon the path of passage from ζ to z, so that the path from ζ to z can be deformed at will, provided it remains always within the domain. Let ζ' be any point in the domain; then the values of the integral and its first $m-1$ derivatives at ζ' are uniquely determined by the initial conditions at ζ, and they can themselves be taken as a new set of initial conditions for a new origin ζ'. Accordingly, construct the domain of ζ'; and, with the values at ζ' taken as a new set of initial values, form the synectic integral which they determine. As the new initial values are themselves dependent upon the initial values at ζ, the synectic integral in the domain of ζ' may be denoted by $P_1(z-\zeta', \zeta)$.

If the domain of ζ' lies entirely within that of ζ (it then will touch the boundary of the domain of ζ internally), the series $P_1(z-\zeta', \zeta)$ must give the same value as $P(z-\zeta)$: for every point z in the domain of ζ' is then within the domain of ζ, and it is known that the synectic integral is unique within the original domain.

If part of the domain of ζ' lies without that of ζ, then in the remainder (which is common to the two domains) the series P_1 must give the same value as P. But in that part which is outside, the series P_1 defines a synectic integral in a region where

P does not exist; it therefore extends our knowledge of the integral, and it is a continuation of the synectic integral out of the original domain.

Let Z be any point in the plane; and join Z to ζ by any curve, drawn so as not to approach infinitesimally near any of the singularities of the coefficients in the differential equation. Beginning with ζ, construct the domains of a succession of points along this curve, choosing the points so that each lies in the domain of a preceding point and each new domain includes some portion of the plane not included by any previous domain. Owing to the way in which the curve is drawn, this choice is always possible and, after the construction of a limited number of domains, it will bring Z within a selected region. With each domain we associate its own series: so that there is a succession of series, each contributing a continuation of its predecessor. We can thus obtain at Z a synectic integral of the equation, which is uniquely determined by the initial values at ζ and by the path from ζ to Z.

Further, taking the values of the integral and its first $m-1$ derivatives at Z as a set of new initial values, and taking the preceding curve reversed as a path from Z to ζ, we obtain at ζ the original set of assigned initial values. To establish this statement, it is sufficient to choose the succession of points along the curve in the preceding construction, so that the centre of any domain lies within the succeeding domain, and to pass back from centre to centre. Stating the proposition briefly, we may say that *the reversal of any path restores the initial values.*

By imagining all possible paths drawn from any initial point ζ to all possible points z that are not singular, we can construct the whole region of continuity of the integral, as defined by the differential equation and by the initial values arbitrarily assigned at ζ: moreover, we shall thus have deduced all possible values of the integral at z, as determined by the initial values at ζ. It is clear, from the construction of the domain of any point and after the establishment of a synectic integral in that domain, which can be continued outside the domain (unless the boundary of the domain is a line of singularity, and this has been assumed not to be the case), that the region of continuity of the integral is bounded by the singularities of the coefficients. As has already

been remarked, these singularities are called the *singularities of the equation.* Thus all the critical points of the integral are fixed points; and if the equation be taken in the form

$$q_0 \frac{d^m w}{dz^m} = q_1 \frac{d^{m-1} w}{dz^{m-1}} + \ldots + q_m w,$$

where the functions $q_0, \ldots, q_m$ are holomorphic over the finite part of the plane and have no common factor, these critical points are included among the roots of q_0, with possibly $z = \infty$ also as a critical point. The value of the integral at an ordinary point near a singularity has been obtained as a synectic function valid over the domain of the point, which excludes the singularity. In later investigations, other expressions for the integral at the point will be determined, when the point belongs to a different domain that includes the singularity.

8. Any path from ζ to z can be deformed in an unlimited number of ways: and it is not inconceivable that these deformations should lead to an unlimited number of values of the integral at z, as determined by a given set of initial values: but the number is not completely unlimited, because *all paths from ζ to z lead to the same final value at z with a given set of initial values at ζ, provided they are deformable into one another without crossing any of the singularities.* To prove this, consider a path from ζ to z, drawn so that no point of it is within an infinitesimal distance of a singularity, and draw a second path between the same two points obtained by an infinitesimal deformation of the first; no point of the second path can therefore be within an infinitesimal distance of a singularity. On the first path, take a succession of points $z_1, z_2, \ldots$, so that z_1 lies within the domains of ζ and of z_2, z_2 within the domains of z_1 and z_3, and so on. On the second path, take a similar succession of points $z_1', z_2', \ldots$, near $z_1, z_2, \ldots$ respectively, in such a way that z_1' lies in the part common to the domains of ζ and z_1, while z_1 is in the domain of z_1'; z_2' in the part common to the domains of z_1 and z_2, while z_2 is in the domain of z_2'; and so on. Join $z_1z_1', z_2z_2', \ldots$ by short arcs in the form of straight lines.

Now we have seen that, in any domain, the path from the centre to a point can be deformed without affecting the value of the integral at the point, provided every deformed path lies within

the domain. Hence in the domain of ζ, the path ζz_1 gives at z_1 the same integral as the path $\zeta z_1' z_1$. This integral furnishes a set of initial values for the domain of z_1; and then the path $z_1 z_2$ gives at z_2 the same integral as the path $z_1 z_1' z_2' z_2$. Consequently the path $\zeta z_1 z_2$ gives at z_2 the same integral as the path $\zeta z_1' z_1$, followed by $z_1 z_1' z_2' z_2$. But the effect of $z_1' z_1$ followed at once by $z_1 z_1'$ is nul, because a reversed path restores the values at the beginning of the path; and therefore the path $\zeta z_1 z_2$ gives at z_2 the same integral as the path $\zeta z_1' z_2' z_2$. And so on, from portion to portion: the last point on the first path is z, which also is the last point on the second path; and therefore the path $\zeta z_1 z_2 \ldots z$ gives at z the same integral as the path $\zeta z_1' z_2' \ldots z$.

Now take any two paths between ζ and z, such that the closed contour formed by them encloses no singularity of the equation. Either of them can be changed into the other by a succession of infinitesimal deformations: each intermediate path gives at z the same integral as its immediate predecessor: and therefore the initial path and the final path from ζ to z give the same integral at z; which is the required result.

If however two paths between ζ and z are such that the closed contour formed by them encloses a singularity of the equation, then at some stage in the intermediate deformation the curve will pass through the singularity, and we cannot infer the continuation along the curve or the deformation into a consecutive curve as above. It may or may not be the case that the two paths from ζ to z give at z one and the same integral determined by a given set of initial values; but we cannot assert that it is the case.

Accordingly, we may deform a given path without affecting the integral at the final point, provided no singularity is crossed in the process. Moreover, in order to take account of different paths not so deformable into one another, it will be necessary to consider the relation of the singularities to the function representing the integral: this will be effected in a later investigation.

When two paths can be deformed into one another, without crossing any singularity, they are called *reconcileable*; when they cannot so be deformed, they are called *irreconcileable*. If two irreconcileable paths lead at z to different integrals from the same initial values at ζ, the closed circuit made up of the two paths leads at ζ to a set of values different from the initial values.

These new values can be taken as a new set of initial values: when the same circuit is described, they are not restored, so that either the old initial values or a further set of values will be obtained: and so on, for repeated descriptions of the circuit. By this process, we may obtain any number, perhaps even an unlimited number, of sets of values at ζ deduced from a given initial set; and thus there may be any number, perhaps even an unlimited number, of values of the integral at any point z.

Consider any path from ζ to z; and without crossing any of the singularities, let it be deformed into loops, drawn from ζ to the singularities and back, (these loops coming in appropriate succession), followed by a simple path (say a straight line) from ζ to z. The final value of the integral at z is determined by the values at ζ at the beginning of the straight line, and these values are deducible from the initial values originally assigned. Hence *the generality of the integral at z is not affected by taking any particular path from ζ to z, provided complete generality be reserved for the initial values:* and therefore, from this aspect, it will be sufficient to discuss the complete system of integrals as arising from completely arbitrary systems of initial values at an ordinary point. This investigation relates to properties of the integrals, which will be found useful in discussing the effect of a singularity upon a given integral; it will accordingly be undertaken at once.

9. It has already been remarked that the synectic integral, determined by the arbitrary constants which are assigned as the initial values of the function and its derivatives, is linear and homogeneous in those constants: so that, if $\mu_{11}, \mu_{12}, \ldots, \mu_{1m}$ denote the arbitrary constants, and w_1 denotes the synectic integral which they determine in the domain of an ordinary point ζ, we have

$$w_1 = \mu_{11}u_1 + \mu_{12}u_2 + \ldots + \mu_{1m}u_m,$$

where $u_1, u_2, \ldots, u_m$ are holomorphic functions of $z - \zeta$, not involving any of the arbitrary coefficients μ. Take other $m-1$ sets of arbitrary constants μ, such that the determinant

$$\begin{vmatrix} \mu_{11}, & \mu_{12}, & \ldots, & \mu_{1m} \\ \mu_{21}, & \mu_{22}, & \ldots, & \mu_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ \mu_{m1}, & \mu_{m2}, & \ldots, & \mu_{mm} \end{vmatrix}, \ = \Delta(\zeta) \text{ say},$$

is different from zero. Each set of m constants, regarded as a set of initial values, determines a synectic integral in the domain of ζ; as the quantities $u_1, u_2, \ldots, u_m$ in the expression for w_1 do not involve the arbitrary constants determining w_1, it is clear that the expressions for these other $m-1$ integrals are

$$w_s = \mu_{s1}u_1 + \mu_{s2}u_2 + \ldots + \mu_{sm}u_m, \qquad (s = 2, \ldots, m).$$

Let M_{st} denote the minor of μ_{st} in the non-vanishing determinant $\Delta(\zeta)$; then from the expressions for the m integrals $w_1, \ldots, w_m$ in terms of $u_1, \ldots, u_m$, we have

$$\Delta(\zeta)\, u = M_{1t}w_1 + M_{2t}w_2 + \ldots + M_{mt}w_m, \qquad (t = 1, \ldots, m).$$

Now any other synectic integral, determined in the domain of ζ by assigned initial values $\theta_1, \theta_2, \ldots, \theta_m$, is given by

$$\begin{aligned} w &= \theta_1 u_1 + \theta_2 u_2 + \ldots + \theta_m u_m \\ &= \vartheta_1 w_1 + \vartheta_2 w_2 + \ldots + \vartheta_m w_m, \end{aligned}$$

where the constants ϑ are given by

$$\vartheta_r = \frac{1}{\Delta(\zeta)} \sum_{t=1}^{m} \theta_t M_{rt}, \qquad (r = 1, \ldots, m).$$

These constants ϑ cannot all vanish, when the constants $\theta_1, \theta_2, \ldots, \theta_m$ are not simultaneous zeros: for the determinant of the minors M_{rt} is $\{\Delta(\zeta)\}^{m-1}$, and therefore is not zero. Accordingly, any integral can be expressed as a linear combination of any m integrals, provided the determinant of the initial values of those m integrals and their first $m-1$ derivatives does not vanish. But it is not yet clear that the integrals $w_1, \ldots, w_m$ are linearly independent of one another; until this property is established, we cannot affirm that the expression obtained is the simplest obtainable.

Consider therefore, more generally, the determinant of the m integrals and their first $m-1$ derivatives, not solely at ζ but for any value of z in the domain of ζ, say

$$\Delta(z) = \begin{vmatrix} \dfrac{d^{m-1}w_1}{dz^{m-1}}, & \dfrac{d^{m-2}w_1}{dz^{m-2}}, & \ldots, & w_1 \\ \dfrac{d^{m-1}w_2}{dz^{m-1}}, & \dfrac{d^{m-2}w_2}{dz^{m-2}}, & \ldots, & w_2 \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{d^{m-1}w_m}{dz^{m-1}}, & \dfrac{d^{m-2}w_m}{dz^{m-2}}, & \ldots, & w_m \end{vmatrix}.$$

When $z=\zeta$, it becomes the determinant of initial values denoted by $\Delta(\zeta)$. We have

$$\frac{d\Delta(z)}{dz}=\begin{vmatrix} \dfrac{d^m w_1}{dz^m}, & \dfrac{d^{m-2}w_1}{dz^{m-2}}, & \dots, & w_1 \\ \dfrac{d^m w_2}{dz^m}, & \dfrac{d^{m-2}w_2}{dz^{m-2}}, & \dots, & w_2 \\ \dots & \dots & \dots & \dots \\ \dfrac{d^m w_m}{dz^m}, & \dfrac{d^{m-2}w_m}{dz^{m-2}}, & \dots, & w_m \end{vmatrix}$$

$$=p_1\Delta(z),$$

on substituting for $\dfrac{d^m w_1}{dz^m}, \dots, \dfrac{d^m w_m}{dz^m}$ their values in terms of the derivatives of lower orders as given by the equation. Hence

$$\Delta(z)=\Delta(\zeta)\,e^{\int_\zeta^z p_1 dz}.$$

Now within the domain of ζ, the function p_1 is regular, being of the form $P_1(z-\zeta)$; hence the integral in the exponent of e is of the form $R(z-\zeta)$, where R is a regular function that vanishes when $z=\zeta$. Consequently the exponential term on the right-hand side does not vanish at any point in the domain of ζ; also $\Delta(\zeta)$ is not zero; so that $\Delta(z)$ has no zero within the domain of ζ. Moreover, each of the quantities $w_1, \dots, w_m$ is a holomorphic function of $z-\zeta$ in that domain, so that $\Delta(z)$ is holomorphic also; hence $\Delta(z)$ has no zero and no infinity within the domain of the ordinary point ζ.

As a matter of fact, the only points where $\Delta(z)$ may vanish or may become infinite are the singularities of p_1. For in any region of common existence of the functions $w_1, \dots, w_m$, we have

$$\frac{\Delta(z)}{\Delta(\zeta)}=e^{\int_\zeta^z p_1 dz},$$

the path from ζ to z lying within that region, while z is not now necessarily in the domain of ζ. If a be one of the singularities of p_1, the expression of p_1 in any part of an annular region round a as centre is of the form

$$p_1=g'(z)+\frac{a_1}{z-a}+\frac{a_2}{(z-a)^2}+\dots,$$

where the number of terms in negative powers of $z-a$ is finite or infinite, according as the singularity is accidental or essential; and $g'(z)$ is holo-

morphic in the vicinity of a. Taking the simplest case as an instance, let $a_2 = a_3 = \ldots = 0$; then

$$e^{\int_\zeta^z p_1 dz} = \left(\frac{z-a}{\zeta-a}\right)^{a_1} e^{g(z)-g(\zeta)},$$

shewing that a is a zero of $\Delta(z)$ if the real part of a_1 be positive, and that it is an infinity of $\Delta(z)$ if the real part of a_1 be negative. More generally, the nature of $\Delta(z)$ in the vicinity of any singularity a depends upon the character of p_1 in that vicinity: in the case of the above more general form, a is an essential singularity of $\Delta(z)$.

Fundamental Systems of Integrals.

10. The linear independence of $w_1, \ldots, w_m$, and the property that $\Delta(z)$ has a finite non-zero value at any point in the plane which is not a singularity of the equation, are involved each in the other.

It is easily seen that, if a homogeneous linear relation between $w_1, \ldots, w_m$ of the form

$$c_1 w_1 + \ldots + c_m w_m = 0$$

were to exist, the quantities $c_1, \ldots, c_m$ being constants, then $\Delta(z)$ would vanish for all values of z. The inference is at once established by forming the $m-1$ derived equations

$$c_1 \frac{d^r w_1}{dz^r} + \ldots + c_m \frac{d^r w_m}{dz^r} = 0, \quad (r = 1, \ldots, m-1),$$

and eliminating the m constants $c_1, \ldots, c_m$ between the m equations which involve them linearly; the result of the elimination is

$$\Delta(z) = 0.$$

Hence if, for any set of integrals $w_1, \ldots, w_m$, the determinant $\Delta(z)$ does not vanish (except possibly at the singularities of the equation), no homogeneous linear relation between the integrals exists.

To establish the inference that, if $\Delta(z)$ does vanish for all values of z, a homogeneous linear relation between $w_1, \ldots, w_m$ exists, we proceed as follows.

In the first place, suppose that some minor of a constituent in the first column of $\Delta(z)$, e.g. the minor of $\dfrac{d^{m-1} w_m}{dz^{m-1}}$ in $\Delta(z)$, say

$\Delta_1(z)$, does not vanish for all ordinary values of z; and take m quantities $y_1, \ldots, y_m$, the ratios of which are defined by the relations

$$\begin{aligned}
y_1 w_1 + \ldots + y_m w_m &= 0, \\
y_1 \frac{dw_1}{dz} + \ldots + y_m \frac{dw_m}{dz} &= 0, \\
\cdots\cdots\cdots\cdots\cdots\cdots&\cdots\cdots \\
y_1 \frac{d^{m-2}w_1}{dz^{m-2}} + \ldots + y_m \frac{d^{m-2}w_m}{dz^{m-2}} &= 0.
\end{aligned}$$

From the hypotheses that $\Delta(z) = 0$ and that Δ_1 does not vanish, it follows that

$$y_1 \frac{d^{m-1}w_1}{dz^{m-1}} + \ldots + y_m \frac{d^{m-1}w_m}{dz^{m-1}} = 0.$$

Because of the assumption that Δ_1 does not vanish, the ratios

$$\frac{y_1}{y_m}, \; \frac{y_2}{y_m}, \; \ldots, \; \frac{y_{m-1}}{y_m}$$

are determinate finite functions of z.

Differentiate the first of the relations: then, using the second, we have

$$y_1' w_1 + \ldots + y_m' w_m = 0,$$

where y_r' denotes dy_r/dz, for the n values of r. Differentiating the second of the relations, and using the third, we have

$$y_1' \frac{dw_1}{dz} + \ldots + y_m' \frac{dw_m}{dz} = 0;$$

and so on, up to

$$y_1' \frac{d^{m-2}w_1}{dz^{m-2}} + \ldots + y_m' \frac{d^{m-2}w_m}{dz^{m-2}} = 0,$$

obtained by differentiating the last of the postulated relations and by using the deduced relation. We thus have $m-1$ relations, homogeneous and linear in the quantities $y_1', \ldots, y_m'$; in form, they are precisely the same as the $m-1$ relations, which are homogeneous and linear in the quantities $y_1, \ldots, y_m$. Hence, as Δ_1 does not vanish, we have

$$\frac{y_r}{y_m} = \frac{y_r'}{y_m'}, \qquad (r = 1, 2, \ldots, m-1),$$

that is,

$$\frac{d}{dz}\left(\frac{y_r}{y_m}\right) = 0,$$

so that

$$\frac{y_r}{y_m} = \text{constant} = \frac{\lambda_r}{\lambda_m}, \quad (r = 1, 2, \ldots, m-1),$$

where $\lambda_1, \ldots, \lambda_{m-1}, \lambda_m$ are simultaneous values of $y_1, \ldots, y_{m-1}, y_m$ for any particular value of z: that is, the quantities λ are constants. This particular value of z is at our disposal; we may assume that λ_m is different from zero, because the ratios of $y_1, \ldots, y_{m-1}$ to y_m are determinate and finite. Now

$$y_1 w_1 + \ldots + y_m w_m = 0;$$

hence

$$\lambda_1 w_1 + \ldots + \lambda_m w_m = 0,$$

that is, a linear relation exists among the quantities w, if $\Delta(z)$ is zero, and some minor of a constituent in the first column does not vanish.

Next, suppose that the minor of every constituent in the first column vanishes: in particular, let $\Delta_1(z) = 0$, for all ordinary values of z. Then $\Delta_1(z)$ is a determinant of $m-1$ rows and columns, constructed from $m-1$ quantities $w_1, \ldots, w_{m-1}$ in the same way as $\Delta(z)$, a determinant of m rows and columns, is constructed from the m quantities $w_1, \ldots, w_m$. The preceding analysis shews that, if some minor of a constituent in the first column of $\Delta_1(z)$ does not vanish for all ordinary values of z, then a relation

$$\kappa_1 w_1 + \ldots + \kappa_{m-1} w_{m-1} = 0,$$

where $\kappa_1, \ldots, \kappa_{m-1}$ are constants, is satisfied: so that a linear relation exists among the quantities w, and it happens not to involve w_m.

Let the process of passing from $\Delta(z)$ to $\Delta_1(z)$, from $\Delta_1(z)$ to a corresponding minor, and so on, be continued: the successive steps are effected by removing the successive columns in $\Delta(z)$ beginning from the left and by removing a corresponding number of rows. At some stage, we must reach some minor which is not zero for all ordinary values of z: so that

$$\begin{vmatrix} \dfrac{d^{m-s-1}w_1}{dz^{m-s-1}}, & \ldots, & w_1 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \dfrac{d^{m-s-1}w_{m-s}}{dz^{m-s-1}}, & \ldots, & w_{m-s} \end{vmatrix}$$

vanishes when $s=0, 1, \dots, r$, but is different from zero when $s=r+1$. Then the earlier analysis shews that a linear relation of the form

$$\rho_1 w_1 + \dots + \rho_{m-s} w_{m-s} = 0$$

exists, where $\rho_1, \dots, \rho_{m-s}$ are constants: in effect, a linear homogeneous relation among the quantities $w_1, \dots, w_m$ which happens not to involve $w_{m-s+1}, \dots, w_m$. Hence, *if the determinant* $\Delta(z)$, *constructed from the* m *integrals* $w_1, \dots, w_m$, *vanishes for all ordinary values of* z, *there is a homogeneous linear relation between these integrals.*

Integrals are sometimes called *independent* when they are linearly independent, that is, connected by no homogeneous linear relation; but the independence is not functional, because all the integrals are functions of the one variable z. A set of m linearly independent integrals w is called a *fundamental system*; and each integral of the set is called an *element* or a *member* of the system. The determinant $\Delta(z)$, constructed out of a set of m integrals, is called *the determinant of the system*; so that the preceding results may be stated in the form:—

If the determinant of a set of m *integrals vanishes for ordinary (that is, non-singular) values of the variable, the set cannot constitute a fundamental system; and the determinant of a fundamental system does not vanish for any non-singular value of the variable.*

11. We now have the important proposition:—

Every integral, which is determined by assigned initial values, can be expressed as a homogeneous linear combination of the elements of a fundamental system.

Let W denote the integral determined by the assigned values at ζ, taken to be an ordinary point of all the coefficients in the differential equation; and let $w_1, \dots, w_m$ be a fundamental system. Let constants $c_1, \dots, c_m$ be deduced such that, when $z=\zeta$, we have

$$\left.\begin{aligned} W &= \sum_{\lambda=1}^{m} c_\lambda w_\lambda \\ \frac{dW}{dz} &= \sum_{\lambda=1}^{m} c_\lambda \frac{dw_\lambda}{dz} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{d^{m-1}W}{dz^{m-1}} &= \sum_{\lambda=1}^{m} c_\lambda \frac{d^{m-1}w_\lambda}{dz^{m-1}} \end{aligned}\right\}.$$

This deduction is uniquely possible; because the determinant of the quantities c on the right-hand sides is the determinant of a fundamental system, and therefore does not vanish when $z=\zeta$. Thus $W-\sum_{\lambda=1}^{m} c_\lambda w_\lambda$ is an integral of the equation; this integral and its first $m-1$ derivatives vanish when $z=\zeta$; so that it vanishes everywhere (Cor. I, § 5), and therefore

$$W=\sum_{\lambda=1}^{m} c_\lambda w_\lambda,$$

the constants c being properly determined as above.

COR. I. *Between any $m+1$ branches of the general solution, there must be a homogeneous linear relation.* For if m of them be linearly independent, the remaining branch can be regarded as another integral: by the proposition, it is expressible linearly in terms of the other m.

COR. II. *Any system of integrals $u_1, \ldots, u_m$ is fundamental if no relation exists of the form*

$$A_1u_1+\ldots+A_mu_m=0,$$

where $A_1, \ldots, A_m$ are constants. For taking a fundamental system $w_1, \ldots, w_m$, we can express each of the solutions u in the form

$$u_r=a_{1r}w_1+\ldots+a_{mr}w_m, \qquad (r=1, 2, \ldots, m),$$

where the coefficients a are constants. If C denote the determinant of these m^2 coefficients, C must be different from zero: for otherwise, on solving the m equations to express w_1 in terms of $u_1, \ldots, u_m$, we should have a relation of the form

$$A_1u_1+\ldots+A_mu_m=Cw_1=0;$$

and no such relation can exist. If, then, $\Delta_u(z)$ denote the determinant of the set of integrals u, and if $\Delta_w(z)$ denote that of the fundamental system $w_1, \ldots, w_m$, we have

$$\Delta_u(z)=C\Delta_w(z),$$

by the properties of determinants. Now C does not vanish, nor does $\Delta_w(z)$ at any ordinary point in the plane; hence $\Delta_u(z)$ does not vanish at any ordinary point in the plane, and therefore $u_1, \ldots, u_m$ are a fundamental system of integrals.

The result may be stated also as follows: *If m integrals u be given by equations*

$$u_r = a_{1r}w_1 + \dots + a_{mr}w_m, \qquad (r = 1, \dots, m),$$

where the determinant of the coefficients a is not zero, and the integrals w are a fundamental system, then the system of integrals u is also fundamental.

12. One particular fundamental system for the differential equation can be obtained as follows. Let w_1 be a special integral of the equation, that is, an integral determined by any special set of initial conditions, and substitute

$$w = w_1 \int v\, dz$$

in the equation; then v is determined by the equation

$$\frac{d^{m-1}v}{dz^{m-1}} = q_1 \frac{d^{m-2}v}{dz^{m-2}} + \dots + q_{m-1}v,$$

where

$$q_1 = p_1 - \frac{m}{w_1}\frac{dw_1}{dz}.$$

Similarly, let v_1 be a special integral of this new equation, with the appropriate conditions; then substituting

$$v = v_1 \int u\, dz,$$

we find that the equation, which determines u, is of the form

$$\frac{d^{m-2}u}{dz^{m-2}} = r_1 \frac{d^{m-3}u}{dz^{m-3}} + \dots + r_{m-2}u,$$

where

$$r_1 = q_1 - \frac{m-1}{v_1}\frac{dv_1}{dz}.$$

And so on.

It is manifest that the quantities

$$w_1, \quad w_1 \int v_1 dz, \quad w_1 \int \left(v_1 \int u_1 dz\right) dz, \dots$$

are integrals of the original equation. Moreover, they constitute a fundamental system; for, otherwise, they would be linearly connected by a relation of the form

$$c_1 w_1 + c_2 w_1 \int v_1 dz + c_3 w_1 \int \left(v_1 \int u_1 dz\right) dz + \dots = 0,$$

that is,

$$c_1 + c_2 \int v_1 dz + c_3 \int \left(v_1 \int u_1 dz\right) dz + \dots = 0.$$

When this is differentiated, it gives

$$c_2 v_1 + c_3 v_1 \int u_1 dz + \ldots = 0,$$

that is,

$$c_2 + c_3 \int u_1 dz + \ldots = 0.$$

Effecting $m-1$ repetitions of this operation of differentiating and removing a non-zero factor, we find

$$c_m = 0$$

as the result at the last stage. Using this in connection with the equation at the last stage but one, we have

$$c_{m-1} = 0.$$

And so on, from the equations at the various stages, we find that all the coefficients c vanish. The homogeneous linear relation therefore does not exist: the system of integrals, obtained in the preceding manner, is a fundamental system.

As an immediate corollary from the analysis, we infer that

$$v_1, \quad v_1 \int u_1 dz, \ldots$$

constitute a fundamental system for the equation in v; and so for each of the equations in succession.

The determinant of this particular fundamental system is simple in expression. Denoting it by Δ, and denoting by Δ_1 the determinant of the fundamental system of the equation in v, we have, as in § 9,

$$\frac{1}{\Delta} \frac{d\Delta}{dz} = p_1,$$

$$\frac{1}{\Delta_1} \frac{d\Delta_1}{dz} = q_1 = p_1 - \frac{m}{w_1} \frac{dw_1}{dz},$$

so that

$$\frac{1}{\Delta} \frac{d\Delta}{dz} - \frac{1}{\Delta_1} \frac{d\Delta_1}{dz} = \frac{m}{w_1} \frac{dw_1}{dz};$$

hence

$$\frac{\Delta}{\Delta_1} = \lambda_1 w_1{}^m,$$

where λ_1 is a constant. Similarly, if Δ_2 denote the determinant of the fundamental system of the equation in u, we have

$$\frac{\Delta_1}{\Delta_2} = \lambda_2 v_1{}^{m-1};$$

and so on. The last determinant of all is the actual integral of the last of the equations; hence

$$\Delta = Cw_1^{\,m} v_1^{\,m-1} u_1^{\,m-2} \ldots,$$

where C is a constant. Moreover, Δ is the determinant of a particular system, so that C is a determinate constant. It is not difficult to prove that

$$\lambda_1 = (-1)^{m-1}, \quad \lambda_2 = (-1)^{m-2}, \ldots,$$

and therefore

$$C = (-1)^{\frac{1}{2}m(m-1)};$$

consequently,

$$\Delta = (-1)^{\frac{1}{2}m(m-1)} w_1^{\,m} v_1^{\,m-1} u_1^{\,m-2} \ldots.$$

Ex. Verify the last result, as to the form of Δ, in the case of

(i) Legendre's equation:

(ii) the equation of the hypergeometric series:

(iii) Bessel's equation.

CHAPTER II.

General Form and Properties of Integrals near a Singularity.

13. We have seen that, within the domain of an ordinary point, a synectic integral of a linear differential equation is uniquely determined by a set of assigned initial values; and that the said integral can be continued beyond that domain, remaining unique for all paths between the initial and the final values of the variable which are reconcileable with one another. When the variable is permitted to pass out of its initial domain though returning to it for a final value, or when two paths between the initial and the final values are not reconcileable, the various propositions that have been established are not necessarily valid under the modified hypothesis: it is therefore desirable to consider the influence of irreconcileable paths upon an integral, still more upon a set of fundamental integrals. Remembering that any path is deformable without affecting the integral if, in the deformation, it does not pass over a singularity, we shall manifestly obtain the effect of a singularity, that renders two paths irreconcileable, by making the variable describe a simple circuit, which passes from the point z round the singularity and returns to that point z, and which encloses no other singularity.

Let a be the singularity round which the simple closed circuit is completely described by the variable. Let $w_1, \ldots, w_m$ denote a fundamental system at z; and suppose that the effect of the

circuit is to change the m integrals into $w_1', \ldots, w_m'$ respectively. That the set of m new integrals thus obtained is a fundamental system can be seen as follows. If it were not a fundamental system, some relation of the form

$$\sum_{r=1}^{m} k_r w_r' = 0$$

would exist, with constant coefficients k, for all values of z in the immediate vicinity. In that case, the quantity $\Sigma\, k_r w_r'$ (which is an integral) is zero everywhere, together with all its derivatives, as it is continued with the variable moving in the ordinary part of the plane. Accordingly, let the integral be continued from z along the closed circuit reversed until it returns to z where, by what has been stated, it is zero. The effect of the reversal is (§ 7) to change w_r' into w_r: and so the integral after the reversed circuit has been described is $\Sigma\, k_r w_r$, so that we should have

$$\sum_{r=1}^{m} k_r w_r = 0,$$

contrary to the fact that $w_1, \ldots, w_m$ constitute a fundamental system. The initial hypothesis from which this result is deduced is therefore untenable: there is no homogeneous linear relation among the quantities $w_1', \ldots, w_m'$, which therefore form a fundamental system.

Since the system $w_1, \ldots, w_m$ is fundamental, each of the integrals $w_1', \ldots, w_m'$ is expressible linearly in terms of the elements of that system; so that we have equations of the form

$$w_s' = a_{s1} w_1 + \ldots + a_{sm} w_m, \qquad (s = 1, \ldots, m),$$

where the coefficients a are constants. As the system w_s' is fundamental, the determinant of these coefficients is different from zero: this being necessary in order to ensure the property that $w_1, \ldots, w_m$ are expressible linearly in terms of $w_1', \ldots, w_m'$, a fundamental system.

Take any arbitrary linear combination of the system, say

$$\rho_1 w_1 + \ldots + \rho_m w_m,$$

where the coefficients ρ are disposable constants; and denote this integral by u. When the variable describes the complete closed

circuit round the singularity, let u' denote the modified value of u, so that

$$\begin{aligned} u' &= \rho_1 w_1' + \dots + \rho_m w_m' \\ &= \rho_1 \sum_{r=1}^{m} \alpha_{1r} w_r + \dots + \rho_m \sum_{r=1}^{m} \alpha_{mr} w_r. \end{aligned}$$

It is conceivable that the coefficients ρ could be chosen so that the integral reproduces itself except as to a possible constant factor; a relation

$$u' = \theta u$$

would then be satisfied, θ being a constant quantity. This relation, in terms of $w_1, \dots, w_m$, is

$$\begin{aligned} \rho_1 \sum_{r=1}^{m} \alpha_{1r} w_r + \dots + \rho_m \sum_{r=1}^{m} \alpha_{mr} w_r \\ = \theta (\rho_1 w_1 + \dots + \rho_m w_m), \end{aligned}$$

which, as it involves only the members of a fundamental system linearly, must be an identity: the coefficients of $w_1, \dots, w_m$ must therefore be equal on the two sides. Hence we have

$$\left.\begin{array}{ll} \rho_1 (\alpha_{11} - \theta) + \rho_2 \alpha_{21} + \dots + \rho_m \alpha_{m1} & = 0 \\ \rho_1 \alpha_{12} + \rho_2 (\alpha_{22} - \theta) + \dots + \rho_m \alpha_{m2} & = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ \rho_1 \alpha_{1m} + \rho_2 \alpha_{2m} \qquad + \dots + \rho_m (\alpha_{mm} - \theta) & = 0 \end{array}\right\}.$$

If, therefore, θ be determined as a root of the equation

$$A = \begin{vmatrix} \alpha_{11} - \theta, & \alpha_{21} & , \dots, & \alpha_{m1} \\ \alpha_{12} & , \alpha_{22} - \theta, & \dots, & \alpha_{m2} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ \alpha_{1m} & , \alpha_{2m} & , \dots, & \alpha_{mm} - \theta \end{vmatrix} = 0,$$

the preceding relations then lead to values for the ratios of the constants ρ for each such root. It is to be noted that, in this equation, the term, which is independent of θ, does not vanish, for it is the determinant of the coefficients α; hence the equation has no zero root.

As the equation definitely possesses roots θ, it follows that integrals exist which, after a description of the simple contour round a, reproduce themselves save as to a constant factor. If it should happen that the constant factor is unity, then the effect of

description of the contour upon the integral is merely to leave it unchanged: in other words, such an integral is uniform in the vicinity of the singularity.

Properties of the Fundamental Equation.

14. The special significance of the equation, in relation to the singularity a, lies in the proposition that *the coefficients of the various powers of θ in $A = 0$ are independent of the fundamental system initially chosen for discussion.* To prove* the statement, it will be sufficient to shew that the same equation is obtained when another fundamental system is initially chosen. For this purpose, let $y_1, \ldots, y_m$ denote some other fundamental system; and suppose that, by the simple closed contour round a described by the variable, the members of the system become $y_1', \ldots, y_m'$ respectively. Then, as both these systems are fundamental, there are relations of the form

$$y_s' = \beta_{s1} y_1 + \ldots + \beta_{sm} y_m, \qquad (s = 1, \ldots, m),$$

where the determinant of the coefficients β is not zero. The equation $B = 0$, corresponding to $A = 0$ for the determination of the factor θ, is formed from the coefficients β in the same way as A from the coefficients α, so that the expression for B is

$$B = \begin{vmatrix} \beta_{11} - \theta, & \beta_{21} & , \ldots, & \beta_{m1} \\ \beta_{12} & , \beta_{22} - \theta, & \ldots, & \beta_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{1m} & , \beta_{2m} & , \ldots, & \beta_{mm} - \theta \end{vmatrix}.$$

Because each of the sets $w_1, \ldots, w_m$; $y_1, \ldots, y_m$; is a fundamental system, the members are connected by relations of the form

$$y_r = \gamma_{r1} w_1 + \gamma_{r2} w_2 + \ldots + \gamma_{rm} w_m, \qquad (r = 1, \ldots, m),$$

where the determinant of the coefficients, which may be denoted by Γ, is different from zero. The quantity

$$y_r - (\gamma_{r1} w_1 + \ldots + \gamma_{rm} w_m)$$

is zero everywhere in the vicinity of z; and it is an integral, which accordingly is zero everywhere in its continuations over the

* The proof adopted is due to Hamburger, *Crelle*, t. LXXVI (1873), pp. 113—125.

ordinary part of the plane. When it is continued along the simple contour round a, the variable returning to z, the integral is zero there; that is,

$$y_r' = \gamma_{r1} w_1' + \ldots + \gamma_{rm} w_m'.$$

Hence

$$\beta_{r1} y_1 + \ldots + \beta_{rm} y_m = \sum_{t=1}^{m} \sum_{s=1}^{m} \gamma_{rs} \alpha_{st} w_t,$$

and therefore

$$\sum_{t=1}^{m} \sum_{s=1}^{m} \beta_{rs} \gamma_{st} w_t = \sum_{t=1}^{m} \sum_{s=1}^{m} \gamma_{rs} \alpha_{st} w_t.$$

This relation involves only the members of a fundamental system linearly; hence it must be an identity. We therefore have

$$\begin{aligned} \sum_{s=1}^{m} \beta_{rs} \gamma_{st} &= \sum_{s=1}^{m} \gamma_{rs} \alpha_{st} \\ &= \delta_{rt}, \end{aligned}$$

say, the relation among the constants holding for all values of r and t. Now forming the product of the determinants Γ and A, we have

$$\Gamma A = \begin{vmatrix} \gamma_{11}, & \gamma_{12}, & \gamma_{13}, & \cdots \\ \gamma_{21}, & \gamma_{22}, & \gamma_{23}, & \cdots \\ \gamma_{31}, & \gamma_{32}, & \gamma_{33}, & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} \begin{vmatrix} \alpha_{11} - \theta, & \alpha_{21} \quad, & \alpha_{31} \quad, & \cdots \\ \alpha_{12} \quad, & \alpha_{22} - \theta, & \alpha_{32} \quad, & \cdots \\ \alpha_{13} \quad, & \alpha_{23} \quad, & \alpha_{33} - \theta, & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

$$= \begin{vmatrix} \delta_{11} - \gamma_{11}\theta, & \delta_{12} - \gamma_{12}\theta, & \cdots \\ \delta_{21} - \gamma_{21}\theta, & \delta_{22} - \gamma_{22}\theta, & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix} = D,$$

say; and similarly, forming the product of B and Γ, we have

$$B\Gamma = \begin{vmatrix} \beta_{11} - \theta, & \beta_{12} \quad, & \beta_{13} \quad, & \cdots \\ \beta_{21} \quad, & \beta_{22} - \theta, & \beta_{23} \quad, & \cdots \\ \beta_{31} \quad, & \beta_{32} \quad, & \beta_{33} - \theta, & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} \begin{vmatrix} \gamma_{11}, & \gamma_{21}, & \gamma_{31}, & \cdots \\ \gamma_{12}, & \gamma_{22}, & \gamma_{32}, & \cdots \\ \gamma_{13}, & \gamma_{23}, & \gamma_{33}, & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

$$= \begin{vmatrix} \delta_{11} - \gamma_{11}\theta, & \delta_{12} - \gamma_{12}\theta, & \cdots \\ \delta_{21} - \gamma_{21}\theta, & \delta_{22} - \gamma_{22}\theta, & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix} = D,$$

so that

$$\Gamma A = B\Gamma,$$

identically. Also Γ does not vanish; hence

$$A = B,$$

for all values of θ.

Accordingly, the equation $A = 0$ is invariantive for all fundamental systems in regard to the effect of the singularity a upon the members of the system: it is called* *the fundamental equation belonging to the singularity* a. We note that its degree is equal to the order of the differential equation.

While the equation is thus invariantive for all fundamental systems, the actual invariance of one of its coefficients is put in evidence, either when the differential equation of § 2 is initially devoid of the term involving $\dfrac{d^{m-1}w}{dz^{m-1}}$, or after the equation has been transformed by the relation

$$w = We^{\frac{1}{m}\int p_1 dz},$$

so as to be devoid of the term involving $\dfrac{d^{m-1}W}{dz^{m-1}}$. *In* $A = 0$, *the term which is independent of* θ *is equal to unity*, a property first noted by Poincaré†. For when p_1 is zero, the determinant Δ of the fundamental system is a constant, for (§ 9) its derivative vanishes; it therefore is unchanged when the variable describes a simple closed circuit round the singularity. The effect of such a circuit upon Δ is to multiply it by the term in A which is independent of θ: accordingly, that term is unity.

The linear equation can always be modified so that the term involving the derivative of the dependent variable next to the highest is absent; and the necessary linear modification of the dependent variable leaves the independent variable unaltered. This change does not influence the law giving the effect, upon the integrals, of a description of a loop round the singularity; and the fundamental equation is independent of the choice of the fundamental system. Accordingly, the coefficients of the various powers of θ (except the highest, which has a coefficient $(-1)^m$, and the lowest, which has a coefficient unity) are frequently called the *invariants* of the singularity: they are $m-1$ in number.

* Sometimes also the *characteristic equation.*

† *Acta Math.*, t. IV (1884), p. 202.

15. There is a further important invariantive property of the determinants $A(\theta)$, $B(\theta)$, viz.: *If all minors of order n (and therefore all minors of lower order) in $A(\theta)$ vanish for a particular value of θ, but not all those of order $n+1$, then all minors of order n in $B(\theta)$ also vanish for that value of θ, but not all those of order $n+1$.*

A minor of order n is obtained by suppressing n rows and n columns; accordingly, the number of them is

$$\left\{\frac{m!}{(m-n)!\;n!}\right\}^2, = \mu^2,$$

say. Let them be denoted by a_{ij}, b_{ij}, c_{ij}, d_{ij} when formed from $A(\theta)$, $B(\theta)$, Γ, D respectively, where i and j have the values $1, \ldots, \mu$, these numbers corresponding to the various suppressions of the rows and the columns. Then, regarding D as the product of A and Γ, we have*

$$d_{ij} = c_{i1}a_{j1} + c_{i2}a_{j2} + \ldots + c_{i\mu}a_{j\mu};$$

and regarding D as the product of B and Γ, we have

$$d_{ij} = b_{i1}c_{j1} + b_{i2}c_{j2} + \ldots + b_{i\mu}c_{j\mu}.$$

All the quantities a_{ij} are supposed to vanish for a particular value of θ; hence for that value all the quantities d_{ij} vanish. Assigning to j all the values $1, \ldots, \mu$ in turn, we therefore have

$$\left.\begin{array}{l} 0 = c_{11}b_{i1} + c_{12}b_{i2} + \ldots + c_{1\mu}b_{i\mu} \\ 0 = c_{21}b_{i1} + c_{22}b_{i2} + \ldots + c_{2\mu}b_{i\mu} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 = c_{\mu1}b_{i1} + c_{\mu2}b_{i2} + \ldots + c_{\mu\mu}b_{i\mu} \end{array}\right\}.$$

The determinant of the coefficients of $b_{i1}, b_{i2}, \ldots, b_{i\mu}$ is equal to†

$$\Gamma^\lambda,$$

where

$$\lambda = \frac{(m-1)!}{(m-n-1)!\;n!};$$

that is, the determinant does not vanish. Accordingly, we must have

$$b_{i1} = 0,\quad b_{i2} = 0, \ldots, b_{i\mu} = 0;$$

as this holds for all values of i, it follows that all the minors of $B(\theta)$ of order n vanish for the particular value of θ.

* Scott's *Determinants*, p. 53. † *ib.*, p. 61.

The minors of $B(\theta)$, which are of order $n+1$, cannot all vanish for the value of θ; for then, by applying the result just obtained, all those of $A(\theta)$, which are of order $n+1$, would vanish, contrary to hypothesis.

16. A more general inference can be made. Leaving θ arbitrary and not restricting it to be a root of the fundamental equation, the two expressions for d_{ij} give

$$\sum_{r=1}^{\mu} c_{ir} a_{jr} = \sum_{s=1}^{\mu} c_{js} b_{is},$$

holding for all values of i and j. Taking this equation for any one value of j and for all the μ values of i, we have μ equations in all, expressing $a_{j1}, a_{j2}, \ldots, a_{j\mu}$ linearly in terms of b_{pq}. The determinant of coefficients on the left-hand side is Γ^{λ}, as before, and does not vanish; so that each of the quantities a_{jr} is expressible linearly in terms of the quantities b_{pq}, the coefficients involving only the constituents of Γ. Similarly, taking the equation for any one value of i and for all the μ values of j, we find that each of the quantities b_{pq} is expressible linearly in terms of the quantities a_{jr}, the coefficients involving only the constituents of Γ. If therefore all the quantities a_{jr} have a common factor $\theta-\theta_1$, and if that factor be of multiplicity σ, then all the quantities b_{pq} also have that factor common and of the same multiplicity σ; and conversely.

These results associate themselves at once with Weierstrass's theory of elementary divisors*. If $(\theta-\theta_1)^{\sigma}$ is the highest power of $\theta-\theta_1$ in $A(\theta)$, if $(\theta-\theta_1)^{\sigma_1}$ is the highest power of that quantity common to all its minors of the first order, if $(\theta-\theta_1)^{\sigma_2}$ is the highest power common to all its minors of the second order, and so on, then (as will be proved immediately)

$$\sigma > \sigma_1 > \sigma_2 > \ldots;$$

and

$$(\theta-\theta_1)^{\sigma-\sigma_1}, \quad (\theta-\theta_1)^{\sigma_1-\sigma_2}, \ldots$$

are called elementary divisors of the determinant $A(\theta)$. It follows from the preceding investigation that *the elementary divisors of the fundamental equation are invariantive*, as well as the equation

* *Berl. Monatsber.*, (1868), pp. 310—338; *Ges. Werke*, t. II, pp. 19—44. See also a memoir by Sauvage, *Ann. de l'Éc. Norm.*, 2e Sér., t. VIII (1891), pp. 285—340; and a treatise by Muth, *Elementartheiler*, (Leipzig, 1899).

itself; for they are independent of the particular choice of a fundamental system.

If the earliest set of minors of the same order that do not all vanish when $\theta = \theta_1$ is of order ρ, so that they are of degree $m - \rho$ in the coefficients in A, then the elementary divisors are

$$(\theta - \theta_1)^{\sigma - \sigma_1},\ (\theta - \theta_1)^{\sigma_1 - \sigma_2},\ \ldots,\ (\theta - \theta_1)^{\sigma_{\rho-2} - \sigma_{\rho-1}},\ (\theta - \theta_1)^{\sigma_{\rho-1}},$$

being ρ in number: and then ρ is one of the invariantive numbers associated with the particular singularity of the equation.

As two of the properties of the invariantive equation, associated with the elementary divisors, are required, they will be proved here: for full discussion of other properties, reference may be made to the authorities quoted.

It is easy to obtain the result

$$\sigma > \sigma_1 > \sigma_2 > \ldots,$$

just stated above. For

$$\frac{\partial A}{\partial \theta} = -\sum_{r=1}^{m} A_{rr},$$

where A_{rr} is the minor of $a_{rr} - \theta$. In A_{rr} there is a factor $(\theta - \theta_1)^{\sigma_1}$, for each of the quantities A_{rr} is a first minor: therefore that factor occurs in their sum and, owing to the combination of terms, it may have an even higher index than σ_1. On the left, the factor in $\theta - \theta_1$ has the index $\sigma - 1$; hence

$$\sigma - 1 \geqslant \sigma_1,$$

that is,

$$\sigma > \sigma_1.$$

Similarly for the other inequalities.

Again, we know* that any minor of degree p which can be formed out of the first minors of $A(\theta)$ is equal to the product of $A^{p-1}(\theta)$ by the complementary of the corresponding minor of $A(\theta)$. Hence, taking $p = 2$, we have relations of the form

$$A_1 B_2 - A_2 B_1 = AC,$$

where A_1, A_2, B_1, B_2 are minors of the first order, and C is a minor of the second order. Choose a minor of the second order which is divisible by no higher power of $\theta - \theta_1$ than $(\theta - \theta_1)^{\sigma_2}$; the left-hand side is certainly divisible by $(\theta - \theta_1)^{2\sigma_1}$, and it may be divisible by a higher power if the terms combine: hence

$$2\sigma_1 \leqslant \sigma + \sigma_2,$$

that is,

$$\sigma - \sigma_1 \geqslant \sigma_1 - \sigma_2.$$

Similarly, we have the other inequalities of the set

$$\sigma - \sigma_1 \geqslant \sigma_1 - \sigma_2 \geqslant \sigma_2 - \sigma_3 \geqslant \ldots \geqslant \sigma_{\rho-1},$$

so that the indices of the elementary divisors, as arranged above, form a series of decreasing numbers.

* Scott's *Determinants*, p. 58.

Association of Differential Equations with Algebraic Functions.

17. Before considering the roots of the fundamental equation, it is worth while establishing a converse* of the propositions in § 13, as follows:

Let $y_1, \ldots, y_m$ be m linearly independent functions of z, which are uniform over any simply-connected area not including any critical point of the functions: let the critical points be isolated and let each of them be such that, when a simple contour enclosing it is described, the values of the functions at the completion of the contour are given by relations of the form

$$y_r' = \alpha_{r1} y_1 + \ldots + \alpha_{rm} y_m, \qquad (r = 1, \ldots, m),$$

where the determinant of the coefficients α is not zero, and the constants may change from one critical point to another: then the m functions are a fundamental system of integrals for a linear differential equation of order m with uniform coefficients.

It is clear that, if the functions are integrals of such an equation, they form a fundamental system because they are linearly independent. On account of this linear independence, the determinant

$$\Delta, = \begin{vmatrix} \dfrac{d^{m-1}y_1}{dz^{m-1}}, & \dfrac{d^{m-2}y_1}{dz^{m-2}}, & \ldots, & y_1 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{d^{m-1}y_m}{dz^{m-1}}, & \dfrac{d^{m-2}y_m}{dz^{m-2}}, & \ldots, & y_m \end{vmatrix},$$

does not vanish for all values of z. Let Δ_s denote the determinant which is derived from Δ by changing the sth column into $\dfrac{d^m y_1}{dz^m}$, $\ldots, \dfrac{d^m y_m}{dz^m}$; and consider the quantity

$$p_s = \frac{\Delta_s}{\Delta}.$$

For any contour that encloses no critical point, Δ and Δ_s are uniform, so that p_s is uniform for such a contour. For a simple contour, which encloses the critical point a and no other, the

* It is given by Tannery, *Ann. de l'Éc. Norm.*, Sér. 2^me, t. IV (1875), p. 130.

determinant Δ after a single description acquires a constant factor R, where R is the (non-zero) determinant of the coefficients in the set of relations

$$y_r' = \alpha_{r1}y_1 + \ldots + \alpha_{rm}y_m, \qquad (r = 1, \ldots, m).$$

The determinant Δ_s acquires the same factor R, in the same circumstances; and therefore p_s is unchanged in value by a description of the contour, that is, it is uniform for such a contour. As this holds for each contour, it follows that p_s is uniform over the plane.

The m quantities $y_1, \ldots, y_m$ evidently are special integrals of the equation

$$\frac{d^m y}{dz^m} = p_1 \frac{d^{m-1}y}{dz^{m-1}} + p_2 \frac{d^{m-2}y}{dz^{m-2}} + \ldots + p_m y,$$

which is linear and the coefficients in which have been proved uniform functions of z.

COROLLARY. *If all the critical points of the functions are of an algebraic character, that is, of the same nature as the critical points of a function defined by an algebraic equation, and are limited in number, then the uniform coefficients p in the differential equation are rational functions of z.* For as p_s is uniform, the critical point a is either an infinity, or an ordinary value (including zero). If it is an infinity, it can be only of finite multiplicity; for the critical point is one, where Δ and Δ_s can vanish only to finite order because of the hypothesis as to the nature of the critical point: that is, the point is then a pole of finite order. Likewise, if it is a zero, the multiplicity of the zero is finite. This holds at each of the critical points of the functions $y_1, \ldots, y_m$; and the number of such points is finite. Moreover, every point that is ordinary for each of the functions is ordinary for Δ and Δ_s and, in particular, Δ cannot vanish there: so that no such point can be a pole of any of the coefficients p. It therefore follows* that each of these coefficients is a rational meromorphic function of z.

The converse of the corollary is not necessarily (nor even generally) true: it raises the question as to the tests sufficient and necessary to secure that the integrals of a linear equation with rational coefficients should be algebraic functions of the variable. This discussion must be deferred.

* *T. F.*, § 48.

Ex. 1. The most conspicuous instance arises when the dependent variable w is an algebraic function of z, defined by an algebraic equation

$$f(w, z) = 0,$$

of degree m in w. Each branch of the function so defined is uniform in the vicinity of an ordinary point; in the vicinity of a branch-point, the branches divide themselves into groups; and any linear combination of them is subject to the foregoing laws of change (which take a particularly simple form in this case) when z describes a circuit round a branch-point.

To obtain the homogeneous linear equation of order m which is satisfied by every root of $f=0$, we can proceed as follows. Let $\phi(z)=0$ be the eliminant of $f=0$ and $\dfrac{\partial f}{\partial w}=0$; so that* all the branch-points of the algebraic function are included among the roots of $\phi=0$, though not every root is a branch-point. By a result† in the theory of elimination, we know that the resultant of two quantics u and v of degree m and n respectively in a variable to be eliminated is of the form

$$uv_1 + vu_1,$$

where u_1 and v_1 are of degrees $m-1$, $n-1$ respectively in that variable; and therefore

$$\phi(z) = Uf + V\frac{\partial f}{\partial w},$$

where U is of degree $m-2$ in w and V is of degree $m-1$ in w. But f is permanently equal to zero for all the values of w considered; hence

$$\phi(z) = V\frac{\partial f}{\partial w}.$$

* *T. F.*, ch. VIII.

† It is most easily derivable from Sylvester's dialytic form of the eliminant, as follows. Let

$$u = a_0x^m + a_1x^{m-1} + a_2x^{m-2} + \dots,$$
$$v = c_0x^n + c_1x^{n-1} + c_2x^{n-2} + \dots;$$

the eliminant is

$$E = \begin{vmatrix} a_0, & a_1, & a_2, & \dots \\ 0, & a_0, & a_1, & \dots \\ 0, & 0, & a_0, & \dots \\ \dots & \dots & \dots & \dots \\ c_0, & c_1, & c_2, & \dots \\ 0, & c_0, & c_1, & \dots \\ 0, & 0, & c_0, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}.$$

To the last column, add the first column multiplied by x^{m+n-1}, the second multiplied by x^{m+n-2}, and so on: a change which does not affect the value of E. The constituents in the new last column are

$$x^{n-1}u,\ x^{n-2}u,\ \dots,\ xu,\ u,\ x^{m-1}v,\ x^{m-2}v,\ \dots,\ xv,\ v;$$

expanding E by taking every term in this last column with its minor, collecting all the terms involving u into one set and those involving v into another, we have

$$E = uv_1 + vu_1,$$

where v_1 is of degree $n-1$ in x and u_1 is of degree $m-1$ in x.

Now

$$\frac{dw}{dz} = -\frac{\frac{\partial f}{\partial z}}{\frac{\partial f}{\partial w}} = -\frac{V\frac{\partial f}{\partial z}}{\phi(z)}.$$

By means of $f=0$ which is of degree m in w, we can reduce $V\frac{\partial f}{\partial z}$ so that it contains no power of w higher than the $(m-1)$th, say

$$\frac{dw}{dz} = \frac{P_1}{\phi(z)},$$

where P_1 is a polynomial in w of degree not higher than $m-1$. (If the highest term in f has unity for its coefficient, then P_1 is a polynomial in z also.) Again,

$$\frac{d^2w}{dz^2} = \frac{P_1}{\phi^2(z)}\frac{\partial P_1}{\partial w} + \frac{1}{\phi(z)}\frac{\partial P_1}{\partial z} - \frac{P_1}{\phi^2(z)}\frac{\partial \phi}{\partial z}$$
$$= \frac{P_2}{\phi^2(z)},$$

on reducing to a common denominator; by means of $f=0$, the polynomial P_2 can be made of degree not higher than $m-1$ in w, and its coefficients are uniform functions of z. And so on, up to

$$\frac{d^mw}{dz^m} = \frac{P_m}{\phi^m(z)},$$

where P_m is a polynomial in w of degree not higher than $m-1$, the coefficients being uniform functions of z. We thus have

$$P_1 = \frac{dw}{dz}\phi,$$

$$P_2 = \frac{d^2w}{dz^2}\phi^2,$$

$$\cdots\cdots\cdots\cdots$$

$$P_m = \frac{d^mw}{dz^m}\phi^m.$$

Among these m equations we can, by a linear combination, eliminate the $m-1$ quantities $w^0, w^2, \ldots, w^{m-1}$ from the left-hand sides; and the result has the form

$$Q_0 w = Q_1\frac{dw}{dz}\phi + Q_2\frac{d^2w}{dz^2}\phi^2 + \ldots + Q_m\frac{d^mw}{dz^m}\phi^m,$$

where $Q_0, Q_1, \ldots, Q_m$ are uniform functions of z. This is satisfied for every root w of the algebraic equation: and it is of order m.

Corollary. There is one special case, when the differential equation is of order $m-1$, viz., when the algebraic equation is

$$f = w^m + a_2 w^{m-2} + \ldots + a_m = 0,$$

so that the term in w^{m-1} is absent. We then have

$$w_1 + w_2 + \ldots + w_m = 0,$$

so that one of the m branches w can be expressed linearly in terms of the others; Tannery's result shews that *the differential equation is then of order not higher than* $m-1$. In that case, it would be sufficient to take only the $m-1$ equations

$$P_r = \frac{d^r w}{dz^r}\phi^r, \qquad (r=1, \ldots, m-1).$$

For instance, consider the algebraic equation

$$w^3 + 3w = u,$$

where u is any function of z; it is to be expected that the linear differential equation satisfied by each of the three branches of the function defined by this cubic equation will be of the second order, say

$$\frac{d^2w}{dz^2} + A\frac{dw}{dz} + Bw = 0,$$

where A and B are functions of z. We have

$$(w^2+1)\frac{dw}{dz} = \tfrac{1}{3}u',$$

$$(w^2+1)\frac{d^2w}{dz^2} + 2w\left(\frac{dw}{dz}\right)^2 = \tfrac{1}{3}u'';$$

so, substituting in

$$(w^2+1)\frac{d^2w}{dz^2} + A(w^2+1)\frac{dw}{dz} + B(w^3+w) = 0,$$

and using $w^3+3w=u$, we have

$$B(u-2w) + \tfrac{1}{3}Au' + \tfrac{1}{3}u'' = 2w\left(\frac{dw}{dz}\right)^2.$$

Multiplying the right-hand side by $(w^2+1)^2$, and the left-hand side by its equivalent $1+wu-w^2$, we have

$$\begin{aligned} 2w\tfrac{1}{9}u'^2 &= (1+wu-w^2)\{\tfrac{1}{3}Au' + \tfrac{1}{3}u'' + B(u-2w)\} \\ &= (1+wu-w^2)(\tfrac{1}{3}Au' + \tfrac{1}{3}u'') + B\{3u + w(u^2-8) - 3w^2u\}, \end{aligned}$$

on reduction by the original algebraic equation. This will hold for each of the three roots of that equation, if

$$\left.\begin{aligned} \tfrac{2}{9}u'^2 &= u(\tfrac{1}{3}Au' + \tfrac{1}{3}u'') + B(u^2-8) \\ 0 &= \tfrac{1}{3}Au' + \tfrac{1}{3}u'' + 3Bu \end{aligned}\right\}.$$

These conditions give the values of A and B; and the equation for w is easily found to be

$$\frac{d^2w}{dz^2} + \left(\frac{uu'}{u^2+4} - \frac{u''}{u'}\right)\frac{dw}{dz} = \tfrac{1}{9}\frac{u'^2}{u^2+4}w,$$

where u' and u'' are the first and the second derivatives of u. The equation is of the second order as indicated.

Note 1. When the algebraic equation of degree m in w is of quite general form, the linear differential equation satisfied by its roots is of order m. But when the algebraic equation has very special forms, though still irresoluble, the differential equation may be of order less than m; for the

elimination of various powers of w may not require derivatives up to that of order m. The most conspicuously simple case is that in which the algebraic equation is

$$w^m = R(z),$$

where R is a rational function of z; the differential equation is

$$\frac{dw}{dz} + \frac{1}{m}\frac{R'(z)}{R(z)} w = 0,$$

only of the first order.

Other cases occur hereafter, in Chapter v, where quantities connected with the roots of algebraic equations of degree higher than two satisfy linear differential equations of the second order.

Note 2. The differential equations considered have, in each case, been homogeneous. If we admit non-homogeneous linear differential equations, viz. those which have a term independent of w and its derivatives, then in the general case, where $f(w, z)$ has a term in w^{m-1}, *the differential equation is of order* $m-1$ *only*. This can be seen at once from the elimination of $w^2, w^3, \ldots, w^{m-1}$ between

$$\left.\begin{aligned} P_1 &= \frac{dw}{dz}\phi \\ &\vdots \\ P_{m-1} &= \frac{d^{m-1}w}{dz^{m-1}}\phi^{m-1} \end{aligned}\right\},$$

leading to a (non-homogeneous) linear equation of order $m-1$. This result appears to have been first stated by Cockle*: it is the initial result in the formal theory of differential resolvents†.

Ex. 2. Shew that, when the algebraic equation is

$$w^2 - 2zw - z^4 = 0,$$

the two linear differential equations, homogeneous and non-homogeneous respectively, are

$$\frac{d^2w}{dz^2} - \frac{3+2z^2}{z+z^3}\frac{dw}{dz} + \frac{3+2z^2}{z^2+z^4} w = 0,$$

$$\frac{dw}{dz} - \frac{1+2z^2}{z+z^3} w = -\frac{z^2}{1+z^2}.$$

Ex. 3. Obtain the differential equations satisfied by each root of

(i) $w^3 - 3w^2 + z^6 = 0$;

(ii) $w^3 - 3zw + z^3 = 0$.

Ex. 4. Shew that any root of the equation

$$y^n - ny = (n-1)x$$

* *Phil. Mag.*, t. XXI (1861), pp. 379—383.

† For references, see a paper by Harley, *Manch. Lit. and Phil. Memoirs*, t. v (1892), pp. 79—89.

(n being greater than 2) satisfies the equation

$$\frac{d^{n-1}y}{dx^{n-1}} = a^{n-1}x^{2-n}\frac{d^{n-1}(x^{a-1}y)}{d(x^{-a})^{n-1}},$$

where $a = 1 - \frac{1}{n}$. What is the form for $n = 2$? (Heymann.)

Ex. 5. Shew that any root of the equation

$$y^n - nxy = n - 1$$

(n being greater than 2) satisfies the equation

$$\frac{d^{n-1}y}{dx^{n-1}} = (-1)^{n-1}x\sum_{r=0}^{n-1} a_r \frac{d^r y}{d(\log x)^r},$$

where the constants a_r arise as the coefficients in the algebraic equation

$$\sum_{r=0}^{n-1} (-1)^r a_r \lambda^r = 0,$$

when the roots are

$$\lambda = (k - z)\frac{n}{n-1} + 1,$$

for $k = 1, \ldots, n-1$, and $a_{n-1} = 1$. (Heymann.)

Ex. 6. Prove that, if

$$y^5 - 5y^3 + 5y - 4x + 2 = 0,$$

then

$$\frac{d^2y}{dx^2} + \frac{2x-1}{2x(x-1)}\frac{dy}{dx} - \frac{y}{25x(x-1)} = 0;$$

and explain the decrease in the order of the differential equation.

(Math. Trip., Part II, 1900.)

Fundamental System of Integrals associated with a Fundamental Equation.

18. We now proceed to the consideration of the fundamental equation $A = 0$ appertaining to the singularity a.

The simplest case is that in which the m roots of that equation are distinct from one another, say $\theta_1, \theta_2, \ldots, \theta_m$. Not all the minors of the first order vanish for any one of the roots: if they did vanish, the root would be multiple for the original equation. Hence each root θ_r determines ratios of coefficients $c_{r1}, c_{r2}, \ldots, c_{rm}$ uniquely, such that an integral of the equation exists, having the value

$$u_r = c_{r1}w_1 + \ldots + c_{rm}w_m,$$

and possessing the property that

$$u_r' = \theta_r u_r,$$

where u_r' is the value of u_r after z has described a complete simple contour round a. We thus obtain a set of m integrals.

These m integrals constitute a fundamental system: otherwise a permanent relation of the form

$$\kappa_1 u_1 + \kappa_2 u_2 + \ldots + \kappa_m u_m = 0$$

would exist. This quantity $\Sigma \kappa_r u_r$ is an integral: as it is zero and all its derivatives are zero at and near z, it is zero everywhere when continued over the regular part of the plane. Accordingly, let z describe a simple closed contour round a: when it has returned to its initial position, the zero-integral is $\Sigma \kappa_r u_r'$, that is,

$$\kappa_1 \theta_1 u_1 + \kappa_2 \theta_2 u_2 + \ldots + \kappa_m \theta_m u_m = 0.$$

Similarly, after a second description of the simple closed contour, we have

$$\kappa_1 \theta_1^2 u_1 + \kappa_2 \theta_2^2 u_2 + \ldots + \kappa_m \theta_m^2 u_m = 0.$$

Let $m-1$ descriptions of the contour be made in this way: we have

$$\kappa_1 \theta_1^r u_1 + \kappa_2 \theta_2^r u_2 + \ldots + \kappa_m \theta_m^r u_m = 0,$$

for $r = 0, 1, \ldots, m-1$. Unless all the coefficients $\kappa_1, \ldots, \kappa_m$ are zero, we have

$$\begin{vmatrix} 1 & , 1 & , \ldots, & 1 \\ \theta_1 & , \theta_2 & , \ldots, & \theta_m \\ \cdots & \cdots & \cdots & \cdots \\ \theta_1^{m-1}, & \theta_2^{m-1}, & \ldots, & \theta_m^{m-1} \end{vmatrix} = 0,$$

that is, the product of the differences of the roots is zero. This is impossible when the roots are distinct from one another; hence the coefficients $\kappa_1, \ldots, \kappa_m$ vanish, and there is no homogeneous linear relation among the integrals $u_1, \ldots, u_m$, which accordingly constitute a fundamental system.

The general functional character of these integrals is easily found. Let

$$\theta_\mu = e^{2\pi i r_\mu},$$

so that r_μ is a new constant, which is determinate save as to any additive integer; as the roots $\theta_1, \ldots, \theta_m$ are unequal, no two of the m constants $r_1, \ldots, r_m$ can differ by an integer. Now the quantity

$$(z-a)^{r_\mu}$$

acquires a factor $e^{2\pi i r_\mu}$, that is, θ_μ, when z describes the simple complete circuit round a. Hence the quantity

$$u_\mu (z-a)^{-r_\mu}$$

returns to its initial value after the variable has described the simple complete circuit round a; and therefore it is a uniform function of z in the immediate vicinity of a, say ϕ_μ, so that

$$u_\mu = (z-a)^{r_\mu} \phi_\mu.$$

As this holds for each of the integers μ, it follows that we have *a system of fundamental integrals in the form*

$$(z-a)^{r_1}\phi_1,\ (z-a)^{r_2}\phi_2,\ \ldots,\ (z-a)^{r_m}\phi_m,$$

where $\phi_1, \phi_2, \ldots, \phi_m$ *are uniform functions of* z *in the vicinity of* a, *the quantities* r_μ *are given by the relations*

$$r_\mu = \frac{1}{2\pi i} \log \theta_\mu,$$

and the roots $\theta_1, \ldots, \theta_m$ *of the fundamental equation are supposed distinct from one another*, no one of them being zero.

As regards this result, it must be noted that the functions ϕ are merely uniform in the vicinity of a: they are not necessarily holomorphic there. Each such function can be expressed in the form of a series of positive and negative powers of $z-a$, converging in an annular space bounded by two circles having a for a common centre and enclosing no other singularity of the equation. There may be no negative powers of $z-a$, in which case the function ϕ is holomorphic at a; or there may be a limited number of negative powers, in which case a is a pole of ϕ; or there may be an unlimited number of negative powers, in which case a is an essential singularity. Moreover, r_μ is only determinate save as to additive integers: it will, where possible (that is, when a is not an essential singularity), be rendered determinate hereafter; so that, in the meanwhile, the result obtained is chiefly important as indicating the precise kind of multiform character possessed by the integrals near a singularity.

19. Now consider the case in which the fundamental equation $A=0$ appertaining to the singularity a has repeated roots, say λ_1 roots equal to θ_1, λ_2 roots equal to θ_2, and so on, where $\theta_1, \theta_2, \ldots$ are unequal quantities, and $\lambda_1 + \lambda_2 + \ldots = m$. It will appear that

a group of linearly independent integrals is associated with each such root, the number in the group being equal to the multiplicity of the root; that each such group can be arranged in a number of sub-groups, the extent and the number of which are determined by the elementary divisors connected with the root; and that the aggregate of the various groups of integrals, associated with the respective roots of the fundamental equation, constitutes a fundamental system.

Group of Integrals associated with a Multiple Root of the Fundamental Equation.

Let κ denote any such root of multiplicity σ, and let the elementary divisors of $A(\theta)$ in its determinantal form be

$$(\theta-\kappa)^{\sigma-\sigma_1},\ (\theta-\kappa)^{\sigma_1-\sigma_2},\ \ldots,\ (\theta-\kappa)^{\sigma_{\tau-2}-\sigma_{\tau-1}},\ (\theta-\kappa)^{\sigma_{\tau-1}};$$

then the minors of order τ (and consequently of degree $m-\tau$ in the coefficients of A) are the earliest in increasing order which do not all vanish when $\theta=\kappa$. Consequently, in the set of equations

$$\rho_1\alpha_{1r}+\rho_2\alpha_{2r}+\ldots+\rho_m\alpha_{mr}=\rho_r\kappa, \qquad (r=1,\ \ldots,\ m),$$

τ of them are linearly dependent upon the rest; hence taking $m-\tau$ which are independent, we can express $m-\tau$ of the constants ρ linearly in terms of the other τ, which thus remain arbitrary. Let the latter be $\rho_1, \ldots, \rho_\tau$; then the integral, given by

$$u=\rho_1 w_1+\rho_2 w_2+\ldots+\rho_m w_m,$$

becomes

$$u=\rho_1 W_1+\rho_2 W_2+\ldots+\rho_\tau W_\tau,$$

where

$$\left.\begin{aligned}
W_1&=w_1+k_{\tau+1,1}w_{\tau+1}+\ldots+k_{m,1}w_m\\
W_2&=w_2+k_{\tau+1,2}w_{\tau+1}+\ldots+k_{m,2}w_m\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
W_\tau&=w_\tau+k_{\tau+1,\tau}w_{\tau+1}+\ldots+k_{m,\tau}w_m
\end{aligned}\right\};$$

and the determinate constants k are given by

$$\left.\begin{aligned}
\rho_{\tau+1}&=k_{\tau+1,1}\rho_1+k_{\tau+1,2}\rho_2+\ldots+k_{\tau+1,\tau}\rho_\tau\\
\rho_{\tau+2}&=k_{\tau+2,1}\rho_1+k_{\tau+2,2}\rho_2+\ldots+k_{\tau+2,\tau}\rho_\tau\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\rho_m&=k_{m,1}\rho_1+k_{m,2}\rho_2+\ldots+k_{m,\tau}\rho_\tau
\end{aligned}\right\},$$

being the expressions for the $m-\tau$ quantities ρ in terms of the τ quantities ρ which remain arbitrary.

Evidently each of the quantities W is an integral of the equation: and they have the property

$$W_r' = \kappa W_r,$$

for $r=1, \ldots, \tau$. Moreover, they are linearly independent; any non-evanescent relation of the form

$$E_1 W_1 + \ldots + E_\tau W_\tau = 0$$

would lead to a relation between $w_1, \ldots, w_m$ which would be homogeneous, linear, and non-evanescent, a possibility excluded by the fact that $w_1, \ldots, w_m$ constitute a fundamental system.

The only case, in which $\tau = \sigma$, occurs when the indices $\sigma - \sigma_1$, $\sigma_1 - \sigma_2, \ldots, \sigma_{\tau-1}$ of the elementary divisors are each unity. In that case, we have obtained a set of integrals, in number equal to the multiplicity of the root.

20. We shall therefore assume that $\tau < \sigma$; and we then use the integrals $W_1, \ldots, W_\tau$ to modify the original fundamental system $w_1, \ldots, w_m$, substituting them for $w_1, \ldots, w_\tau$. When the variable z describes a simple closed contour round a, the effect upon the elements of the modified system is to change them into $W_1', W_2', \ldots, W_\tau', w'_{\tau+1}, \ldots, w'_m$, where

$$\begin{aligned} W_r' &= \kappa W_r, \\ w_s' &= \beta_{s1} W_1 + \ldots + \beta_{s\tau} W_\tau + \beta_{s,\tau+1} w_{\tau+1} + \ldots + \beta_{s,m} w_m, \end{aligned}$$

for $r = 1, \ldots, \tau$, and $s = \tau + 1, \ldots, m$. The fundamental equation derived from this system for the singularity a is

$$A(\Omega) = 0,$$

where

$$A(\Omega) = \begin{vmatrix} \kappa - \Omega, & 0 & ,\ldots, & 0 & , & 0 & , & 0 & ,\ldots, & 0 \\ 0 & , \kappa - \Omega & ,\ldots, & 0 & , & 0 & , & 0 & ,\ldots, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & , 0 & ,\ldots, & \kappa - \Omega & , & 0 & , & 0 & ,\ldots, & 0 \\ \beta_{\tau+1,1}, & \beta_{\tau+1,2} & ,\ldots, & \beta_{\tau+1,\tau} & , & \beta_{\tau+1,\tau+1} - \Omega & , & \beta_{\tau+1,\tau+2} & ,\ldots, & \beta_{\tau+1,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta_{m1} & , \beta_{m2} & ,\ldots, & \beta_{m\tau} & , & \beta_{m,\tau+1} & , & \beta_{m,\tau+2} & ,\ldots, & \beta_{mm} - \Omega \end{vmatrix}$$

$$= (\kappa - \Omega)^\tau A_1(\Omega),$$

where

$$A_1(\Omega) = \begin{vmatrix} \beta_{\tau+1,\tau+1} - \Omega, & \beta_{\tau+1,\tau+2}, & \dots, & \beta_{\tau+1,m} \\ \dots & \dots & \dots & \dots \\ \beta_{m,\tau+1} \quad , & \beta_{m,\tau+2} \quad , & \dots, & \beta_{mm} - \Omega \end{vmatrix}.$$

As κ is a root of $A(\Omega)$ of multiplicity σ, it is root of $A_1(\Omega)$ of multiplicity $\sigma - \tau$; and a question arises as to the elementary divisors of $A_1(\Omega)$ associated with κ.

The elementary divisors of $A_1(\Omega)$, which are powers of $\kappa - \Omega$, are

$$(\Omega - \kappa)^{\sigma - \sigma_1 - 1}, \quad (\Omega - \kappa)^{\sigma_1 - \sigma_2 - 1}, \quad (\Omega - \kappa)^{\sigma_2 - \sigma_3 - 1}, \dots$$

being, in each instance, of index less by unity than those of $A(\Omega)$. This result, which is due to Casorati*, follows from the property that $A_1(\Omega)$ is divisible by $(\Omega - \kappa)^{\sigma - \tau}$; its first minors are divisible by $(\Omega - \kappa)^{\sigma_1 - (\tau - 1)}$ and not simultaneously by any higher power; its second minors are divisible by $(\Omega - \kappa)^{\sigma_2 - (\tau - 2)}$ and not simultaneously by any higher power; and so on.

This property, that all the minors of $A_1(\Omega)$ of order μ are divisible by $(\kappa - \Omega)^{\sigma_\mu - (\tau - \mu)}$ and not simultaneously by any higher power, can be proved as follows†.

Any minor of order μ of $A(\Omega)$ must contain at least $m - \tau - \mu$ of the last $m - \tau$ columns: let it contain $m - \tau - \mu + a$ of these columns, where a can range from 0 to μ. It then must contain $\tau - a$ of the first τ columns. Similarly, it must contain at least $m - \tau - \mu$ of the last $m - \tau$ rows: let it contain $m - \tau - \mu + a'$ of these rows, where a' can range from 0 to μ. It then must contain $\tau - a'$ of the first τ rows. The minor may be identically zero: if not, then, owing to the early columns and early rows that are retained, it is divisible by $(\kappa - \Omega)^{\tau - \mu}$, and possibly by a higher power of $\kappa - \Omega$. Consequently, some among these minors are expressible as the product of $(\kappa - \Omega)^{\tau - \mu}$ by a linear combination of minors of $A_1(\Omega)$ which are of order μ; the coefficients in the combination are composed of the constants, which occur in the first $\tau - \mu$ columns and the last $m - \tau$ rows, and thus are independent of Ω. But a minor of order μ of $A(\Omega)$ is not necessarily divisible by a power of $\kappa - \Omega$ with an index higher than σ_μ; thus

$$(\kappa - \Omega)^{\sigma_\mu} . \text{ polynomial in } \Omega = (\kappa - \Omega)^{\tau - \mu} . \text{ sum of minors of } A_1(\Omega).$$

It therefore follows that the power of $\kappa - \Omega$ common to all those minors of $A_1(\Omega)$ is of index not higher than $\sigma_\mu - (\tau - \mu)$.

* *Comptes Rendus*, t. XCII (1881), p. 177.

† Heffter, *Einleitung in die Theorie der linearen Differentialgleichungen*, pp. 250—256.

Next, we know that there are some minors of the original $A(\Omega)$ of order τ, which do not vanish when $\Omega=\kappa$ and which therefore are not divisible by $\kappa-\Omega$. Clearly they cannot contain any of the first τ rows in $A(\Omega)$; and thus they must be composed of sets of $m-\tau$ columns selected among the last $m-\tau$ rows. Take the minors of order μ of any one of these non-vanishing determinants, their number being N^2, where

$$\frac{(m-\tau)\,!}{(m-\tau-\mu)\,!\,\mu\,!}=N;$$

and denote these minors by

$$M_{hk}, \qquad (h, k=1, \ldots, N),$$

the integers h and k corresponding to the obliteration of a set of μ columns and a set of μ rows out of the non-vanishing determinant of order $m-\tau$. Let m_{hk} be the complementary of M_{hk} in its own determinant.

Now take the minors of $A_1(\Omega)$ which are of order μ: their number is N^2, and they may be denoted by a_{ij}, for $i, j=1, \ldots, N$, with the same significance in the integers as for M_{hk}. Construct an expression

$$m_{h1}\,a_{i1}+m_{h2}\,a_{i2}+\ldots+m_{hN}\,a_{iN}, \;=J_h,$$

say, where J_h is a determinant of order $m-\tau$. Then either (i), J_h vanishes identically, owing to identities of rows or columns: or (ii), J_h is equal to $\pm A_1(\Omega)$ and therefore is divisible by $(\kappa-\Omega)^{\sigma-\tau}$, that is, certainly divisible by $(\kappa-\Omega)^{\sigma_\mu-(\tau-\mu)}$, for (§ 16) we have

$$\sigma-\sigma_1 \geqslant \sigma_1-\sigma_2 \geqslant \ldots \geqslant \sigma_{\tau-1} \geqslant 1;$$

or (iii) J_h, when bordered by $\tau-\mu$ of the first rows, and the first columns in $A(\Omega)$, is a minor of order μ of $A(\Omega)$ and is therefore divisible by $(\kappa-\Omega)^{\sigma_\mu}$, so that the equivalence of the two expressions for the minor of $A(\Omega)$ gives

$$(\kappa-\Omega)^{\sigma_\mu}\,.\,\text{polynomial in }\Omega=(\kappa-\Omega)^{\tau-\mu}\,.\,J_h,$$

and therefore J_h is divisible by $(\kappa-\Omega)^{\sigma_\mu-(\tau-\mu)}$. It thus follows that J_h is divisible by $(\kappa-\Omega)^{\sigma_\mu-(\tau-\mu)}$, in every case when it is not zero: and this holds for all values of h. Taking then

$$m_{h1}\,a_{i1}+m_{h2}\,a_{i2}+\ldots+m_{hN}\,a_{iN}=J_h,$$

for $h=1, \ldots, N$ and for one particular value of i, we have a series of N linear equations in the quantities $a_{i1}, \ldots, a_{iN}$. The determinant of their coefficients is a power of the non-vanishing determinant of order $m-\tau$, for it is a determinant of all its minors of one order: and therefore it does not vanish. Hence, so far as powers of $\kappa-\Omega$ are concerned, each of the minors $a_{i1}, \ldots, a_{iN}$ is a linear combination of $J_1, \ldots, J_N$: all of these are divisible by $(\kappa-\Omega)^{\sigma_\mu-(\tau-\mu)}$, and therefore each of the minors $a_{i1}, \ldots, a_{iN}$ is certainly divisible by that power. The result holds for each of the values of i.

It has been seen that the power of $\kappa-\Omega$, common to all these minors of $A_1(\Omega)$, has an index not greater than $\sigma_\mu-(\tau-\mu)$; combining the results, we infer that the highest power of $\kappa-\Omega$, common to all the minors of $A_1(\Omega)$ of order μ, has its index equal to $\sigma_\mu-(\tau-\mu)$.

21. The indices of the elementary divisors of $A_1(\Omega)$ are

$$\sigma - \sigma_1 - 1,\ \ \sigma_1 - \sigma_2 - 1,\ \ \sigma_2 - \sigma_3 - 1,\ \ldots;$$

let there be τ' of them, where $\tau' \leqslant \tau$, so that the last $\tau - \tau'$ of the indices of those of $A(\Omega)$ are equal to unity, on account of the property

$$\sigma - \sigma_1 \geqslant \sigma_1 - \sigma_2 \geqslant \sigma_2 - \sigma_3 \geqslant \ldots \geqslant \sigma_{\tau-1} \geqslant 1.$$

Then the minors of A_1 of order τ' (and consequently of degree $m - \tau - \tau'$ in the coefficients of A_1) are the earliest in successively increasing order, which do not all vanish when $\Omega = \kappa$; consequently, in the set of equations

$$\rho_1'\beta_{r,\tau+1} + \rho_2'\beta_{r,\tau+2} + \ldots + \rho'_{m-\tau}\beta_{r,m} = \kappa\rho_r', \quad (r = \tau + 1, \ldots, m),$$

τ' of them are linearly dependent upon the rest. Hence taking $m - \tau - \tau'$ of the equations which are independent, we can express $m - \tau - \tau'$ of the constants ρ' in terms of the other τ', which thus remain arbitrary and which may be taken to be $\rho_1', \ldots, \rho'_{\tau'}$.

Now take an integral

$$v = \rho_1' w_{\tau+1} + \ldots + \rho'_{m-\tau} w_m,$$

and substitute for the various coefficients ρ' in terms of $\rho_1', \ldots, \rho'_{\tau'}$. The integral becomes

$$v = \rho_1' W_{11} + \rho_2' W_{12} + \ldots + \rho'_{\tau'} W_{1\tau'},$$

where, writing $\lambda = \tau + \tau'$, we have

$$W_{1r} = w_{\tau+r} + l_{\lambda+1,r} w_{\lambda+1} + \ldots + l_{m,r} w_m,$$

for $r = 1, \ldots, \tau'$; and the determinate constants l are given by

$$\rho_{\tau'+s} = l_{\lambda+s,1}\rho_1' + \ldots + l_{\lambda+s,\tau'}\rho'_{\tau'},$$

for $s = 1, \ldots, m - \lambda$, being the expressions of the constants ρ' in terms of $\rho_1', \ldots, \rho'_{\tau'}$.

Clearly each of the quantities $W_{11}, W_{12}, \ldots, W_{1\tau'}$ is an integral of the equation. Moreover, they are linearly independent of one another and of $W_1, \ldots, W_\tau$; for any non-evanescent linear relation of the form

$$F_1 W_1 + \ldots + F_\tau W_\tau + F_1' W_{11} + \ldots + F'_{\tau'} W_{1\tau'} = 0$$

would lead, after substitution for $W_1, \ldots, W_\tau, W_{11}, \ldots, W_{1\tau'}$ in terms of the original fundamental system $w_1, \ldots, w_m$, to a non-evanescent homogeneous linear relation among the members of that system—a possibility that is excluded.

As regards the effect, which is caused upon each of these newly obtained integrals by the description of a simple contour round the singularity, we have

$$\begin{aligned} W_{1r}' &= w'_{\tau+r} + l_{\lambda+1,r} w'_{\lambda+1} + \ldots + l_{m,r} w_m' \\ &= \kappa W_{1r} + V_r, \end{aligned}$$

where V_r denotes a homogeneous linear combination of $W_1, \ldots, W_\tau$. Now no one of the quantities V_r can be evanescent, nor can any linear combination of the form

$$\gamma_1 V_1 + \ldots + \gamma_{\tau'} V_{\tau'}$$

be evanescent: for in the former case, we should have

$$W_{1r}' = \kappa W_{1r},$$

and in the latter

$$(\gamma_1 W_{1r} + \ldots + \gamma_{\tau'} W_{1\tau'})' = \kappa (\gamma_1 W_{11} + \ldots + \gamma_{\tau'} W_{1\tau'}).$$

As W_{1r} and $\gamma_1 W_{11} + \ldots + \gamma_{\tau'} W_{1\tau'}$ in the respective cases are linearly independent of $W_1, \ldots, W_\tau$, we should thus have a new integral of the same type as $W_1, \ldots, W_\tau$; and then, instead of having some of the minors of order τ in $A(\Omega)$ different from zero when $\Omega = \kappa$, all of them of that order would be zero, and we should only be able to declare that some of order $\tau + 1$ are different from zero: in other words, the number of elementary divisors of $A(\Omega)$ would be $\tau + 1$ instead of τ. The quantities $V_1, \ldots, V_{\tau'}$ are thus linearly equivalent to τ' of the quantities $W_1, \ldots, W_\tau$, say to $W_1, \ldots, W_{\tau'}$; hence constructing the linear combinations of $V_1, \ldots, V_{\tau'}$ which are equal to $W_1, \ldots, W_{\tau'}$ respectively, and denoting by $w_{11}, \ldots, w_{1\tau'}$ the linear combinations of $W_{11}, \ldots, W_{1\tau'}$ with the same coefficients as occur in these combinations of $V_1, \ldots, V_{\tau'}$, we have a set of τ' integrals $w_{11}, \ldots, w_{1\tau'}$, such that

$$w_{1r}' = \kappa w_{1r} + W_r, \qquad (r = 1, \ldots, \tau').$$

These integrals are linearly independent of one another, and also of $W_1, \ldots, W_\tau$, before obtained. They constitute the aggregate of linearly independent integrals of this type; for if there were another linearly independent of them, it would imply that $A_1(\Omega)$ had $\tau' + 1$ elementary divisors instead of only τ'.

As regards the two sets of integrals already obtained, it may be noted, (i), that the set $W_1, \ldots, W_\tau$ can be linearly combined among themselves, without affecting the characteristic equation

$$W_r' = \kappa W_r;$$

(ii), that to each integral of the set $w_{11}, \ldots, w_{1\tau'}$ there may be added any linear combination of the integrals of the set $W_1, \ldots, W_\tau$, without affecting the characteristic equation

$$w_{1r}' = \kappa w_{1r} + W_r.$$

If the index of each of the elementary divisors of $A_1(\Omega)$ is unity, then $\tau' = \sigma - \tau$, so that the number $\tau + \tau'$ of integrals obtained is then equal to σ, the multiplicity of the root of $A(\Omega) = 0$ in question. In every other case, $\tau' + \tau < \sigma$.

22. When $\tau' + \tau < \sigma$, so that τ' is less than the degree of $A_1(\Omega)$, we use the integrals $w_{11}, \ldots, w_{1\tau'}$ to modify the fundamental system $W_1, \ldots, W_\tau, w_{\tau+1}, \ldots, w_m$, substituting them for $w_{\tau+1}, \ldots, w_{\tau+\tau'}$ in that system. When the variable z describes a simple closed contour round a, the effect upon the elements of the modified fundamental system is to change them into $W_1', \ldots, W_\tau'$, $w_{11}', \ldots, w'_{1\tau'}, w'_{\lambda+1}, \ldots, w_m'$, where $\tau + \tau' = \lambda$, and

$$\begin{aligned} W_r' &= \kappa W_r, \\ W_{1s}' &= \kappa w_{1s} + W_s, \\ w_t' &= \gamma_{t1} W_1 + \ldots + \gamma_{t\tau} W_\tau \\ &\quad + \gamma_{t,\tau+1} w_{11} + \ldots + \gamma_{t\lambda} w_{1\tau'} + \gamma_{t,\lambda+1} w_{\lambda+1} + \ldots + \gamma_{t,m} w_m, \end{aligned}$$

for $r = 1, \ldots, \tau$; $s = 1, \ldots, \tau'$; $t = \lambda + 1, \ldots, m$. The fundamental equation derived through this system is

$$A(\Omega) = (\kappa - \Omega)^{\tau+\tau'} A_2(\Omega) = 0,$$

where

$$A_2(\Omega) = \begin{vmatrix} \gamma_{\lambda+1,\lambda+1} - \Omega, & \gamma_{\lambda+1,\lambda+2}, & \ldots, & \gamma_{\lambda+1,m} \\ \cdots & \cdots & \cdots & \cdots \\ \gamma_{m,\lambda+1} & , \gamma_{m,\lambda+2} & , \ldots, & \gamma_{m,m} - \Omega \end{vmatrix}.$$

Also κ is of a root of $A_2(\Omega)$ of multiplicity $\sigma - \tau - \tau'$. By a further application of the proposition (§ 20) connecting the elementary divisors of $A(\Omega)$ and $A_1(\Omega)$, the indices of the elementary divisors of $A_2(\Omega)$, which are powers of $\kappa - \Omega$, are seen to be

$$\sigma - \sigma_1 - 2, \;\; \sigma_1 - \sigma_2 - 2, \;\; \sigma_2 - \sigma_3 - 2, \;\; \ldots,$$

say τ'' in number.

The procedure from the equation $A_2(\Omega) = 0$ to the corresponding sub-group of integrals is similar to that adopted in the case of the equation $A_1(\Omega) = 0$; and the conclusion is that there

exists a sub-group of τ'' integrals $w_{21}, w_{22}, \ldots, w_{2\tau''}$, characterised by the equations

$$w_{2t}' = \kappa w_{2t} + w_{1t},$$

for $t = 1, 2, \ldots, \tau''$.

And so on, for the sub-groups in succession. Combining these results, we have the theorem*:

When a root κ of the fundamental equation $A(\Omega) = 0$ is of multiplicity σ, and when the elementary divisors of $A(\Omega)$ associated with that root are

$$(\kappa - \Omega)^{\sigma - \sigma_1}, \ (\kappa - \Omega)^{\sigma_1 - \sigma_2}, \ \ldots, \ (\kappa - \Omega)^{\sigma_{\tau-1}},$$

a group of σ linearly independent integrals is associated with that root: this group consists of a number $\sigma - \sigma_1$ of sub-groups, which satisfy the equations

$$w_r' = \kappa w_r, \text{ for } r = 1, \ldots, \tau,$$
$$w_{1s}' = \kappa w_{1s} + w_s, \text{ for } s = 1, \ldots, \tau',$$
$$w_{2t}' = \kappa w_{2t} + w_{1t}, \text{ for } t = 1, \ldots, \tau'',$$

and so on. The integer τ is the number of elementary divisors of $A(\Omega)$; τ' is the number of those divisors with an index greater than unity; τ'' is the number of those divisors with an index greater than two; and so on.

The group of σ integrals, and $m - \sigma$ other integrals, all linearly independent of one another, make up a fundamental system: the $m - \sigma$ other integrals being associated with the $m - \sigma$ roots of $A(\Omega) = 0$ other than $\Omega = \kappa$. When these roots are taken in turn, we have a single integral associated with each simple root, and a group of integrals of the preceding type associated with each multiple root, the number in the group being equal to the order of multiplicity of the root. We thus have a system of integrals of the original differential equation distributed among the roots of the fundamental equation associated with the

* That part of the theorem, which establishes the existence of the group of integrals associated with a multiple root, is due to Fuchs, *Crelle*, t. LXVI (1866), p. 136: but the initial expression given to the members of the group was much more complicated. The part which arranges the group in sub-groups, each with its own characteristic equation, is due to Hamburger, *Crelle*, t. LXXVI (1873), p. 121; he takes it in an arrangement, which will be found in the next section. The association of the sub-groups with the elementary divisors of $A(\Omega)$ is due to Casorati, *Comptes Rendus*, t. XCII (1881), p. 177.

singularity: that the system is fundamental is manifest from the facts, that the initial system was fundamental, and that all modifications introduced have been such as to leave it fundamental.

Ex. 1. Two independent integrals of the equation

$$z^2(z+1)\frac{d^2w}{dz^2}-z^2\frac{dw}{dz}+\tfrac{1}{4}(3z+1)w=0$$

are given by

$$w_1=z^{\frac{1}{2}}, \quad w_2=z^{\frac{3}{2}}+z^{\frac{1}{2}}\log z.$$

Hence when the variable describes a simple closed contour round the origin in the positive direction, we have

$$\begin{aligned} w_1' &= -w_1, \\ w_2' &= -2\pi i w_1 - w_2 ; \end{aligned}$$

and therefore the fundamental equation belonging to the origin (which is a singularity of the equation) is

$$\begin{vmatrix} -1-\theta, & 0 \\ -2\pi i\ , & -1-\theta \end{vmatrix}=0,$$

that is, it is

$$(\theta+1)^2=0.$$

Similarly, two independent integrals of the equation

$$z^2\frac{d^2w}{dz^2}-\tfrac{5}{6}z\frac{dw}{dz}+\tfrac{2}{3}w=0$$

are given by

$$w_1=z^{\frac{1}{2}}, \quad w_2=z^{\frac{4}{3}}.$$

Hence after a simple closed contour round the origin, we have

$$w_1'=-w_1, \quad w_2'=aw_2,$$

where a is $e^{\frac{2}{3}\pi i}$; the fundamental equation belonging to the origin is

$$\begin{vmatrix} -1-\theta, & 0 \\ 0\ , & a-\theta \end{vmatrix}=0,$$

that is,

$$(\theta+1)(\theta-e^{\frac{2}{3}\pi i})=0.$$

Ex. 2. Construct the linear differential equation of the third order, having

$$z^{\frac{13}{18}}, \quad z^{\frac{13}{18}}\log z, \quad z^{\frac{14}{9}}$$

for three linearly independent integrals; obtain the fundamental equation appertaining to the origin as a singularity; and from the form of the differential equation, verify Poincaré's theorem (§ 14) that the product of the three roots of this fundamental equation is unity.

HAMBURGER'S RESOLUTION OF A GROUP OF INTEGRALS INTO SUB-GROUPS.

23. In the case when the roots of the fundamental equation are all distinct from one another, the general analytical character of each of the integrals of the fundamental system in the vicinity of the singularity has been obtained (§ 18). We proceed to the corresponding investigation of the general analytical character of the group of integrals in the vicinity of the singularity, when the group is associated with a multiple root of the fundamental equation.

We have seen that the group of linearly independent integrals can be arranged in sub-groups of the form

$$\begin{array}{l} W_1, \ W_2, \ \ldots, \ W_\tau \ ; \\ w_{11}, \ w_{12}, \ \ldots, \ w_{1\tau'} \ ; \\ w_{21}, \ w_{22}, \ \ldots, \ w_{2\tau''} \ ; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

the members of each sub-group being arranged in a line and satisfying an equation characteristic of the line. Let these be rearranged in the form*

$$\begin{array}{l} W_1, \ w_{11}, \ w_{21}, \ w_{31}, \ \ldots \\ W_2, \ w_{12}, \ w_{22}, \ w_{32}, \ \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ; \end{array}$$

each of the integrals in the new line satisfies an equation, and the set of characteristic equations for any line is, in sequence, the same as for any other line, so far as the members extend. When any such line is taken in the form

$$u_1, \ u_2, \ \ldots, \ u_\mu,$$

where the integer μ changes from line to line, the set of the characteristic equations is

$$\left.\begin{array}{l} u_1' = \kappa u_1 \\ u_2' = \kappa u_2 + u_1 \\ u_3' = \kappa u_3 + u_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ u_\mu' = \kappa u_\mu + u_{\mu-1} \end{array}\right\}.$$

* These are Hamburger's sub-groups; see note, p. 60. Their number is equal to the number of elementary divisors of $A(\Omega)$ connected with the multiple root.

Let

$$2\pi i\alpha = \log \kappa;$$

we have

$$[(z-a)^\alpha]' = \kappa (z-a)^\alpha,$$

and therefore

$$[u_1 (z-a)^{-\alpha}]' = u_1 (z-a)^{-\alpha}.$$

Thus $u_1 (z-a)^{-\alpha}$ is unaltered by the description of a simple closed contour round a; it therefore is uniform in the vicinity of a, but it cannot be declared holomorphic in that vicinity, for a might be a pole or an essential singularity of $u_1 (z-a)^{-\alpha}$. Denoting this uniform function of $z-a$ by ψ_1, we have

$$u_1 = (z-a)^\alpha \psi_1.$$

To obtain expressions for the other integrals, Hamburger* proceeds as follows. Introduce the function L, defined by the relation

$$L = \frac{1}{2\pi i} \log (z-a);$$

then, after the description of a simple contour, we have

$$L' = L + 1.$$

We consider an expression

$$F(L) = F = \psi_\mu + \binom{\mu-1}{1} \psi_{\mu-1} L + \binom{\mu-1}{2} \psi_{\mu-2} L^2 + \dots$$
$$\dots + \binom{\mu-1}{1} \psi_2 L^{\mu-2} + \psi_1 L^{\mu-1},$$

where

$$\binom{\mu-1}{r} = \frac{(\mu-1)!}{(\mu-1-r)!\; r!},$$

and the functions $\psi_1, \dots, \psi_\mu$ are uniform functions of $z-a$. Then if, for all values of n, we take

$$y_{\mu-n} = (z-a)^\alpha \kappa^n \Delta^n F,$$

where the symbolical operator Δ is defined by the relation

$$\Delta F = F(L+1) - F(L) = F' - F,$$

we have

$$\begin{aligned} y'_{\mu-n} &= (z-a)^\alpha \kappa^{n+1} \Delta^n F' \\ &= (z-a)^\alpha \kappa^{n+1} (\Delta^n F + \Delta^{n+1} F) \\ &= \kappa y_{\mu-n} + y_{\mu-n-1}, \end{aligned}$$

* *Crelle*, t. LXXVI (1873), p. 122.

holding for all values of n. These are the characteristic equations of the modified sub-group; and therefore we can write

$$u_{\mu-n} = (z-a)^{\alpha} \kappa^n \Delta^n F,$$

with the above notations. This is Hamburger's functional form for the integrals.

24. The integrals $u_1, \ldots, u_\mu$ are a linearly independent set out of the fundamental system; and the system will remain fundamental if $u_1, \ldots, u_\mu$ are replaced by μ other functions, linearly independent of one another and linearly equivalent to $u_1, \ldots, u_\mu$. A modification of this kind, leading to simpler expressions for the sub-group of integrals, can be obtained. In association with F, take a series of quantities, defined by the relations

$$\begin{aligned}
v_1 &= \psi_1,\\
v_2 &= \psi_2 + \psi_1 L,\\
v_3 &= \psi_3 + 2\psi_2 L + \psi_1 L^2,\\
v_4 &= \psi_4 + 3\psi_3 L + 3\psi_2 L^2 + \psi_1 L^3,\\
&\vdots\\
F = v_\mu &= \psi_\mu + \binom{\mu-1}{1}\psi_{\mu-1}L + \binom{\mu-1}{2}\psi_{\mu-2}L^2 + \ldots\\
&\qquad \ldots + \binom{\mu-1}{1}\psi_2 L^{\mu-2} + \psi_1 L^{\mu-1}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\Delta F &= a_{11} v_{\mu-1} + a_{12} v_{\mu-2} + a_{13} v_{\mu-3} + \ldots + a_{1,\mu-1} v_1,\\
\Delta^2 F &= a_{21} v_{\mu-2} + a_{22} v_{\mu-3} + \ldots + a_{2,\mu-2} v_1,\\
\Delta^3 F &= a_{31} v_{\mu-3} + a_{32} v_{\mu-4} + \ldots + a_{3,\mu-3} v_1,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\Delta^{\mu-2} F &= a_{\mu-2,1} v_2 + a_{\mu-2,2} v_1,\\
\Delta^{\mu-1} F &= a_{\mu-1,1} v_1,
\end{aligned}$$

where the constants a are non-vanishing numbers, the exact expressions for which are not needed for the present purpose.

Then $(z-a)^{\alpha} v_1$ is a constant multiple of $(z-a)^{\alpha}\Delta^{\mu-1}F$, that is, of u_1; and it therefore is an integral of the differential equation.

By the last two of the above equations, $(z-a)^{\alpha} v_2$ is a linear combination of $(z-a)^{\alpha}\Delta^{\mu-2}F$ and $(z-a)^{\alpha}\Delta^{\mu-1}F$, that is, it is a linear combination of u_2 and u_1; it therefore is an integral of the differential equation.

By the last three of the above equations, $(z-a)^{\alpha} v_3$ is a linear combination of $(z-a)^{\alpha}\Delta^{\mu-3}F$, $(z-a)^{\alpha}\Delta^{\mu-2}F$, $(z-a)^{\alpha}\Delta^{\mu-1}F$, that is,

it is a linear combination of u_1, u_2, u_3; it therefore is an integral of the differential equation.

Proceeding in this way, we obtain μ integrals of the form

$$(z-a)^a v_1,\ (z-a)^a v_2,\ \ldots,\ (z-a)^a v_\mu.$$

Moreover, these are linearly independent, and so are linearly equivalent to $u_1, \ldots, u_\mu$; for, having regard to the expressions of $\Delta F, \ldots, \Delta^{\mu-1}F$, we see at once that any homogeneous linear relation among the quantities $v_1, \ldots, v_\mu$ would imply a homogeneous linear relation among the quantities $F, \Delta F, \ldots, \Delta^{\mu-1}F$, that is, among $u_1, \ldots, u_\mu$; and no such linear relation exists. Hence Hamburger's sub-group of integrals is equivalent to (and can be replaced by) the sub-group

$$(z-a)^a v_1,\ (z-a)^a v_2,\ \ldots,\ (z-a)^a v_\mu.$$

Accordingly, we now can enunciate the following result as giving the general analytical expression of the group of integrals, associated with a multiple root κ of the fundamental equation*:—

When a root κ of the fundamental equation $A(\theta)=0$ is of multiplicity σ, the group of σ integrals associated with that root can be arranged in sub-groups; the number of these sub-groups is equal to the number of elementary divisors of $A(\theta)$ which are powers of $\kappa-\theta$; the number of integrals in any sub-group is determined by means of the exponents of the elementary divisors; and a sub-group, which contains μ integrals, is linearly equivalent to the μ quantities

$$(z-a)^a v_1,\ (z-a)^a v_2,\ \ldots,\ (z-a)^a v_\mu,$$

where $2\pi i a = \log \kappa$, and the μ quantities v are of the form

$$\begin{aligned}
v_1 &= \psi_1,\\
v_2 &= \psi_2 + \psi_1 L,\\
v_3 &= \psi_3 + 2\psi_2 L + \psi_1 L^2,\\
v_4 &= \psi_4 + 3\psi_3 L + 3\psi_2 L^2 + \psi_1 L^3,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
v_\mu &= \psi_\mu + \binom{\mu-1}{1}\psi_{\mu-1}L + \binom{\mu-1}{2}\psi_{\mu-2}L^2 + \ldots\\
&\qquad \ldots + \binom{\mu-1}{1}\psi_2 L^{\mu-2} + \psi_1 L^{\mu-1},
\end{aligned}$$

* This form of expression for the group of integrals appears to have been given first by Jürgens, *Crelle*, t. LXXX (1875), p. 154. See also a memoir by Fuchs, *Berl. Sitzungsber.*, 1901, pp. 34—48.

where $L = \frac{1}{2\pi i} \log (z-a)$, $\binom{\mu-1}{r}$ *denotes* $\frac{(\mu-1)!}{(\mu-1-r)!\, r!}$, *and the* μ *quantities* $\psi_1, \ldots, \psi_\mu$ *are uniform (but not necessarily holomorphic) functions of* $z-a$ *in the vicinity of the singularity.*

Differential Equation of Lower Order satisfied by a Sub-Group of Integrals.

25. The preceding form of the integrals in each sub-group of a group, associated with a multiple root of the fundamental equation, has been inferred on the supposition that the coefficients of the linear equation are uniform functions.

It will be noticed that the coefficient of the highest power of L in each of the members of the sub-group is the same, being an integral of the equation,—a result which is a special case of a more general theorem. Moreover, it is of course possible to verify that each member of the sub-group satisfies the differential equation; and it happens that the kind of analysis subsidiary to this purpose leads to the more general theorem above indicated, as well as to a result of importance which will be useful in the subsequent discussion of the reducibility of a given equation. We proceed to establish the following theorem*, which is of the nature of a converse to the theorem just established:

If an expression for a quantity u *be given in the form*

$$u = \phi_n + \phi_{n-1} L + \phi_{n-2} L^2 + \ldots + \phi_2 L^{n-2} + \phi_1 L^{n-1},$$

where $L = \frac{1}{2\pi i} \log (z-a)$, *and each of the quantities* ϕ *is of the form*

$$\phi = (z-a)^\alpha \text{ . uniform function of } z-a,$$

α *being a constant, then* u *satisfies a homogeneous linear differential equation of order* n, *the coefficients of which are functions of* z *uniform in the vicinity of* $z=a$; *moreover,*

$$\frac{\partial u}{\partial L}, \frac{\partial^2 u}{\partial L^2}, \ldots, \frac{\partial^{n-1} u}{\partial L^{n-1}}$$

are integrals of the same equation and, taken together with u, *they constitute a fundamental system for the equation.*

* Fuchs, in the memoir quoted on the preceding page.

(It is clear that $\frac{\partial^{n-1}u}{\partial L^{n-1}}$ is a numerical multiple of ϕ_1, and that the coefficient of the highest power of L in each of the announced integrals is, save as to a numerical constant, the same for all; it is a multiple of ϕ_1, which is an integral of the equation.)

It is convenient to make a slight modification in the form of u; we take

$$u = \psi_n + \binom{n-1}{1}\psi_{n-1}L + \binom{n-1}{2}\psi_{n-2}L^2 + \dots$$
$$\dots + \binom{n-1}{1}\psi_2 L^{n-2} + \psi_1 L^{n-1},$$

where

$$\binom{n-1}{r}\psi_{n-r} = \phi_{n-r},$$

so that the character of the functions ψ and their form (except as to a mere numerical constant) are the same as those of the functions ϕ. Further, no change, either in the property that

$$\frac{\partial u}{\partial L}, \ \frac{\partial^2 u}{\partial L^2}, \ \dots$$

are integrals of the equation or in the property that, taken together with u, they constitute a fundamental system, will be caused if they are multiplied by constants: so that, if the theorem can be established for $u_1, \dots, u_{n-1}$, where

$$u_1 = \frac{1}{(n-1)!}\frac{\partial^{n-1}u}{\partial L^{n-1}} = \psi_1,$$

$$u_2 = \frac{1\,!}{(n-1)!}\frac{\partial^{n-2}u}{\partial L^{n-2}} = \psi_2 + \psi_1 L,$$

$$u_3 = \frac{2\,!}{(n-1)!}\frac{\partial^{n-3}u}{\partial L^{n-3}} = \psi_3 + 2\psi_2 L + \psi_1 L^2,$$

....

$$u_{n-1} = \frac{(n-2)!}{(n-1)!}\frac{\partial u}{\partial L} = \psi_{n-1} + \binom{n-2}{1}\psi_{n-2}L + \dots$$
$$\dots + \binom{n-2}{1}\psi_2 L^{n-3} + \psi_1 L^{n-2},$$

the theorem holds for the quantities as given in the enunciation of the theorem.

26. Merely in order to abbreviate the analysis, we take $n=4$: with the above forms, it will be found that the analysis for any particular case such as $n=4$ is easily amplified into the analysis for the general case. Accordingly, we deal with quantities u, u_1, u_2, u_3, where

$$\begin{aligned} u &= \psi_4 + 3\psi_3 L + 3\psi_2 L^2 + \psi_1 L^3, \\ u_3 &= \psi_3 + 2\psi_2 L + \psi_1 L^2, \\ u_2 &= \psi_2 + \psi_1 L, \\ u_1 &= \psi_1. \end{aligned}$$

If u can be an integral of a linear equation of the fourth order with coefficients that are uniform functions of $z-a$ in the vicinity of a, let the equation be

$$\Delta = \frac{d^4}{dz^4} + P\frac{d^3}{dz^3} + Q\frac{d^2}{dz^2} + R\frac{d}{dz} + S = 0.$$

Let the variable z describe a simple contour round a; this leaves the differential equation (if it exists) unaltered, and so the new form of u is an integral, say u', where

$$u' = \kappa\psi_4 + 3\kappa\psi_3(L+1) + 3\kappa\psi_2(L+1)^2 + \kappa\psi_1(L+1)^3,$$

where κ is the factor common to all the functions ψ after the description of the circuit. As u and u' are integrals of a homogeneous linear equation, so also is

$$\begin{aligned} v, &= \frac{1}{\kappa}u' - u \\ &= 3u_3 + 3u_2 + u_1. \end{aligned}$$

Hence v' also is an integral, and it is given by

$$\begin{aligned} v' = 3\{\kappa\psi_3 + 2\kappa\psi_2(L+1) + \kappa\psi_1(L+1)^2\} \\ + 3\{\kappa\psi_2 + \kappa\psi_1(L+1)\} + \kappa\psi_1: \end{aligned}$$

and therefore

$$w, = \tfrac{1}{6}\left(\frac{1}{\kappa}v' - v\right) = u_2 + u_1,$$

is also an integral. Hence w' is also an integral, and it is given by

$$w' = \kappa\psi_2 + \kappa\psi_1(L+1) + \kappa\psi_1,$$

and therefore

$$t, = \frac{1}{\kappa}w' - w = \psi_1 = u_1,$$

is also an integral.

Thus integrals are given by

$$t, \quad = u_1,$$

$$w - t, \quad = u_2,$$

$$\tfrac{1}{3}(v - 3w + 2t), \quad = u_3,$$

which proves one part of the theorem, viz. that u, u_1, u_2, u_3 are simultaneous integrals of the linear equation if it exists.

27. In order to establish the property that u, u_1, u_2, u_3 constitute a fundamental system of the equation if it exists, a preliminary lemma will be useful; viz. if A, B, C, D be functions free from logarithms and if they be such that a simple closed contour round a restores their initial values, except as to a constant factor the same for all, then no identical relation of the kind

$$\alpha A + \beta BL + \gamma CL^2 + \delta DL^3 = 0$$

can exist, in which $\alpha, \beta, \gamma, \delta$ are constants different from zero. For let the simple contour be described any number, N, of times in succession; and let f be the constant factor acquired by the functions A, B, C, D after a single description of the simple contour. Then we should have the relation

$$f^N[\alpha A + \beta B(L+N) + \gamma C(L+N)^2 + \delta D(L+N)^3] = 0,$$

and consequently the relation

$$\alpha A + \beta B(L+N) + \gamma C(L+N)^2 + \delta D(L+N)^3 = 0,$$

valid for all integer values of N. Consequently, the coefficients of the various powers of N must vanish: hence

$$\begin{aligned} 0 &= \delta D, \\ 0 &= 3\delta DL + \gamma C, \\ 0 &= 3\delta DL^2 + 2\gamma CL + \beta B, \\ 0 &= \delta DL^3 + \gamma CL^2 + \beta BL + \alpha A, \end{aligned}$$

the last of which is the original postulated relation. From the first of these relations, it follows that

$$\delta = 0;$$

then, from the second, that

$$\gamma = 0;$$

then, from the third, that

$$\beta = 0;$$

and so, from the original relation, that

$$\alpha = 0.$$

The lemma is thus established.

It may also be proved that, if A, B, C, D be functions free from logarithms, and if they be such that a simple closed contour round a restores their initial values, except as to constant factors which are not the same for all, then no identical relation of the kind

$$\alpha A + \beta BL + \gamma CL^2 + \delta DL^3 = 0$$

can exist, in which α, β, γ, δ are constants different from zero. The proof is left as an exercise.

It is an immediate inference from the course of the lemma that no relation of the form

$$\alpha' u + \beta' u_3 + \gamma' u_2 + \delta' u_1 = 0$$

can exist, in which α', β', γ', δ' are constants different from zero; for proceeding as before, it would require

$$\begin{aligned}
0 &= \alpha'\psi_1, \\
0 &= 3\alpha'\psi_2 + \beta'\psi_1, \\
0 &= 3\alpha'\psi_3 + 2\beta'\psi_2 + \gamma'\psi_1, \\
0 &= \alpha'\psi_4 + \beta'\psi_3 + \gamma'\psi_2 + \delta'\psi_1,
\end{aligned}$$

which clearly are satisfied only if $\alpha' = \beta' = \gamma' = \delta' = 0$. Hence there is no homogeneous linear relation among the quantities u, u_1, u_2, u_3; and they therefore constitute a fundamental system for the linear equation if it exists.

28. If the equation exists, we must have

$$\Delta u = 0, \quad \Delta u_3 = 0, \quad \Delta u_2 = 0, \quad \Delta u_1 = 0;$$

and in the operator Δ, the functions P, Q, R, S are to be uniform functions of z in the vicinity of a. Let Z denote any function of z with the same characteristic properties as ψ_1, ψ_2, ψ_3, ψ_4; then with such an operator Δ, we have

$$\begin{aligned}
\Delta(ZL) &= L\Delta Z + Z', \\
\Delta(ZL^2) &= L^2\Delta Z + 2LZ' + Z'', \\
\Delta(ZL^3) &= L^3\Delta Z + 3L^2Z' + 3LZ'' + Z''',
\end{aligned}$$

where Z', Z'', Z''' are functions of the same characteristic properties as Z, that is, as ψ_1, ψ_2, ψ_3, ψ_4, and they are free from logarithms. Now as $\Delta u_1 = 0$, we have

$$\Delta\psi_1 = 0.$$

As $\Delta u_2 = 0$, we have

$$\Delta\psi_2 + L\Delta\psi_1 + \psi_1' = 0,$$

that is,

$$\Delta\psi_2 + \psi_1' = 0.$$

As $\Delta u_3 = 0$, we have

$$\Delta\psi_3 + 2(L\Delta\psi_2 + \psi_2') + L^2\Delta\psi_1 + 2L\psi_1' + \psi_1'' = 0,$$

that is, by using the two preceding relations,

$$\Delta\psi_3 + 2\psi_2' + \psi_1'' = 0.$$

As $\Delta u = 0$, we have

$$\Delta\psi_4 + 3(L\Delta\psi_3 + \psi_3') + 3(L^2\Delta\psi_2 + 2L\psi_2' + \psi_2'')$$
$$+ L^3\Delta\psi_1 + 3L^2\psi_1' + 3L\psi_1'' + \psi_1''' = 0,$$

that is, by using the three preceding relations,

$$\Delta\psi_4 + 3\psi_3' + 3\psi_2'' + \psi_1''' = 0.$$

Thus there are four equations; each of them involves the coefficients P, Q, R, S linearly and not homogeneously. The required inferences will be obtained if the equations determine P, Q, R, S as functions of z, uniform in the vicinity of a.

Now each of the functions ψ is such that $(z-a)^{-\alpha}\psi$ is a uniform function of $z-a$ in the vicinity of a; accordingly, let

$$(z-a)^{-\alpha}\psi_\mu = \theta_\mu, \qquad (\mu = 1, 2, 3, 4),$$

where each of the θ's denotes a uniform function. Substituting $(z-a)^\alpha\theta_\mu$ for ψ_μ in each of the four equations, the factor $(z-a)^\alpha$ can be removed after the differential operations have been performed; and then all the coefficients of P, Q, R, S, and the term independent of them, are uniform functions of z in the vicinity of a. Solving these four equations of the first degree for P, Q, R, S, we obtain expressions for them as uniform functions of $z-a$ in the vicinity of a. (In general, this point is a singularity for each of the expressions.) It follows that, for these values of P, Q, R, S, the four quantities u_1, u_2, u_3, u are integrals of the linear differential equation of the fourth order.

As already remarked, similar analysis leads to the establishment of the result for the general case; and thus the theorem is proved.

COROLLARY I. It is an obvious inference from the preceding theorem that, when a group of integrals is associated with a multiple root of the fundamental equation, any (Hamburger) sub-group, containing (say) n of the integrals, is a fundamental system of a linear equation of order n with uniform coefficients. Further, it is at once inferred that the n' members of that sub-group, which contain the lowest powers of the logarithm, constitute a fundamental system for a linear equation of order n' with uniform coefficients.

COROLLARY II. Similarly it may be established that one (Hamburger) sub-group containing n integrals, and another sub-group containing p integrals, constitute together a fundamental system for a linear equation of order $n+p$ with uniform coefficients. And so on, for combinations of the sub-groups generally.

Ex. Prove that if the linear equation in w has a sub-group of n integrals which, in the vicinity of a singularity a, have the form

$$\begin{aligned} w_1 &= \psi_1, \\ w_2 &= \psi_2 + \psi_1 L, \\ w_3 &= \psi_3 + 2\psi_2 L + \psi_1 L^2, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

where $2\pi i L = \log(z-a)$, and each of the functions ψ is such that $(z-a)^{-a}\psi$ is uniform, where $e^{2\pi i a}$ is a multiple root of the fundamental equation with which the sub-group of integrals is associated, then if the linear equation for v be constructed, where

$$w = w_1 \int v\,dz,$$

that linear equation has a corresponding sub-group of $n-1$ integrals of the form

$$\begin{aligned} v_1 &= \phi_1, \\ v_2 &= \phi_2 + \phi_1 L, \\ v_3 &= \phi_3 + 2\phi_2 L + \phi_1 L^2, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

where the functions ϕ are of the same character as the functions ψ.

CHAPTER III.

Regular Integrals; Equation having all its Integrals regular near a Singularity.

29. The general character of a fundamental system of integrals in the vicinity of a singularity has now been ascertained. For this purpose, the main property of the linear equation which has been used is that a is a singularity of the uniform coefficients; the precise nature of the singularity has not entered into the discussion. On the other hand, the functions ϕ which occur in the integrals are merely uniform in the vicinity of a: no knowledge as to the nature of the point a in relation to these functions has been derived, so that it might be an ordinary point, or a pole, or an essential singularity. Moreover, the index r in the expressions for the integrals is not definite; being equal to $\frac{1}{2\pi i}\log\theta$, it can have any one of an unlimited number of values differing from one another by integers. Hence, merely by changing r into one of the permissible alternatives, the character of a for the changed functions ϕ may be altered, if originally a were either an ordinary point or a pole: that character would not be altered, if a originally were an essential singularity.

It is obvious that the character of a for the integral is bound up with the nature of a as a singularity of the differential equation, each of them affecting, and possibly determining, the other. Accordingly, we proceed to the consideration of those linear equations of order m such that no singularity of the equation can be an essential singularity of any of the functions ϕ, which occur in the expression of the integrals in its vicinity. In this case, the functions ϕ, which are uniform in the vicinity of

a and therefore, by Laurent's theorem, can be expanded in a series of positive and negative integral powers of $z-a$ converging within an annulus round a, will at the utmost contain only a limited number of negative powers. To render r definite, we absorb all these negative indices into r by selecting that one among its values which makes the function ϕ in an expression

$$(z-a)^r \phi$$

finite (but not zero) when $z=a$.

An integral of the form

$$u = (z-a)^r [\phi_0 + \phi_1 \log(z-a) + \ldots + \phi_\kappa \{\log(z-a)\}^\kappa],$$

where $\phi_0, \phi_1, \ldots, \phi_\kappa$ are uniform functions having the point a either an ordinary point or a zero, is called* *regular* near a. When a value of r is chosen, such that $(z-a)^{-r}u$ is not zero and (if infinite) is only logarithmically infinite like

$$c_0 + c_1 \log(z-a) + \ldots + c_\kappa \{\log(z-a)\}^\kappa,$$

the integral is said to *belong* to the *index* (or *exponent*) r: the coefficients c being constants and not all of them zero. Similarly, when the singularity a is at infinity, and there is an integral

$$z^{-p}\left[\psi_0 + \psi_1 \log\frac{1}{z} + \ldots + \psi_\kappa\left(\log\frac{1}{z}\right)^\kappa\right],$$

where $\psi_0, \psi_1, \ldots, \psi_\kappa$ are uniform functions having $z=\infty$ for an ordinary point or a zero, the integral is said to belong to the index or exponent p.

It will be possible later to consider one class of integrals that do not answer to this definition of regularity: but it is clear that regular integrals, as a class, are the simplest class of integrals, and that the first attempt at obtaining integrals would be directed towards the regular integrals, if any. Accordingly, we proceed to consider the characteristics of linear differential equations which possess regular integrals: and in the first place, we shall consider equations all of whose integrals are regular in the vicinity of one of its singularities, in order to determine the form of equation in that vicinity.

* After Thomé, *Crelle*, t. LXXV (1873), p. 266. The use of this name for a function, which is not regular in the variable, may seem anomalous: but it is now wide-spread, and confusion might be caused by the introduction of another name. See footnote, p. 4.

As subsidiary to the investigation, one or two simple properties, associated with the indices to which the functions belong, will first be proved.

If a regular function u (in the present sense of the term) *belong to an index* r *and another* v *to an index* s, *then* $u \div v$ *belongs to the index* $r-s$: as is obvious from the definition.

If a regular function u *belong to an index* r, *then* $\frac{du}{dz}$ *belongs to the index* $r-1$. To prove this, let

$$u=(z-a)^r \sum_{\kappa=0}^{n} \phi_\kappa \{\log (z-a)\}^\kappa ;$$

then

$$\frac{du}{dz}=(z-a)^{r-1}\left[\sum_{\kappa=0}^{n} \{r\phi_\kappa+(z-a)\,\phi_\kappa'\}\{\log (z-a)\}^\kappa+\kappa\phi_\kappa \{\log (z-a)\}^{\kappa-1}\right],$$

so that $\frac{du}{dz}$ can only belong to the index $r-1$, if some at least of the coefficients of powers of $\log (z-a)$ are different from zero when $z=a$. These coefficients in succession are

$$\begin{gathered} r\phi_0+\phi_1, \\ r\phi_1+2\phi_2, \\ r\phi_2+3\phi_3, \\ \dots\dots\dots \\ r\phi_{n-1}+n\phi_n, \\ r\phi_n, \end{gathered}$$

when $z=a$ is substituted: they cannot all vanish, for then $\phi_0, \phi_1, \dots, \phi_n$ would vanish when $z=a$, so that $c_0, c_1, \dots, c_\kappa$ would all be zero, and then u would not belong to the index r. Thus $\frac{du}{dz}$ belongs to the index $r-1$.

There is one slight exception, viz. when u is uniform and the index r is zero; then $\frac{du}{dz}$ is also uniform, and it may even vanish when $z=a$; so that, if u were said to belong to the index 0, $\frac{du}{dz}$ could be said to belong to an index not less than 0.

Form of the Differential Equation when all the Integrals are regular near a Singularity.

30. As a first step towards the determination of the form of a differential equation that has all its integrals regular, we shall obtain the index to which the determinant of a fundamental system belongs. Let the system be $w_1, w_2, \dots, w_m$: and let the

indices of the members be r_1; r_2, ..., r_m respectively. We take the determinant in the form

$$Cw_1^m v_1^{m-1} u_1^{m-2} \dots$$

of § 12, where C is a constant.

The quantity v_1 is a solution of an equation, a fundamental system for which is given by

$$v_1 = \frac{d}{dz}\left(\frac{w_2}{w_1}\right), \quad v_2 = \frac{d}{dz}\left(\frac{w_3}{w_1}\right), \dots.$$

It is clear that, if w_1, w_2, ... are all free from logarithms, then v_1, v_2, ... are also free from them. If however there be a group or a sub-group of integrals associated with a repeated root θ of the fundamental equation, we may take (§ 23)

$$w_1' = \theta w_1, \quad w_2' = w_1 + \theta w_2,$$

so that

$$v_1' = \frac{d}{dz}\left(\frac{w_2'}{w_1'}\right) = \frac{d}{dz}\left(\frac{1}{\theta} + \frac{w_2}{w_1}\right) = v_1;$$

thus v_1 is uniform and therefore free from logarithms. Similarly, u_1 and all the quantities used in the special form of the determinant are free from logarithms.

The indices to which v_1, v_2, ... respectively belong are

$$r_2 - r_1 - 1, \quad r_3 - r_1 - 1, \quad r_4 - r_1 - 1, \dots$$

unless it should happen that, for instance, $r_2 = r_1$. In that case, we replace w_2 by $w_2 + \alpha w_1$, choosing α so as to make the new integral belong to an index higher than r_2 or r_1: this change will be supposed made in each case where it is required.

Again, the quantity u_1 is a solution of an equation, a fundamental system for which is given by

$$u_1 = \frac{d}{dz}\left(\frac{v_2}{v_1}\right), \quad u_2 = \frac{d}{dz}\left(\frac{v_3}{v_1}\right), \dots.$$

The index to which u_1 belongs is

$$r_3 - r_1 - 1 - (r_2 - r_1 - 1) - 1, \; = r_3 - r_2 - 1,$$

and so for u_2, ... ; that is, their indices are

$$r_3 - r_2 - 1, \quad r_4 - r_2 - 1, \dots.$$

And so on, down the series.

Hence the index to which

$$C w_1^m v_1^{m-1} u_1^{m-2} \ldots$$

belongs is

$$= m r_1 + (m-1)(r_2 - r_1 - 1) + (m-2)(r_3 - r_2 - 1) + \ldots$$
$$\ldots + 1\,(r_m - r_{m-1} - 1)$$
$$= r_1 + r_2 + \ldots + r_m - \tfrac{1}{2} m (m-1);$$

so that, denoting the determinant of the fundamental system by $\Delta(z)$ as in § 9, it follows that, in the vicinity of the singularity a, we have

$$\Delta(z) = (z-a)^{r_1 + r_2 + \ldots + r_m - \frac{1}{2} m (m-1)} R(z-a),$$

where R is a holomorphic function of its argument in that immediate vicinity, and does not vanish at a.

31. This result enables us to infer the form of the differential equation in the vicinity of the singularity a. Manifestly, the equation is

$$\frac{d^m w}{dz^m} = p_1 \frac{d^{m-1} w}{dz^{m-1}} + p_2 \frac{d^{m-2} w}{dz^{m-2}} + \ldots,$$

if

$$p_\kappa = \frac{\Delta_\kappa}{\Delta}, \qquad (\kappa = 1, \ldots, m),$$

where Δ is the determinant of the m integrals in the fundamental system, and Δ_κ is the determinant that is obtained from Δ on replacing the column $\dfrac{d^{m-\kappa} w_s}{dz^{m-\kappa}}$, for $s = 1, \ldots, m$, by the column $\dfrac{d^m w_s}{dz^m}$, for $s = 1, \ldots, m$.

Now consider a simple closed path round a. After it has been described, Δ and Δ_κ resume their initial values multiplied by the same constant factor, which is the non-vanishing determinant of the coefficients α (§ 13) in the expressions for the transformed integrals; thus p_κ is uniform for the circuit. Hence, when the expressions for the regular integrals are substituted in Δ and Δ_κ, all the terms involving powers of $\log(z-a)$ disappear. Moreover, Δ belongs to the index

$$r_1 + \ldots + r_m - \tfrac{1}{2} m (m-1);$$

and so far as concerns the index to which Δ_κ belongs, it contains a column of derivatives of order κ, $= m - (m - \kappa)$, higher than the corresponding column in Δ, so that Δ_κ belongs to the index

$$r_1 + \ldots + r_m - \tfrac{1}{2} m (m-1) - \kappa.$$

Hence p_κ belongs to the index $-\kappa$ and therefore, in the immediate vicinity of a, the form of p_κ is given by

$$p_\kappa = \frac{P_\kappa(z-a)}{(z-a)^\kappa},$$

where, at a and in the immediate vicinity of a, the function $P_\kappa(z-a)$ is a holomorphic function which, in the most general instance, does not vanish when $z=a$, though it may do so in special instances. As this result holds for $\kappa = 1, \ldots, m$, we conclude that, *when a homogeneous linear differential equation of order m has all its integrals regular in the vicinity of a singularity a, the equation is of the form*

$$\frac{d^m w}{dz^m} = \frac{P_1}{z-a}\frac{d^{m-1}w}{dz^{m-1}} + \frac{P_2}{(z-a)^2}\frac{d^{m-2}w}{dz^{m-2}} + \ldots + \frac{P_m}{(z-a)^m} w$$

in that vicinity, where $P_1, P_2, \ldots, P_m$ are holomorphic functions of $z-a$ in a region round a that encloses no other singularity of the equation.

Construction of Regular Integrals, by the Method of Frobenius.

32. The argument establishing this result, which is due to Fuchs*, is somewhat general, being directed mainly to the deduction of the uniform meromorphic character of the coefficients of the derivatives of w in the equation. No account is taken of the constants in the integrals: and it is conceivable that they might require the existence of relations among the constants in the functions $P_1, \ldots, P_m$. Hence for this reason alone, even if for no other, the converse of the above proposition cannot be assumed without an independent investigation. The conditions, which have been shewn to apply to the form of the equation, are necessary for the converse: their sufficiency has to be discussed. Accordingly, we now consider the integrals of the equation in the vicinity of the singularity†.

Denoting the singularity by a, we write

$$z - a = x, \quad P_r(z-a) = p_r(x) = p_r, \qquad (r = 1, \ldots, m);$$

* *Crelle*, t. LXVI (1866), p. 146.

† The following method is due to Frobenius; references will be given later.

so that the equation can be taken in the form

$$Dw = x^m \frac{d^m w}{dx^m} - \left(x^{m-1} p_1 \frac{d^{m-1} w}{dx^{m-1}} + \dots + x p_{m-1} \frac{dw}{dx} + p_m w \right) = 0,$$

valid in the vicinity of $x = 0$.

If regular integrals exist in this vicinity, they are of the form indicated in §§ 18, 24, the simplest of them being of the form

$$\begin{aligned} w = x^\rho \sum_{\nu=0}^{\infty} g_\nu x^\nu &= \sum_{\nu=0}^{\infty} g_\nu x^{\rho+\nu} \\ &= g(x, \rho), \end{aligned}$$

say; should this be an integral, it must satisfy the equation identically. We have

$$\begin{aligned} Dx^\sigma &= \{\sigma(\sigma-1)\dots(\sigma-m+1) - \sigma(\sigma-1)\dots(\sigma-m+2)p_1 - \dots \\ &\qquad\qquad \dots - p_m\} x^\sigma \\ &= x^\sigma f(x, \sigma), \end{aligned}$$

say. Here, $f(x, \sigma)$ is a holomorphic function of x in the vicinity of the x-origin and is a polynomial of degree m in σ, the coefficient of σ^m being unity: so that, if it be arranged as a power-series in x, we have

$$f(x, \sigma) = f_0(\sigma) + x f_1(\sigma) + x^2 f_2(\sigma) + \dots,$$

where $f_0(\sigma)$ is a polynomial in σ of degree m, and $f_1(\sigma)$, $f_2(\sigma)$, ... are polynomials in σ of degree not higher than $m-1$. Then

$$\begin{aligned} Dg(x, \rho) &= \sum_{\nu=0}^{\infty} g_\nu D x^{\rho+\nu} \\ &= \sum_{\nu=0}^{\infty} g_\nu x^{\rho+\nu} f(x, \rho+\nu) \\ &= \sum_{\nu=0}^{\infty} x^{\rho+\nu} \{g_\nu f_0(\rho+\nu) + g_{\nu-1} f_1(\rho+\nu-1) + \dots + g_0 f_\nu(\rho)\}. \end{aligned}$$

If the postulated expression for w is to satisfy the equation, the coefficients of the various powers of x on the right-hand side must vanish: hence

$$\begin{aligned} 0 &= g_0 f_0(\rho), \\ 0 &= g_0 f_1(\rho) + g_1 f_0(\rho+1), \\ 0 &= g_0 f_2(\rho) + g_1 f_1(\rho+1) + g_2 f_0(\rho+2), \end{aligned}$$

and so on. These equations shew that the values of ρ, which are to be considered, are the roots of the algebraical equation

$$f_0(\rho) = 0$$

of degree m in ρ: and that, for each such value of ρ,

$$g_\nu = \frac{g_0}{f_0(\rho+1)\, f_0(\rho+2) \ldots f_0(\rho+\nu)} h_\nu(\rho),$$

where $-h_\nu(\rho)$ is the value of the determinant

$$\begin{vmatrix} f_0(\rho+1), & 0 , & 0 , & \ldots, & 0 , & f_1(\rho) \\ f_1(\rho+1), & f_0(\rho+2), & 0 , & \ldots, & 0 , & f_2(\rho) \\ f_2(\rho+1), & f_1(\rho+2), & f_0(\rho+3), & \ldots, & 0 , & f_3(\rho) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ f_{\nu-2}(\rho+1), & f_{\nu-3}(\rho+2), & f_{\nu-4}(\rho+3), & \ldots, & f_0(\rho+\nu-1), & f_{\nu-1}(\rho) \\ f_{\nu-1}(\rho+1), & f_{\nu-2}(\rho+2), & f_{\nu-3}(\rho+3), & \ldots, & f_1(\rho+\nu-1), & f_\nu(\rho) \end{vmatrix},$$

so that $h_\nu(\rho)$ is a polynomial in ρ.

If no two of the roots of the equation $f_0(\rho)=0$ differ by whole numbers, then no denominator in the expressions for the successive coefficients g_ν vanishes; the expression $g(x, \rho)$ is formally adequate for an integral, but the convergence of the series must be established to ensure the significance of the expression.

If a group of roots of the equation $f_0(\rho)=0$ differ among one another by whole numbers, let them be

$$\rho,\ \rho+\epsilon_1,\ \ldots,\ \rho+\epsilon,$$

where the real part of ρ is the smallest, and that of $\rho+\epsilon$ is the largest, among the real parts of these roots; equality of roots would be indicated by corresponding equalities among the positive integers $0, \epsilon_1, \ldots, \epsilon$. We then take

$$g_0 = f_0(\rho+1) \ldots f_0(\rho+\epsilon)\, g,$$

and thus secure that no one of the coefficients g_ν becomes infinite. The condition, that the equation shall formally be satisfied, has imposed no limitation upon g_0, which accordingly can be regarded as arbitrary; hence g also can be regarded as arbitrary.

33. In order to deal with both sets of cases simultaneously, the formal expression is constructed in a slightly different manner. A parametric quantity α is introduced and it is made to vary within regions round the roots of $f_0(\rho)=0$, each such region round a root being chosen so as to contain no other root. The quantity g_0 in the first set of cases, and the quantity g in the second set,

are arbitrary; they are made arbitrary functions of α. Quantities $g_1, g_2, \ldots$ are determined by the equations

$$0 = g_1 f_0(\alpha+1) + g_0 f_1(\alpha),$$
$$0 = g_2 f_0(\alpha+2) + g_1 f_1(\alpha+1) + g_0 f_2(\alpha),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

the same in form as the earlier equations other than the first: these quantities g are functions of α. Moreover, we have

$$g_\nu(\alpha) = \frac{g_0(\alpha)}{f_0(\alpha+1) f_0(\alpha+2) \ldots f_0(\alpha+\nu)} h_\nu(\alpha);$$

in consequence of the assumption as to $g_0(\alpha)$ in the second set of cases, and of the regions round the roots of $f_0(\rho)=0$ in which α varies, it follows that the quantities $g_1, g_2, \ldots$ are each of them finite for all variations of α within the regions indicated. We thus have an expression

$$y = g(x, \alpha) = \sum_{\nu=0}^{\infty} g_\nu x^{\alpha+\nu};$$

also

$$\begin{aligned} Dy &= \sum_{\nu=0}^{\infty} g_\nu D x^{\alpha+\nu} \\ &= \sum_{\nu=0}^{\infty} g_\nu x^{\alpha+\nu} f(x, \alpha+\nu) \\ &= \sum_{\nu=0}^{\infty} x^{\alpha+\nu} \{g_\nu f_0(\alpha+\nu) + g_{\nu-1} f_1(\alpha+\nu-1) + \ldots + g_0 f_\nu(\alpha)\} \\ &= g_0(\alpha) f_0(\alpha) x^\alpha, \end{aligned}$$

the coefficient of every power of x except x^α vanishing, in consequence of the law of formation of the quantities g.

34. We proceed next to consider the convergence of the power-series for y, before bringing the equation satisfied by y into relation with the original differential equation. We denote by R the radius of a circle round the x-origin within and upon which the functions $p_1, \ldots, p_m$ are holomorphic: so that the circle lies within the domain of this origin. Then $f(x, \alpha)$ and its derivatives with regard to x are also holomorphic for values of x within the circle and for all values of α considered. As the first of them, say $f'(x, \alpha)$, is of degree in α one less than $f(x, \alpha)$, it is convenient to consider that first derivative: let $M(\alpha)$ be the greatest value of $|f'(x, \alpha)|$ for the values of x and α, so that, as

$$f'(x, \alpha) = \sum_{\nu=0}^{\infty} (\nu+1) f_{\nu+1}(\alpha) x^\nu,$$

we have*

$$|(\nu+1) f_{\nu+1}(\alpha)| \leqslant \frac{M(\alpha)}{R^\nu},$$

and therefore, as $\nu+1$ is a positive integer $\geqslant 1$, also

$$|f_{\nu+1}(\alpha)| \leqslant R^{-\nu} M(\alpha).$$

By the definition of the regions of variation of α and the significance of the integer ϵ, it follows that the quantity $f_0(\alpha+\nu+1)$ is distinct from zero, for all values of $\nu \geqslant \epsilon$ and for all values of α; hence, as

$$g_{\nu+1} = -\frac{1}{f_0(\alpha+\nu+1)} \{g_0 f_{\nu+1}(\alpha) + g_1 f_\nu(\alpha) + \ldots + g_\nu f_1(\alpha+\nu)\}$$

from the equations that define the coefficients g, it follows that

$$\begin{aligned} |g_{\nu+1}| &\leqslant \frac{1}{|f_0(\alpha+\nu+1)|} \{|g_0|\,|f_{\nu+1}(\alpha)| + \ldots + |g_\nu|\,|f_1(\alpha+\nu)|\} \\ &\leqslant \frac{1}{|f_0(\alpha+\nu+1)|} \{|g_0| R^{-\nu} M(\alpha) + |g_1| R^{-\nu+1} M(\alpha+1) + \ldots \\ &\qquad\qquad \ldots + |g_\nu| M(\alpha+\nu)\} \\ &\leqslant \gamma_{\nu+1}, \end{aligned}$$

say, where $\gamma_{\nu+1}$ denotes the expression on the right-hand side. Evidently

$$\begin{aligned} \gamma_{\nu+1} |f_0(\alpha+\nu+1)| - \gamma_\nu |f_0(\alpha+\nu)| R^{-1} &= |g_\nu| M(\alpha+\nu) \\ &\leqslant \gamma_\nu M(\alpha+\nu), \end{aligned}$$

and therefore

$$\gamma_{\nu+1} \leqslant \gamma_\nu \left\{\frac{M(\alpha+\nu)}{|f_0(\alpha+\nu+1)|} + \frac{1}{R}\left|\frac{f_0(\alpha+\nu)}{f_0(\alpha+\nu+1)}\right|\right\}.$$

Let a series of quantities Γ_ν be determined by the equation

$$\Gamma_{\nu+1} = \Gamma_\nu \left\{\frac{M(\alpha+\nu)}{|f_0(\alpha+\nu+1)|} + \frac{1}{R}\left|\frac{f_0(\alpha+\nu)}{f_0(\alpha+\nu+1)}\right|\right\},$$

for values of $\nu \geqslant \epsilon$; and let $\Gamma_\epsilon = \gamma_\epsilon$. Then all the quantities Γ thus determined are positive, and we have

$$|g_{\nu+1}| \leqslant \gamma_{\nu+1} \leqslant \Gamma_{\nu+1}.$$

Consider the series

$$\Gamma_\epsilon x^\epsilon + \Gamma_{\epsilon+1} x^{\epsilon+1} + \ldots + \Gamma_\nu x^\nu + \ldots;$$

* *T. F.*, § 22.

its radius of convergence is determined* as the reciprocal of

$$\operatorname*{Lim}_{\nu=\infty} \frac{\Gamma_{\nu+1}}{\Gamma_\nu}.$$

Now $M(\theta)$ is the greatest value of the modulus of

$$-\theta(\theta-1)\ldots(\theta-m+2)p_1' - \ldots - p_m'$$

within the circle $|x| = R$. As the functions $p_1', \ldots, p_m'$ are holomorphic within the circle, there are finite upper limits to the values of $|p_1'|, \ldots, |p_m'|$ within the region, say $M_1, \ldots, M_m$; then

$$M(\theta) \leqslant \sigma(\sigma+1)\ldots(\sigma+m-2)M_1 + \ldots + M_m \leqslant \psi(\sigma),$$

say, where $|\theta| = \sigma$. Again

$$f_0(\theta) = \theta(\theta-1)\ldots(\theta-m+1) - \theta(\theta-1)\ldots(\theta-m+2)p_1(0) - \ldots \\ \ldots - p_m(0),$$

so that, if

$$\phi(\sigma) = -\sigma^m + \sigma(\sigma+1)\ldots(\sigma+m-1) \\ + \sigma(\sigma+1)\ldots(\sigma+m-2)|p_1(0)| + \ldots + |p_m(0)|,$$

we have

$$|f_0(\theta)| \geqslant |\theta^m| - |f_0(\theta) - \theta^m| \geqslant \sigma^m - |f_0(\theta) - \theta^m|,$$

and

$$|f_0(\theta) - \theta^m| \leqslant \phi(\sigma),$$

the term in θ^m being absent from $f_0(\theta) - \theta^m$, and the term in σ^m being absent from $\phi(\sigma)$. Moreover, as these quantities are required for a limit when ν tends to infinity, the quantities σ and θ will be large where they occur; thus σ^m is greater than $\phi(\sigma)$, which is a polynomial in σ only of degree $m-1$. Hence

$$|f_0(\theta)| \geqslant \sigma^m - \phi(\sigma).$$

Returning now to the expression for $\Gamma_{\nu+1} \div \Gamma_\nu$, let β denote $|\alpha|$; then

$$|\alpha+\nu+1| \geqslant \nu+1-\beta,$$

so that

$$|\alpha+\nu+1|^m \geqslant (\nu+1-\beta)^m.$$

Again,

$$|\alpha+\nu+1| \leqslant \nu+1+\beta,$$

so that

$$\phi(|\alpha+\nu+1|) \leqslant \phi(\nu+1+\beta),$$

and therefore

$$|f_0(\alpha+\nu+1)| \geqslant (\nu+1-\beta)^m - \phi(\nu+1+\beta).$$

* Chrystal's *Algebra*, vol. II, p. 150.

Finally, $|\alpha + \nu| \leqslant \nu + \beta$, and therefore

$$M(\alpha + \nu) \leqslant \psi\,|\alpha + \nu| \leqslant \psi(\nu + \beta);$$

so that

$$\frac{M(\alpha + \nu)}{|f_0(\alpha + \nu + 1)|} \leqslant \frac{\psi(\nu + \beta)}{(\nu + 1 - \beta)^m - \phi(\nu + 1 + \beta)}.$$

Now $\psi(\sigma)$ is a polynomial in σ of degree $m - 1$, as also is $\phi(\sigma)$; hence, owing to the term $(\nu + 1 - \beta)^m$ in the denominator on the right-hand side, we have

$$\operatorname*{Lim}_{\nu=\infty} \frac{M(\alpha + \nu)}{|f_0(\alpha + \nu + 1)|} = 0,$$

for all values of β, that is, for all values of α within its regions of variation. Again, as $f_0(\alpha)$ is a polynomial in α of degree m, it follows that

$$\operatorname*{Lim}_{\nu=\infty} \frac{f_0(\alpha + \nu)}{f_0(\alpha + \nu + 1)} = 1,$$

for all the values of α, and therefore

$$\operatorname*{Lim}_{\nu=\infty} \left| \frac{f_0(\alpha + \nu)}{f_0(\alpha + \nu + 1)} \right| = 1.$$

Using these results, we have

$$\operatorname*{Lim}_{\nu=\infty} \frac{\Gamma_{\nu+1}}{\Gamma_\nu} = \frac{1}{R};$$

and therefore the series

$$\Gamma_\epsilon x^\epsilon + \Gamma_{\epsilon+1} x^{\epsilon+1} + \dots$$

converges within the circle $|x| = R$ and for the values of α: consequently also the series

$$\gamma_\epsilon x^\epsilon + \gamma_{\epsilon+1} x^{\epsilon+1} + \dots$$

converges for the same ranges of variation for x and α. The addition of a limited number of terms that are finite does not affect the convergence: and therefore

$$\sum_{\nu=0}^{\infty} g_\nu x^\nu$$

converges, for values of x within the circle $|x| = R$, and for values of α within its regions of variation.

Let any region for α be defined by the condition $|\alpha - \rho| \leqslant r$. Then the series converges absolutely within the x-circle of radius R and the α-circle of radius r. Let $R' < R$, and $r' < r$; and let κ, κ' denote any finite positive quantities which may be taken

small: then* the series converges uniformly for values of x and α such that

$$|x| < R' - \kappa, \quad |\alpha - \rho| < r' - \kappa'.$$

Thus the series converges uniformly in the vicinity of the x-origin, for all values of α in the regions assigned to that parametric variable.

By a theorem due to Weierstrass†, the uniform convergence of the series, which is a power-series in x and a function-series in α, permits it to be differentiated with regard to α; and the derivatives of the series are the derivatives of the function represented by the series within the α-regions considered.

Significance of the Indicial Equation.

35. We now associate the factor x^α with the preceding series, and then we have

$$g(x, \alpha) = x^\alpha \sum_{\nu=0}^{\infty} g_\nu x^\nu = \sum_{\nu=0}^{\infty} g_\nu x^{\alpha+\nu}$$

as a series, which converges uniformly within a finite region round the x-origin and can be differentiated with regard to α term by term. (It may happen that the origin must be excluded from the region of continuity of $g(x, \alpha)$, as would be the case if the real part of α were negative; the origin must then be excluded from the region of continuity of the derivatives with regard to α, owing to the presence of terms such as $g_0 x^\alpha \log x$.)

The function $g(x, \alpha)$ thus determined has been shewn to satisfy the equation

$$Dg(x, \alpha) = x^\alpha f_0(\alpha) g_0(\alpha).$$

As associated with the original differential equation, this result requires the consideration of the algebraical equation (hereafter called the *indicial equation*)

$$f_0(\rho) = 0$$

of degree m. The preceding analysis indicates that two cases have to be discussed, according as a root does not, or does, belong to a group the members of which differ from one another by

* The uniform convergence with regard to x is known, *T. F.*, § 14, *finis*. The uniform convergence with regard to α is established by means of a theorem due to Osgood, *Bull. Amer. Math. Soc.*, t. III (1897), p. 73; see the *Note*, p. 122, at the end of this chapter.

† *Ges. Werke*, t. II, p. 208; see *T. F.*, §§ 82, 83.

whole numbers (including a difference by zero, so as to take account of equal roots).

Firstly, let ρ be a simple root of $f_0(\rho)=0$, in the sense that it is not equal to any other root and that the difference between ρ and any other root is not a whole number. Then when we take $\alpha=\rho$, all the coefficients $g_1, g_2, \ldots$ in $g(x, \rho)$ are finite; we have

$$Dg(x, \rho)=0,$$

that is,

$$w=g(x, \rho)$$

is an integral of the differential equation: it is associated with the simple root ρ of the equation $f_0(\rho)=0$, and it is a regular integral.

36. Secondly, let $\rho_0, \rho_1, \ldots, \rho_n$ constitute a group of roots of $f_0(\rho)=0$, differing from one another by whole numbers and from each of the other roots by quantities that are not whole numbers; and let them be arranged so that the real parts of the successive roots decrease: thus the real part of ρ_0 is the greatest and that of ρ_n is the least in the group. In order to secure the finiteness of the coefficients $g_1, g_2, \ldots$, it now is necessary to take

$$g_0(\alpha)=f_0(\alpha+1)f_0(\alpha+2)\ldots f_0(\alpha+\epsilon)\,g(\alpha),\quad =f(\alpha)\,g(\alpha),$$

say, where $\epsilon \geqslant \rho_0-\rho_n$, and $g(\alpha)$ is an arbitrary function of α: and now

$$Dg(x, \alpha)=x^\alpha g(\alpha)\prod_{s=0}^{\epsilon} f_0(\alpha+s)=x^\alpha g(\alpha)\,F(\alpha),$$

where

$$F(\alpha)=\prod_{s=0}^{\epsilon}\{f_0(\alpha+s)\}.$$

Further, there may be equalities among the roots in the group: let $\rho_0, \rho_i, \rho_j, \rho_k, \ldots$ be the distinct roots taken from the succession in the group as they occur, so that ρ_0 is a root of multiplicity i, ρ_i of multiplicity $j-i$, ρ_j of multiplicity $k-j$, and so on. Then in $F(\alpha)$, there is a factor $(\alpha-\rho_0)^i$ through its occurrence in $f_0(\alpha)$; there is a factor $(\alpha-\rho_i)^j$, through the occurrence of $(\alpha-\rho_i)^{j-i}$ in $f_0(\alpha)$, and the occurrence of $(\alpha-\rho_i)^i$ in $f_0(\alpha+\rho_0-\rho_i)$; there is a factor $(\alpha-\rho_j)^k$, through the occurrence of $(\alpha-\rho_j)^{k-j}$ in $f_0(\alpha)$, the occurrence of $(\alpha-\rho_j)^{j-i}$ in $f_0(\alpha+\rho_i-\rho_j)$, and the occurrence of $(\alpha-\rho_j)^i$ in $f_0(\alpha+\rho_0-\rho_j)$. Now

$$0<i<j<k<\ldots,$$

so that, for $F(\alpha)$, ρ_0 is a root of multiplicity i, that is, 1 at least: ρ_i is a root of multiplicity j, that is, $i+1$ at least; ρ_j is a root of multiplicity k, that is, $j+1$ at least; and so on. Hence if ρ_κ be a root in the group as arranged, it is a root of $F(\alpha)$ of multiplicity $\kappa+1$ at least; and therefore

$$\left[\frac{\partial^\mu F(\alpha)}{\partial \alpha^\mu}\right]_{\alpha=\rho_\kappa} = 0,$$

for $\mu = 0, 1, \ldots, \kappa$. But

$$Dg(x, \alpha) = x^\alpha g(\alpha) F(\alpha),$$

and $g(x, \alpha)$ can be differentiated with regard to α; hence

$$D\left[\frac{\partial^\mu g(x, \alpha)}{\partial \alpha^\mu}\right]_{\alpha=\rho_\kappa} = \left[\frac{\partial^\mu}{\partial \alpha^\mu}\{x^\alpha g(\alpha) F(\alpha)\}\right]_{\alpha=\rho_\kappa}$$
$$= 0,$$

for $\mu = 0, 1, \ldots, \kappa$ certainly, and for all other integer values of μ less than the multiplicity of ρ_κ as a root of $F(\alpha) = 0$. Consequently, the expression

$$w = \left[\frac{\partial^\mu g(x, \alpha)}{\partial \alpha^\mu}\right]_{\alpha=\rho_\kappa}$$
$$= \frac{\partial^\mu g(x, \rho_\kappa)}{\partial {\rho_\kappa}^\mu},$$

say, for the same values of μ, provides a set of integrals of the equation.

Moreover, each of the distinct roots in the group thus provides a set of integrals; we must therefore enquire how many of the integrals out of this aggregate are linearly independent.

37. We first consider the members of any set; they are furnished by

$$\frac{\partial^\mu g(x, \alpha)}{\partial \alpha^\mu},$$

for a value ρ assigned to α, and for a number of values of μ, say $0, 1, \ldots, \kappa$. Now

$$g(x, \alpha) = x^\alpha \sum_{\nu=0}^{\infty} g_\nu(\alpha) x^\nu;$$

and therefore

$$\frac{\partial^\mu g(x, \alpha)}{\partial \alpha^\mu}$$

$$= x^\alpha \left[\sum_{\nu=0}^{\infty} \frac{\partial^\mu g_\nu}{\partial \alpha^\mu} x^\nu + \mu (\log x) \sum_{\nu=0}^{\infty} \frac{\partial^{\mu-1} g_\nu}{\partial \alpha^{\mu-1}} x^\nu + \ldots + (\log x)^\mu \sum_{\nu=0}^{\infty} g_\nu(\alpha) x^\nu\right],$$

where it will be noticed that the coefficient of the highest power of $\log x$ on the right-hand side is $g(x, \alpha)$. Hence the set is

$$\begin{aligned}
y_0 &= w, \\
&= g(x, \alpha), \\
y_1 &= w \log x + w_1, \\
y_2 &= w (\log x)^2 + 2w_1 \log x + w_2, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
y_\kappa &= w (\log x)^\kappa + \kappa w_1 (\log x)^{\kappa-1} + \tfrac{1}{2}\kappa(\kappa-1) w_2 (\log x)^{\kappa-2} + \dots \\
&\qquad\qquad + \kappa w_{\kappa-1} \log x + w_\kappa,
\end{aligned}$$

where the coefficients w_p are independent of logarithms. From the fact that y_p contains a power of $\log x$ higher than any occurring in $y_0, y_1, \dots, y_{p-1}$, it follows (by the lemma in § 27) that no linear relation of the form

$$c_0 y_0 + c_1 y_1 + \dots + c_\kappa y_\kappa = 0$$

can subsist among the integrals.

38. Next, we consider the sets in turn, associated with the values $\rho_0, \rho_i, \rho_j, \dots$ of α, as arranged in decreasing order of real parts. The earliest of them is given by $\alpha = \rho_0$: and it contains the i members

$$\left[\frac{\partial^\mu g(x, \alpha)}{\partial \alpha^\mu}\right]_{\alpha=\rho_0},$$

for $\mu = 0, 1, \dots, i-1$. Now

$$\begin{aligned}
f_0(\alpha) &= (\alpha - \rho_0)^i (\alpha - \rho_i)^{j-i} (\alpha - \rho_j)^{k-j} \dots (\alpha - \rho_l)^{n+1-l} A_1, \\
F(\alpha) &= (\alpha - \rho_0)^i (\alpha - \rho_i)^j \quad (\alpha - \rho_j)^k \quad \dots (\alpha - \rho_l)^{n+1} \quad A_2,
\end{aligned}$$

and therefore

$$f(\alpha) = \frac{F(\alpha)}{f_0(\alpha)} = (\alpha - \rho_i)^i (\alpha - \rho_j)^j \dots (\alpha - \rho_l)^l A_3,$$

where A_1, A_2, A_3 are quantities which neither vanish nor become infinite for any of the values $\rho_0, \rho_i, \dots, \rho_l$ of α. Also

$$g_0(\alpha) = g(\alpha) f(\alpha),$$

where $g(\alpha)$ is an arbitrary function of α; so that $g_0(\alpha)$ does not vanish for $\alpha = \rho_0$; and therefore the various quantities derived

from $g_0(\alpha)$ for $\alpha=\rho_0$, including $g_0(\alpha)$ itself, given by $\dfrac{\partial^\mu g_0(\alpha)}{\partial\alpha^\mu}$ for $\mu=0, 1, \ldots, i-1$ do not all vanish. Further

$$\left[\frac{\partial^\mu g(x, \alpha)}{\partial\alpha^\mu}\right]_{\alpha=\rho_0}$$

$$= x^{\rho_0}\left[\sum_{\nu=0}^{\infty}\frac{\partial^\mu g_\nu}{\partial\rho_0{}^\mu}x^\nu + \mu(\log x)\sum_{\nu=0}^{\infty}\frac{\partial^{\mu-1} g_\nu}{\partial\rho_0{}^{\mu-1}}x^\nu + \ldots + (\log x)^\mu \sum_{\nu=0}^{\infty} g_\nu x^\nu\right],$$

which is one of the integrals; as the quantities

$$\frac{\partial^\mu g_0}{\partial\rho_0{}^\mu},\ \frac{\partial^{\mu-1} g_0}{\partial\rho_0{}^{\mu-1}},\ \ldots,\ \frac{\partial g_0}{\partial\rho_0},\ g_0$$

do not all vanish, this integral belongs to the index ρ_0; and the coefficient of the highest power of $\log x$ is $g(x, \rho_0)$. The first set thus gives i linearly independent integrals obtained by taking $\mu=0, 1, \ldots, i-1$ in the preceding expression. That which arises from $\mu=0$ is

$$\begin{aligned} w &= g(x, \rho_0) \\ &= x^{\rho_0} g(\rho_0)\{1 + xh_1(\rho_0) + x^2h_2(\rho_0) + x^3h_3(\rho_0) + \ldots\}, \end{aligned}$$

where all the coefficients are finite: thus it is a constant multiple of

$$x^{\rho_0} + x^{\rho_0+1}h_1(\rho_0) + x^{\rho_0+2}h_2(\rho_0) + \ldots,$$

an integral that is uniquely determinate.

Now consider the second set: it is given by $\alpha=\rho_i$, and it contains the members

$$\left[\frac{\partial^\mu g(x, \alpha)}{\partial\alpha^\mu}\right]_{\alpha=\rho_i},$$

for $\mu=0, \ldots, i-1, i, i+1, \ldots, j-1$. The value of $g(x, \alpha)$ is

$$\begin{aligned} g(x, \alpha) &= x^\alpha \sum_{\nu=0}^{\infty} g_\nu(\alpha) x^\nu \\ &= x^\alpha \sum_{\nu=0}^{\rho_0-\rho_i-1} g_\nu(\alpha) x^\nu + x^{\alpha+\rho_0-\rho_i}\sum_{\nu=0}^{\infty} g_{\nu+\rho_0-\rho_i}(\alpha) x^\nu. \end{aligned}$$

As regards the first part of this expression, we note that all the coefficients $g_\nu(\alpha)$ for $\nu=0, 1, \ldots, \rho_0-\rho_i-1$ contain the factor $(\alpha-\rho_i)^i$; and therefore all the derivatives

$$\left[\frac{\partial^\mu}{\partial\alpha^\mu}\left\{x^\alpha \sum_{\nu=0}^{\rho_0-\rho_i-1} g_\nu(\alpha) x^\nu\right\}\right]_{\alpha=\rho_i}$$

for $\mu = 0, 1, \ldots, i-1$, vanish when α is made equal to ρ_i, while they do not necessarily vanish for higher values of μ.

As regards the second part of the expression for $g(x, \alpha)$, we write it in the full form

$$x^{\alpha+\rho_0-\rho_i} g_{\rho_0-\rho_i}(\alpha) + x^{\alpha+\rho_0-\rho_i+1} g_{\rho_0-\rho_i+1}(\alpha) + \ldots;$$

when $\alpha = \rho_i$, this becomes

$$x^{\rho_0} g_{\rho_0-\rho_i}(\rho_i) + x^{\rho_0+1} g_{\rho_0-\rho_i+1}(\rho_i) + \ldots,$$

which accordingly is an integral, and it belongs to the index ρ_0, being free from logarithms. But it has been seen that the integral, which belongs to the index ρ_0 and is free from logarithms, is uniquely determinate, being $g(x, \rho_0)$; hence the foregoing integral, being the non-vanishing part of $g(x, \alpha)$ when α is ρ_i, is a constant multiple of $g(x, \rho_0)$, say $Kg(x, \rho_0)$. It might happen that $K = 0$.

A similar result holds for the derivatives of $g(x, \alpha)$, for the values $\mu = 1, \ldots, i-1$.

Consequently, it follows that the integrals

$$\left[\frac{\partial^\mu}{\partial\alpha^\mu} g(x, \alpha)\right]_{\alpha=\rho_i}$$

for $\mu = 0, 1, \ldots, i-1$, can be compounded from the integrals of the first set; they are i in number, but they provide no integrals additional to those in the first set; and therefore, without limiting the range of their own set, they can be replaced by the i integrals of that set. As for the remainder arising from other values of μ, they are

$$x^{\rho_i}\left[\sum_{\nu=0}^{\infty}\frac{\partial^\mu g_\nu(\rho_i)}{\partial\rho_i^\mu}x^\nu + \mu(\log x)\sum_{\nu=0}^{\infty}\frac{\partial^{\mu-1} g_\nu(\rho_i)}{\partial\rho_i^{\mu-1}}x^\nu + \ldots + (\log x)^\mu\sum_{\nu=0}^{\infty} g_\nu(\rho_i)\, x^\nu\right]$$

for $\mu = i, i+1, \ldots, j-1$. Now

$$g_0(\alpha) = g(\alpha)(\alpha-\rho_i)^i(\alpha-\rho_j)^j \ldots (\alpha-\rho_l)^l A_3,$$

so that the quantities

$$\left[\frac{\partial^{\mu-s} g_0(\alpha)}{\partial\alpha^{\mu-s}}\right]_{\alpha=\rho_i},$$

for the values $s = 0, 1, \ldots, \mu$ in any one integral, and for the values $\mu = i, i+1, \ldots, j-1$ in the different integrals, do not all vanish.

All these integrals therefore belong to the index ρ_i, and they are $j-i$ in number. Moreover, the original set of j integrals, composed of these $j-i$ and of the replaced i integrals, was a set of linearly independent members; and therefore we now have $j-i$ integrals, linearly independent of one another and of the former set of i integrals. Thus our second set provides $j-i$ new integrals, distinct from those of the first set; and each of them belongs to the index ρ_i. The first of them is given by $\mu = i$: it is

$$x^{\rho_i}\left[\sum_{\nu=0}^{\infty} \frac{\partial^i g_\nu(\rho_i)}{\partial \rho_i^{\,i}} x^\nu + i \log x \sum_{\nu=0}^{\infty} \frac{\partial^{i-1} g_\nu(\rho_i)}{\partial \rho_i^{\,i-1}} x^\nu + \dots\right],$$

which certainly contains terms not involving $\log x$; if $j-1>i$, the second of them is

$$x^{\rho_i}\left[\sum_{\nu=0}^{\infty} \frac{\partial^{i+1} g_\nu(\rho_i)}{\partial \rho_i^{\,i+1}} x^\nu + (i+1) \log x \sum_{\nu=0}^{\infty} \frac{\partial^{i} g_\nu(\rho_i)}{\partial \rho_i^{\,i}} x^\nu + \dots\right],$$

which certainly contains terms multiplying the first power of $\log x$; if $j-1>i+1$, the third of them certainly contains terms involving the second power of $\log x$; and so on.

The third set among our integrals is connected with the value $\alpha = \rho_j$, and it is given by

$$\left[\frac{\partial^\mu g(x, \alpha)}{\partial \alpha^\mu}\right]_{\alpha=\rho_j},$$

for $\mu = 0, 1, \dots, k-1$. Now

$$\begin{aligned} g(x, \alpha) &= x^\alpha \sum_{\nu=0}^{\infty} g_\nu(\alpha) x^\nu \\ &= x^\alpha \sum_{\nu=0}^{\rho_i-\rho_j-1} g_\nu(\alpha) x^\nu + x^{\alpha+\rho_i-\rho_j} \sum_{\nu=0}^{\infty} g_{\nu+\rho_i-\rho_j}(\alpha) x^\nu. \end{aligned}$$

The coefficients $g_\nu(\alpha)$ contain $(\alpha-\rho_j)^j$ as factor for all integers ν which are less than $\rho_i - \rho_j$; hence the quantities

$$\left[\frac{\partial^\mu}{\partial \alpha^\mu}\left\{x^\alpha \sum_{\nu=0}^{\rho_i-\rho_j-1} g_\nu(\alpha) x^\nu\right\}\right]_{\alpha=\rho_j}$$

vanish for $\mu = 0, 1, \dots, j-1$, and are different from zero only for $\mu = j, j+1, \dots, k-1$. As in the case of the preceding set, the quantities

$$\left[\frac{\partial^\mu}{\partial \alpha^\mu}\left\{x^{\alpha+\rho_i-\rho_j} \sum_{\nu=0}^{\infty} g_{\nu+\rho_i-\rho_j}(\alpha) x^\nu\right\}\right]_{\alpha=\rho_j},$$

for $\mu = 0, 1, \ldots, i-1$, are linearly expressible in terms of the i integrals of the first set; while for $\mu = i, i+1, \ldots, j-1$, they are linearly expressible in terms of the $j-i$ integrals of the second set, subject to additive linear combinations of the first set. Thus the integrals in the present set which are given by

$$\left[\frac{\partial^\mu g(x, \alpha)}{\partial \alpha^\mu}\right]_{\alpha=\rho_j},$$

for $\mu = 0, 1, \ldots, j-1$, provide no integrals linearly independent of the i integrals of the first set and the $j-i$ integrals of the second set; the j integrals in this new aggregate are linearly expressible in terms of those in the old. Now the present set of integrals, for $\mu = 0, 1, \ldots, j-1, j, j+1, \ldots, k-1$, are linearly independent of one another; and therefore the integrals for

$$\mu = j, j+1, \ldots, k-1$$

are linearly independent of one another, of the i integrals of the first set, and the $j-i$ integrals of the second set. Thus the third set provides $k-j$ new independent integrals, given by the $k-j$ highest values of μ. The first of them, determined by $\mu = j$, is

$$x^{\rho_j}\left[\sum_{\nu=0}^{\infty} \frac{\partial^j g_\nu(\rho_j)}{\partial \rho_j{}^j} x^\nu + j \log x \sum_{\nu=0}^{\infty} \frac{\partial^{j-1} g_\nu(\rho_j)}{\partial \rho_j{}^{j-1}} x^\nu + \ldots\right],$$

which certainly contains terms not involving $\log x$; if $k-1 > j$, the second of them, determined by $\mu = j+1$, is

$$x^{\rho_j}\left[\sum_{\nu=0}^{\infty} \frac{\partial^{j+1} g_\nu(\rho_j)}{\partial \rho_j{}^{j+1}} x^\nu + (j+1) \log x \sum_{\nu=0}^{\infty} \frac{\partial^j g_\nu(\rho_j)}{\partial \rho_j{}^j} x^\nu + \ldots\right],$$

which certainly contains terms multiplying the first power of $\log x$; if $k-1 > j+1$, the third of them certainly contains terms multiplying the second power of $\log x$; and so on. Moreover, it is clear that all these $k-j$ integrals belong to the index ρ_j.

The law of the successive sets is now clear. The last of them, determined by $\alpha = \rho_l$, contains the integrals

$$\left[\frac{\partial^\mu g(x, \alpha)}{\partial \alpha^\mu}\right]_{\alpha=\rho_l},$$

for $\mu = l, l+1, \ldots, n$, which are linearly independent of one another and of all the integrals of the preceding sets already retained. All these integrals, being $n+1-l$, in number, belong to the index ρ_l.

The results thus obtained may be summarised as follows: *When the equation $f_0(\rho)=0$ has a group of roots $\rho_0, \rho_1, \ldots, \rho_n$, which differ from one another by integers (including zero) and differ from all the other roots by quantities that are not integers; when also the distinct roots are arranged in decreasing succession of real parts, so that ρ_0 is a root of multiplicity i, ρ_i is a root of multiplicity $j-i$, ρ_j is a root of multiplicity $k-j$, and so on, where $\rho_0, \rho_i, \rho_j, \ldots$ are distinct from one another and are arranged in decreasing succession of real parts; then, corresponding to that group of roots, there exists a group of $n+1$ linearly independent integrals which are regular in the vicinity of the singularity. This group of integrals is composed of a set of i integrals, which are given by*

$$\left[\frac{\partial^\mu g(x, \alpha)}{\partial \alpha^\mu}\right]_{\alpha=\rho_0},$$

for $\mu=0, 1, \ldots, i-1$, and belong to the index ρ_0; of a set of $j-i$ integrals, which are given by

$$\left[\frac{\partial^\mu g(x, \alpha)}{\partial \alpha^\mu}\right]_{\alpha=\rho_i},$$

for $\mu=i, i+1, \ldots, j-1$, and belong to the index ρ_i; of a set of $k-j$ integrals, which are given by

$$\left[\frac{\partial^\mu g(x, \alpha)}{\partial \alpha^\mu}\right]_{\alpha=\rho_j},$$

for $\mu=j, j+1, \ldots, k-1$, and belong to the index ρ_j; and so on, the last set being composed of $n+1-l$ integrals, which are given by

$$\left[\frac{\partial^\mu g(x, \alpha)}{\partial \alpha^\mu}\right]_{\alpha=\rho_l},$$

for $\mu=l, l+1, \ldots, n$, and belong to the index ρ_l.

The preceding investigation is in substantial agreement with that which is given by Frobenius*.

A different proof is given by Fuchs†: briefly stated, it amounts to the establishment of an integral w_0 belonging to the index ρ_0, to the transformation of the equation of order m by the substitution

$$w=w_0\int v\,dx$$

* *Crelle*, t. LXXVI (1873), pp. 214—224.

† *Crelle*, t. LXVI (1866), pp. 148—154; *ib.*, t. LXVIII (1868), pp. 361—367.

into a linear equation in v of order $m-1$, and to the discussion of this new equation in a manner similar to that in which the equation of order m is discussed. Expositions of the method devised by Fuchs will also be found in memoirs by Tannery* and Fabry†.

39. All the integrals of the differential equation, which has the specified form in the vicinity of the singularity, are regular in that vicinity; their particular characteristics are governed by the roots of the equation $f_0(\rho)=0$, that is,

$$\rho(\rho-1)\ldots(\rho-m+1)-\rho(\rho-1)\ldots(\rho-m+2)p_1(0)-\ldots-p_m(0)=0,$$

the differential equation in the vicinity of the singularity being of the form

$$x^m\frac{d^m w}{dx^m}-\sum_{r=1}^{m}x^{m-r}p_r(x)\frac{d^{m-r}w}{dx^{m-r}}=0.$$

This algebraic equation is of degree m, equal to the order of the differential equation; it is called‡ the *indicial equation of the singularity*, and the function $f(x, \rho)$, of which $f_0(\rho)$ is the term independent of x, is called the *indicial function*. From the form of the integrals which belong to the roots ρ of the indicial equation of a singularity, and those which belong to the roots θ of the (§ 13) fundamental equation of the same singularity, it is clear that the roots of the two equations can be associated in pairs such that

$$\theta=e^{2\pi i\rho}.$$

When the roots of the indicial equation are such that no two of them differ by an integer, the roots of the fundamental equation are different from one another; there is a system of m regular integrals, and the m members belong to the m different values of ρ. When the indicial equation possesses a group of n roots which differ from one another by integers (including zero), the corresponding root of the fundamental equation is of multiplicity n: there is a corresponding group of n regular integrals, the expressions of the members of which in the vicinity of the singularity may (but do not necessarily) involve integer powers of $\log x$. When a root of the indicial equation occurs in multiplicity κ,

* *Ann. de l'Éc. Norm.*, 2e Sér. t. IV (1875), pp. 113–182.

† Thèse, Faculté des Sciences, Paris (1885).

‡ Cayley, *Coll. Math. Papers*, vol. XII, p. 398. The names adopted by Fuchs are *determinirende Fundamentalgleichung*, and *determinirende Function*, respectively.

so that the corresponding root of the fundamental equation occurs in at least multiplicity κ, there is a set of κ associated integrals, the expressions of all but one of which certainly involve integer powers of $\log x$.

40. Having now obtained the form of integral or integrals associated with a root of the indicial equation $f_0(\rho) = 0$, we must shew that the aggregate of the integrals obtained in association with all the roots constitutes a fundamental system.

First, suppose that the roots of the indicial equation are such that no two of them differ by an integer; denoting them by $\rho_1, \rho_2, \ldots, \rho_m$, and the m integrals associated with these roots respectively by $w_1, \ldots, w_m$, we have

$$w_s = (z-a)^{\rho_s} P_s(z-a),$$

where $P_s(z-a)$ is a holomorphic function that does not vanish when $z = a$. No homogeneous linear relation can exist among these integrals: for, otherwise, we should have some equation of the kind

$$c_1 w_1 + c_2 w_2 + \ldots + c_m w_m = 0.$$

Writing

$$\theta_s = e^{2\pi i \rho_s}, \qquad (s = 1, 2, \ldots, m),$$

so that no two of the quantities $\theta_1, \ldots, \theta_m$ are equal to one another, we can, as in § 18, deduce the equation

$$c_1 \theta_1{}^r w_1 + c_2 \theta_2{}^r w_2 + \ldots + c_m \theta_m{}^r w_m = 0,$$

for any number of integer values of r, from the above equation, by making z describe r times a simple contour round a. Taking the latter equation for $r = 1, \ldots, m-1$, the set of m equations can exist with values of $c_1, \ldots, c_m$ differing from simultaneous zeros, only if

$$\begin{vmatrix} 1 & , 1 & , \ldots, & 1 \\ \theta_1 & , \theta_2 & , \ldots, & \theta_m \\ \cdots & \cdots & \cdots & \cdots \\ \theta_1{}^{m-1}, & \theta_2{}^{m-1}, & \ldots, & \theta_m{}^{m-1} \end{vmatrix} = 0,$$

which cannot hold as no two of the quantities θ are equal. Hence we must have $c_1 = 0 = c_2 = \ldots = c_m$, and no homogeneous linear relation exists: the system of integrals is a fundamental system.

Next, suppose that the roots of the indicial equation can be arranged in sets, such that the members contained in each set differ from one another by integers. With each such set of roots a group of integrals is associated, the number of integrals in the group being the same as the number of roots in the set.

It is impossible that any homogeneous linear relation among the members of a group can exist: if it could, it would have the form

$$b_1 w_1 + \ldots + b_n w_n = 0.$$

If $w_1, \ldots, w_n$ involve logarithms, then (§ 27) the aggregate coefficient of the highest power of $\log(z-a)$ must vanish; in the case of each integral in which the logarithm occurs, this coefficient (§ 25) is itself an integral of the equation, and therefore we should have a relation of the form

$$b_r w_r + \ldots + b_s w_s = 0,$$

where the quantities $w_r, \ldots, w_s$ belong to different indices, say $\rho_r, \ldots, \rho_s$, no two of which are the same; and $w_r, \ldots, w_s$ are free from logarithms. Dividing by $(z-a)^{\rho_s}$, we should have an equation of the form

$$b_r (z-a)^{\rho_r - \rho_s} P_r(z-a) + \ldots + b_s P_s(z-a) = 0,$$

where $P_r, \ldots, P_s$ are holomorphic functions of $z-a$, not vanishing when $z=a$. No one of the indices $\rho_r - \rho_s$ is zero: no two are the same: and so the preceding equation can be satisfied identically, only if $b_r = \ldots = b_s$. We therefore remove the corresponding terms from

$$b_1 w_1 + \ldots + b_m w_m = 0,$$

and proceed as before: we ultimately obtain zero as the only possible value of each of the coefficients b.

If $w_1, \ldots, w_m$ do not involve logarithms, the argument, above applied to $w_r, \ldots, w_s$, can be repeated: there is no linear relation. The initial statement is thus established.

If the tale of the groups, the members of each of which are linearly independent among themselves, is not made up of linearly independent integrals, then an equation of the form

$$c_1 w_1 + \ldots + c_m w_m = 0$$

exists. Equating to zero (§ 27) the aggregate coefficient of the highest power of $\log z$ that occurs, we have, as above, a relation of the form

$$c_r w_r + \ldots + c_s w_s + c_p w_p + \ldots + c_q w_q + \ldots = 0,$$

where $w_r, \ldots, w_s$ belong to one group, $w_p, \ldots, w_q$ belong to another group, and so on. Writing

$$c_r w_r + \ldots + c_s w_s = W_1, \quad c_p w_p + \ldots + c_q w_q = W_2, \ldots$$

we have

$$W_1 + W_2 + \ldots = 0.$$

Now let $\theta_1, = e^{2\pi i\rho}$, be the factor which, after description of a loop round a, should be associated with W_1; let θ_2 be the corresponding factor for W_2; and so on: the quantities $\theta_1, \theta_2, \ldots$ being unequal to one another, because $W_1, W_2, \ldots$ belong to different groups. Then, as in § 18, we deduce the equation

$$\theta_1^\lambda W_1 + \theta_2^\lambda W_2 + \ldots = 0,$$

after λ descriptions of the loop; and this would hold for all integer values of λ. As before, taking a sufficient number of these equations for successive values of λ, we infer that

$$W_1 = 0, \quad W_2 = 0, \ldots;$$

if these are not evanescent, they would imply relations among the members of a group, and so they can be satisfied only if

$$c_r = 0 = \ldots = c_s, \quad c_p = 0 = \ldots = c_q, \ldots.$$

Remove therefore the corresponding terms from the relation

$$c_1 w_1 + \ldots + c_m w_m = 0,$$

and proceed as before: we ultimately obtain zero as the only possible value of each of the coefficients c. Hence no homogeneous linear relation exists: the system is fundamental.

Some examples illustrating the preceding method of obtaining the integrals of a linear equation will now be given.

Ex. 1. Consider the integrals of the equation*

$$D(w) = x(2 - x^2) w'' - (x^2 + 4x + 2)\{(1-x) w' + w\} = 0$$

in the vicinity of the origin. To obtain a regular integral, we take

$$w = x^a \sum_{n=0}^{\infty} c_n x^n;$$

* The equation is not in the exact form indicated in the text. We have $m=2$, $p_2(0)=0$, and so a factor x has been removed; also we have multiplied by the factor $2-x^2$.

substituting, we have

$$xDw = 2a(a-2)\,c_0 x^a,$$

provided

$$0 = c_1(a^2-1) - c_0(a+1),$$
$$0 = 2c_2 a(a+2) - 2c_1(a+2) - c_0(a-2)^2,$$

and

$$2(n+a+1)\{(n+a-1)\,c_{n+1} - c_n\} = (n+a-3)\{(n+a-3)\,c_{n-1} - c_{n-2}\},$$

the last holding for $n=2, 3, \ldots$.

The indicial equation is

$$a(a-2)=0,$$

giving (simple) roots $a=2$, $a=0$, so that a factor $a+1$ can be neglected: the relations among the coefficients are equivalent to

$$(a+2n-1)\,c_{2n+1} - c_{2n} = 0,$$
$$(a+2n)\,c_{2n+2} - c_{2n+1} = \frac{c_0(a-2)^2}{2^{n+1}}\,\frac{a}{(a+2n)(a+2n+2)}.$$

Firstly, consider the root $a=2$. We have

$$c_1 = c_0,$$
$$2c_2 = c_1,$$
$$3c_3 = c_2, \ldots$$

so that the integral belonging to the index 2 is

$$c_0 x^2\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right);$$

say the integral is u, where

$$u = x^2 e^x.$$

Secondly, consider the root $a=0$. From the original form of the relations, we have

$$c_1 = -c_0,$$

by the first relation: and the second relation is then identically satisfied, leaving c_2 arbitrary. Using the reduced form of the relations for the higher coefficients, we have

$$c_3 = c_2,$$
$$2c_4 = c_3,$$
$$3c_5 = c_4, \ldots$$

and therefore the integral belonging to the index 0 is

$$c_0(1-x) + c_2 x^2\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right).$$

On subtracting $c_2 u$ from this integral, the remainder is still an integral, and it belongs to the index 0 in the form (say)

$$v, = 1 - x.$$

Thus the system of two integrals, regular in the vicinity of $x=0$, is

$$u, = x^2 e^x; \quad v, = 1 - x.$$

This method of dealing with the root $a=0$ is not quite in accord with the course of the general theory; it happens to be successful because c_2 is left arbitrary. In order to follow the general theory, we note that the coefficient of c_2 in the original difference-equation contains a factor a which vanishes for the present root. Hence, taking

$$c_0 = Ca,$$

we find

$$c_1 = \frac{Ca}{a-1},$$

$$c_2 = \tfrac{1}{2}\,\frac{Ca\,(a^2-5a+10)}{a^2+a-2},$$

$$c_3 = \frac{c_2}{a+1},$$

$$(a+2)\,c_4 = c_3 + a^2 A,$$

where A is finite, and so on; thus

$$w = Cax^a\left(1+\frac{x}{a-1}\right) + c_2 x^{a+2}\left\{1+\frac{x}{a+1}+\frac{x^2}{(a+1)(a+2)}+\dots\right\} + a^2 R(x, a),$$

where $R(x, a)$ is a holomorphic function of x which, by the general theory, is finite when $a=0$. According to the general theory, this quantity should give rise to two integrals, viz.

$$[w]_{a=0}, \quad \left[\frac{dw}{da}\right]_{a=0}.$$

Taking account of the value of c_2, the first of them is zero, thus giving an evanescent integral. The second is

$$C(1-x) - \tfrac{5}{2}Cx^2e^x,$$

or adding to this integral $\tfrac{5}{2}Cu$, which is an integral, we have

$$C(1-x),$$

thus giving $1-x$ as the integral.

Ex. 2. Discuss in a similar manner the regular integrals of the equation

$$xw'' + w' - w = 0$$

in the vicinity of the origin: likewise those of the equation

$$x(1-x)\,w'' - (1+4x+2x^2)\,w' + (3+3x-x^2)\,w = 0$$

in the same vicinity.

Ex. 3. Consider the integrals of the equation

$$Dw = x^2(1+x)\,w'' - (1+2x)(xw'-w) = 0$$

in the vicinity of the origin. Substituting the expression

$$x^a \sum_{n=0}^{\infty} c_n x^n,$$

we have

$$Dw = (a-1)^2\,c_0 x^a,$$

provided

$$(a+n-1)^2 c_n = -(a+n-2)(a+n-3) c_{n-1},$$

for $n=1, 2, 3, \ldots$; these values give

$$w = c_0 x^a \left\{1 - \frac{(a-1)(a-2)}{a^2} x\right\} + (a-1)^2 Y,$$

where Y is a holomorphic function of x which is finite when $a=1$.

The indicial equation has a repeated root $a=1$; hence two regular integrals are

$$[w]_{a=1}, \quad \left[\frac{dw}{da}\right]_{a=1}.$$

The former is $c_0 x$, say the integral is u, where

$$u = x;$$

the latter is $c_0 x \log x + c_0 x^2$, say the integral is v, where

$$v = x \log x + x^2.$$

Both integrals belong to the index 1; and one of them must contain a logarithm, since the index is a repeated root of the indicial equation.

Ex. 4. Consider Bessel's equation for functions of order zero, viz.

$$Dw = xw'' + w' + xw = 0.$$

Substituting

$$w = c_0 x^a + c_1 x^{a+1} + \ldots + c_p x^{a+p} + \ldots,$$

we have

$$xDw = c_0 a^2 x^a,$$

provided

$$c_1 = 0,$$

$$(a+p+1)^2 c_{p+1} + c_{p-1} = 0,$$

the latter holding for $p=1, 2, 3, \ldots$. When these relations are solved, the value of w is

$$w = c_0 x^a \left\{1 - \frac{x^2}{(a+2)^2} + \frac{x^4}{(a+2)^2 (a+4)^2} - \ldots\right\}.$$

The indicial equation is $a^2=0$, so that $a=0$ is a repeated root; thus the integrals of the equation, both of them belonging to the index zero, are

$$[w]_{a=0}, \quad \left[\frac{dw}{da}\right]_{a=0}.$$

The first of them is

$$c_0 \left\{1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} - \ldots\right\};$$

in effect, $J_0(x)$, on making $c_0=1$. The second is

$$c_0 \log x \left\{1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} - \ldots\right\}$$

$$+ c_0 \left\{\frac{x^2}{2^2} - \frac{x^4}{2^2 . 4^2}(1+\tfrac{1}{2}) + \frac{x^6}{2^2 . 4^2 . 6^2}(1+\tfrac{1}{2}+\tfrac{1}{3}) - \ldots\right\}.$$

Denoting this by K_0 when $c_0=1$, we have

$$K_0=J_0\log x+\sum_{p=1}^{\infty}\frac{(-1)^{p-1}}{\{\Pi(p)\}^2}\left(\frac{x}{2}\right)^{2p}\psi(p),$$

where $\psi(p)$ denotes the value of $\frac{d}{dz}\{\log\Pi(z)\}$ when $z=p$. The two integrals, regular in the vicinity of $x=0$, are J_0 and K_0.

Ex. 5. Consider next Bessel's equation for functions of order n, viz.

$$Dw=x^2w''+xw'+(x^2-n^2)w=0.$$

Substituting an expression

$$w=c_0x^a+c_1x^{a+1}+\ldots+c_px^{a+p}+\ldots$$

in the equation, we have

$$Dw=c_0(a^2-n^2)x^a,$$

provided

$$c_1\{(a+1)^2-n^2\}=0,$$

$$c_p\{(a+p)^2-n^2\}+c_{p-2}=0,$$

for $p=1, 2, 3, \ldots$; we thus have

$$w=c_0x^a\left[1-\frac{x^2}{(a+2)^2-n^2}+\frac{x^4}{\{(a+2)^2-n^2\}\{(a+4)^2-n^2\}}-\ldots\right].$$

The roots of the indicial equation are

$$a=+n,\quad a=-n.$$

When n is not an integer, the corresponding integrals are seen to be effectively J_n, J_{-n}.

When n is zero, we have a repeated root; this case has been discussed in the preceding example (Ex. 4).

When n is an integer different from zero, the two roots belong to a group; and for $a=-n$, the coefficient of x^{2n} is formally infinite, so that we have an illustration of the general theory in §§ 36—38. We take the roots in order.

Firstly, let $a=+n$: then the integral is

$$x^n\sum_{p=0}^{\infty}(-1)^p\left(\frac{x}{2}\right)^{2p}\frac{1}{\Pi(p)\,\Pi(n+p)},$$

on taking c_0 equal to $\dfrac{1}{2^n\,\Pi(n)}$. This is the function usually denoted by J_n; and it belongs to the index n.

Secondly, when $a=-n$, one of the coefficients becomes formally infinite through the occurrence of a denominator factor $(a+2n)^2-n^2$. Accordingly, we write

$$c_0=C\{(a+2n)^2-n^2\},\quad (-1)^nC=E\prod_{r=1}^{n-1}\{(a+2r)^2-n^2\};$$

and then

$$w = C\{(a+2n)^2 - n^2\} x^a \left[1 - \frac{x^2}{(a+2)^2 - n^2} + \ldots + (-1)^{n-1} \frac{x^{2n-2}}{\prod\limits_{r=1}^{n-1} \{(a+2r)^2 - n^2\}}\right]$$

$$+ Ex^{a+2n}\left[1 - \frac{x^2}{(a+2n+2)^2 - n^2} + \frac{x^4}{\{(a+2n+2)^2 - n^2\}\{(a+2n+4)^2 - n^2\}} - \ldots\right]$$

$$= w_1 + w_2,$$

say: and now

$$Dw = C(a^2 - n^2)\{(a+2n)^2 - n^2\} x^a.$$

Two integrals arise through this root, viz.

$$[w_1 + w_2]_{a=-n}, \quad \left[\frac{\partial w_1}{\partial a} + \frac{\partial w_2}{\partial a}\right]_{a=-n}.$$

For the first of them, we have

$$[w_1]_{a=-n} = 0, \quad [w_2]_{a=-n} = \frac{E}{2^n \Pi(n)} J_n;$$

so that it provides no new integral. For the second of them, we have

$$\left[\frac{\partial w_1}{\partial a}\right]_{a=-n} = \frac{1}{x^n} \frac{2Cn}{\Pi(n-1)} \sum_{p=0}^{n-1} \left(\frac{x}{2}\right)^{2p} \frac{\Pi(n-1-p)}{\Pi(p)} = W_1,$$

say; and

$$\frac{\partial w_2}{\partial a} = Ex^n (\log x)\left[1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^4(n+1)(n+2)\,1\,.\,2} - \ldots\right]$$

$$+ \tfrac{1}{2} Ex^n \sum_{r=0}^{\infty} \frac{(-1)^{r-1}\Pi(n)}{\Pi(r)\Pi(n+r)} \{\psi(r) + \psi(n+r) - \psi(n)\} \left(\frac{x}{2}\right)^{2r}$$

$$= W_2,$$

say: so that the integral is $W_1 + W_2$.

In W_2, the part represented by

$$\tfrac{1}{2} Ex^n \sum_{r=0}^{\infty} \frac{(-1)^r \Pi(n)\psi(n)}{\Pi(r)\Pi(n+r)} \left(\frac{x}{2}\right)^{2r}$$

is a constant multiple of J_n and therefore can be omitted, owing to the earlier retention of J_n. Rejecting this part, and taking

$$C = -\frac{1}{n} 2^{n-1} \Pi(n-1),$$

so that

$$E = \frac{1}{2^{n-1}} \frac{1}{\Pi(n)},$$

the integral becomes

$$-\left(\frac{2}{x}\right)^n \sum_{p=0}^{n-1} \frac{\Pi(n-p-1)}{\Pi(p)} \left(\frac{x}{2}\right)^{2p}$$

$$+ \left(\frac{x}{2}\right)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{\Pi(r)\Pi(n+r)} \{2\log x - \psi(r) - \psi(n+r)\} \left(\frac{x}{2}\right)^{2r},$$

which differs, only by a constant multiple of J_n, from the expression given by Hankel*.

* *Math. Ann.*, t. I (1869), pp. 469—471, quoted in my *Treatise on Differential Equations*, p. 167.

Ex. 6. Discuss in a similar manner the integrals of the equation

$$x(1-x)w''+\{1-(a+b+1)x\}w'-abw=0$$

in the vicinities of $x=0$, and $x=1$: indicating the form for the latter vicinity when $a+b=1$.

This equation is the differential equation of the hypergeometric series $F(a, b, 1, x)$. When, in Legendre's equation

$$(1-z^2)\frac{d^2w}{dz^2}-2z\frac{dw}{dz}+p(p+1)w=0,$$

the independent variable is transformed to x, where $z=1-2x$, it becomes

$$x(1-x)w''+(1-2x)w'+p(p+1)w=0,$$

which is the special case of the above given by $b=p+1$, $a=-p$. The integrals of Legendre's equation in the vicinity of $x=0$ and of $x=1$, that is, in the vicinity of $z=1$ and of $z=-1$, can be deduced from those of the hypergeometric equation; the actual deduction is left as an exercise.

Ex. 7. Apply the general theory to obtain the integrals of

$$x^3w'''-3x^2w''+7xw'-8w=0,$$

which are regular in the vicinity of $x=0$.

Ex. 8. Consider in the same way the equation

$$D(w)=(1+x)x^3w'''-(2+4x)x^2w''+(4+10x)xw'-(4+12x)w=0.$$

Substituting for w the expression

$$w=c_0x^a+c_1x^{a+1}+\ldots+c_nx^{a+n}+\ldots$$

as in the earlier examples, we have

$$Dw=c_0(a-1)(a-2)^2x^a,$$

provided

$$c_n(n+a-1)(n+a-2)^2+c_{n-1}(n+a-3)^2(n+a-4)=0,$$

for $n=1, 2, 3, \ldots$.

The roots of the indicial equation are 2, repeated, and 1, so that they form a group the members of which differ by integers. Moreover, when $a=1$, the coefficient c_1, which is

$$-c_0\frac{(a-2)^2(a-3)}{a(a-1)^2},$$

is formally infinite; for that root, we shall take

$$c_0=C(a-1)^2.$$

Firstly, for the repeated root $a=2$, we have

$$w=c_0x^a\{1+(a-2)^2R(x, a)\},$$

where R is a holomorphic function of x which remains finite when $a=2$. The two integrals are

$$[w]_{a=2}, \quad \left[\frac{\partial w}{\partial a}\right]_{a=2};$$

it is easy to see that they are constant multiples of

$$u_1 = x^2, \quad u_2 = x^2 \log x,$$

both of which belong to the index 2.

Secondly, for the root $a=1$, we take $c_0 = C(a-1)^2$, and then

$$D(w) = C(a-1)^3 (a-2)^2 x^a,$$

where

$$w = C(a-1)^2 x^a - C\frac{(a-2)^2(a-3)}{a} x^{a+1}$$
$$+ C\frac{(a-1)^2(a-2)^3(a-3)}{a^3(a+1)} x^{a+2} + (a-1)^3 Q(x, a),$$

where $Q(x, a)$ is a holomorphic function of x which remains finite when $a=1$. In connection with this expression, three integrals are derivable, viz.

$$[w]_{a=1}, \quad \left[\frac{\partial w}{\partial a}\right]_{a=1}, \quad \left[\frac{\partial^2 w}{\partial a^2}\right]_{a=1}.$$

The first of these is

$$C\,2x^2,$$

which is $2Cu_1$: it is not a new integral. The second is

$$2Cx^2 \log x - 7Cx^2,$$

which is $2Cu_2 - 7Cu_1$: it is not a new integral. The third is

$$2Cx + 2Cx^2 (\log x)^2 - 14Cx^2 \log x + 22Cx^2 + 2Cx^3;$$

adding to it $14Cu_2 - 22Cu_1$, the new expression is still an integral and is a constant multiple of u_3, where

$$u_3 = x + x^3 + x^2 (\log x)^2,$$

which manifestly belongs to the index 1.

Ex. 9. Obtain the integrals of the equations

(i) $(1+x^2)x^3w''' - (2+4x^2)x^2w'' + (4+10x^2)xw' - (4+12x^2)w = 0;$

(ii) $(1+4x)x^4w'''' - (4+20x)x^3w''' + (14+72x)x^2w''$
$\qquad - (32+168x)xw' + (36+192x)w = 0;$

which are regular in the vicinity of the origin.

Ex. 10. Consider the integrals of

$$Dw = xw''' + (a_1 + b_1x + \ldots)w'' + (a_2 + b_2x + \ldots)w' + (a_3 + b_3x + \ldots)w = 0$$

in the vicinity of $x=0$, the constant a_1 not being an integer. To obtain the regular integrals, we substitute

$$w = x^a \sum_{n=0}^{\infty} c_n x^n,$$

so that

$$x^2 Dw = a(a-1)(a-2+a_1)c_0 x^a,$$

provided

$$(n+a)(n+a-1)(n+a-2+a_1)c_n = f_0 c_{n-1} + f_1 c_{n-2} + \ldots,$$

for values of n greater than zero, no one of the quantities $f_0, f_1, \ldots$ being of degree in n greater than 2.

The roots of the indicial equation are

$$a=0,\ 1,\ 2-a_1.$$

For $a=2-a_1$, the difference-equation determines coefficients c_n, which lead to a series converging for values of $|x|$ within the common region of convergence of the coefficients of w'', w', w.

For $a=0$ or 1, the difference-equation holds for values of n greater than 2 or 1 respectively; the only other conditions are

$$a_1 . 2c_2 + a_2 . c_1 + a_3 c_0 = 0$$

for $a=0$, and

$$a_1 . 2c_1 + a_2 . c_2 = 0$$

for $a=1$. Then the difference-equation again determines coefficients which in each case lead to a series that converges within the same region as the series that belongs to the exponent $2-a_1$. Each of the latter integrals is a holomorphic function of x; and therefore the three integrals of the equation, which are regular in the vicinity of $x=0$, are :—one, a holomorphic function of x belonging to the index 0; a second, likewise a holomorphic function of x belonging to the index 1; and a third, belonging to the index $2-a_1$.

Ex. 11. Discuss the regular integrals of the equation in the preceding example, when a_1 is an integer.

Ex. 12. Prove that the equation

$$x\frac{d^n w}{dx^n} + (a_1 + b_1 x + \ldots)\frac{d^{n-1}w}{dx^{n-1}} + \ldots + (a_n + b_n x + \ldots)\, w = 0$$

has $m-1$ integrals which are holomorphic functions of x in the vicinity of $x=0$, when a_1 is not an integer, the various coefficients $a_r + b_r x + \ldots$ in the differential equation being holomorphic in that vicinity; and discuss the regular integrals when a_1 is an integer. (Poincaré.)

Ex. 13. Shew that the series

$$F(a, \rho, \sigma, \tau, x) = 1 + \frac{a}{\rho\sigma\tau}x + \frac{a(a+1)}{2!\,\rho(\rho+1)\,\sigma(\sigma+1)\,\tau(\tau+1)}x^2 + \ldots$$

satisfies the equation

$$x^3\frac{d^4y}{dx^4} + (\rho+\sigma+\tau+3)\,x^2\frac{d^3y}{dx^3} + (1+\rho+\sigma+\tau+\rho\sigma+\sigma\tau+\tau\rho)\,x\frac{d^2y}{dx^2}$$

$$= (x-\rho\sigma\tau)\frac{dy}{dx} + ay;$$

and obtain the other integrals, regular in the vicinity of $x=0$.

Verify that, when $a=\tau$, the form of the function F, say $G(\rho, \sigma, x)$, satisfies the equation of the third order

$$x^2\frac{d^3G}{dx^3} + (\rho+\sigma+1)\,x\frac{d^2G}{dx^2} + \rho\sigma\frac{dG}{dx} - G = 0;$$

and indicate the relation between the two differential equations.

(Pochhammer.)

Regular Integrals, free from Logarithms.

41. Alike in the general investigation and in the particular examples, it has appeared that the regular integrals are sometimes affected with logarithms, sometimes free from them. Thus if no two of the roots of the indicial equation differ by a whole number, each one of the integrals in the vicinity of the singularity is certainly free from logarithms; if a root of the indicial equation is a repeated root of multiplicity n, then the first $n-1$ powers of $\log x$ certainly appear in the group of n integrals which belong to that root. When a root of the indicial equation, though not a repeated root, belongs to a group the members of which differ from one another by whole numbers, the integral belonging to the root may or may not involve logarithms: we proceed to find the conditions which will secure that every integral belonging to that root is free from logarithms.

Let the group of roots be denoted by $\rho_0, \rho_1, \ldots, \rho_\mu, \ldots$, arranged in descending order of real parts, so that $\rho_\kappa - \rho_\mu$, for $\kappa = 0, 1, \ldots \mu - 1$, is in each case a positive integer: and consider the root ρ_μ, in order to obtain the conditions under which every integral belonging to ρ_μ shall be free from logarithms. In the first place, ρ_μ must be a simple root of the indicial equation. Assuming this to be the case, we know that the integral belonging to ρ_μ is

$$\left[\frac{\partial^\mu g(x, \alpha)}{\partial \alpha^\mu}\right]_{\alpha=\rho_\mu},$$

in the notation of § 38. If we further admit the legitimate possibility that, to this expression, we may add constant linear multiples of the integrals which belong to the earlier roots $\rho_0, \rho_1, \ldots, \rho_{\mu-1}$ and still have an integral belonging to the root ρ_μ, then, in order to secure that every integral belonging to ρ_μ shall be free from logarithms, the integrals belonging to the earlier roots must also be free from logarithms; hence, as further conditions, each of the roots $\rho_0, \rho_1, \ldots, \rho_{\mu-1}$ of the indicial equation must be simple. These conditions also will be assumed to be satisfied.

The full expression for the integral belonging to ρ_μ is the value, when $\alpha = \rho_\mu$, of the expression

$$x^\alpha \left[\sum_{\nu=0}^{\infty} \frac{\partial^\mu g_\nu}{\partial \alpha^\mu} x^\nu + \mu (\log x) \sum_{\nu=0}^{\infty} \frac{\partial^{\mu-1} g_\nu}{\partial \alpha^{\mu-1}} x^\nu + \ldots + (\log x)^\mu \sum_{\nu=0}^{\infty} g_\nu x^\nu\right];$$

in order to be free from logarithms, the quantities

$$\left[\frac{\partial^\sigma g_\nu(\alpha)}{\partial\alpha^\sigma}\right]_{\alpha=\rho_\mu},$$

for $\sigma=0, 1, \ldots, \mu-1$, and for all values $\nu=0, 1, \ldots$ ad inf., must vanish: and if these conditions be satisfied, the above expression will acquire the desired form. The conditions will be satisfied for every value of σ, if $g_\nu(\alpha)$ contains $(\alpha-\rho_\mu)^\mu$ as a factor. But (p. 81)

$$\frac{g_\nu(\alpha)}{g_0(\alpha)}=\frac{h_\nu(\alpha)}{f_0(\alpha+1)f_0(\alpha+2)\ldots f_0(\alpha+\nu)}=H_\nu(\alpha),$$

say; and $g_0(\alpha)$, which (§ 36) is equal to $g(\alpha)f(\alpha)$, contains $(\alpha-\rho_\mu)^\mu$ as a factor on account of its occurrence in $f(\alpha)$; hence it is sufficient that $H_\nu(\alpha)$ should remain finite (that is, not become infinite) when $\alpha=\rho_\mu$, for all values of ν. Moreover, $H_0(\alpha)=1$. Having regard to the equation by which $g_\nu(\alpha)$ is determined, we obtain the relation

$$H_\nu f_0(\alpha+\nu)+H_{\nu-1}f_1(\alpha+\nu-1)+\ldots$$
$$\ldots+H_1 f_{\nu-1}(\alpha+1)+H_0 f_\nu(\alpha)=0.$$

All the quantities $f_1(\alpha+\nu-1), \ldots, f_\nu(\alpha)$ are finite for values of α that are considered; hence $H_\nu f_0(\alpha+\nu)$ is finite if $H_0(=1)$, $H_1, \ldots, H_{\nu-1}$ are finite, and therefore, on the same hypothesis, H_ν will be finite for all values of ν, if it remains finite for those values of the positive integer ν, which make $\rho_\mu+\nu$ a root of the indicial equation $f_0(\theta)=0$. These values are known; in ascending order of magnitude, they are

$$\rho_{\mu-1}-\rho_\mu,\ \rho_{\mu-2}-\rho_\mu,\ \ldots,\ \rho_0-\rho_\mu.$$

Consider them in ascending order. We have

$$H_\nu=\frac{h_\nu(\alpha)}{f_0(\alpha+1)f_0(\alpha+2)\ldots f_0(\alpha+\nu)}.$$

When $\nu=\rho_{\mu-1}-\rho_\mu$, a single factor

$$f_0(\alpha+\nu)$$

in the denominator vanishes when $\alpha=\rho_\mu$; and it vanishes to the first order, because $\rho_{\mu-1}$ is a simple root of the indicial equation. Hence, in order that H_ν may be finite for this value of ν when $\alpha=\rho_\mu$, it is necessary that

$$h_\nu(\rho_\mu)=0,\quad\text{when}\quad \nu=\rho_{\mu-1}-\rho_\mu;$$

and it is sufficient that $h_\nu(\rho_\mu)$ should vanish to the first order.

When $\nu = \rho_{\mu-2} - \rho_\mu$, two factors

$$f_0(\alpha + \nu - \rho_{\mu-2} + \rho_{\mu-1}), \quad f_0(\alpha + \nu)$$

in the denominator vanish when $\alpha = \rho_\mu$; and each of them vanishes to the first order, because $\rho_{\mu-1}$ and $\rho_{\mu-2}$ are simple roots of the indicial equation. Hence, in order that H_ν may be finite for this value of ν when $\alpha = \rho_\mu$, it is necessary and sufficient that $h_\nu(\alpha)$ should vanish to the second order when $\alpha = \rho_\mu$: the analytical conditions are that

$$h_\nu(\alpha) = 0, \quad \frac{\partial h_\nu(\alpha)}{\partial \alpha} = 0,$$

when $\nu = \rho_{\mu-2} - \rho_\mu$ and $\alpha = \rho_\mu$.

When $\nu = \rho_{\mu-3} - \rho_\mu$, then the three factors

$$f_0(\alpha + \nu - \rho_{\mu-3} + \rho_{\mu-1}), \quad f_0(\alpha + \nu - \rho_{\mu-3} + \rho_{\mu-2}), \quad f_0(\alpha + \nu)$$

in the denominator vanish when $\alpha = \rho_\mu$; and each of them vanishes to the first order, because $\rho_{\mu-1}, \rho_{\mu-2}, \rho_{\mu-3}$ are simple roots of the indicial equation. Hence, in order that H_ν may be finite for this value of ν when $\alpha = \rho_\mu$, it is necessary and sufficient that $h_\nu(\alpha)$ should vanish to the third order when $\alpha = \rho_\mu$; the analytical conditions are that

$$h_\nu(\alpha) = 0, \quad \frac{\partial h_\nu(\alpha)}{\partial \alpha} = 0, \quad \frac{\partial^2 h_\nu(\alpha)}{\partial \alpha^2} = 0,$$

when $\nu = \rho_{\mu-3} - \rho_\mu$ and $\alpha = \rho_\mu$.

Proceeding in this way, we obtain the conditions for the successive values of ν that need to be considered: the last set is that

$$\frac{\partial^\sigma h_\nu(\alpha)}{\partial \alpha^\sigma} = 0, \qquad (\sigma = 0, 1, \ldots, \mu - 1),$$

when $\nu = \rho_0 - \rho_\mu$ and $\alpha = \rho_\mu$.

Such is the aggregate of conditions for $\alpha = \rho_\mu$. We have seen that, in order to secure the freedom from logarithms of every integral belonging to ρ_μ, every preceding integral in the set as arranged must similarly be free: and so we have, in addition, all the similar conditions for $\rho_{\mu-1}, \rho_{\mu-2}, \ldots, \rho_1$, there being no condition for the simple root ρ_0. *When all these conditions are satisfied, every integral belonging to ρ_μ is free from logarithms.*

Manifestly these conditions also secure that every integral belonging to the roots $\rho_{\mu-1}, \rho_{\mu-2}, \ldots, \rho_1$ of the indicial equation

is free from logarithms: (one integral, belonging to ρ_0, is always unconditionally free from logarithms): it being assumed that each of the roots $\rho_0, \rho_1, \ldots, \rho_\mu$ is a simple root of the indicial equation. The conditions thus secure that, when each of the $\mu+1$ greatest roots in the group of roots of the indicial equation is simple, the $\mu+1$ integrals belonging to those roots respectively are free from logarithms.

The preceding investigation is based upon the results obtained by Frobenius, *Crelle*, t. LXXVI (1873), pp. 224—226.

A different investigation is given by Fuchs, *Crelle*, t. LXVIII (1868), pp. 361—367, 373—378; see also Tannery, *Ann. de l'Éc. Norm.*, t. IV (1875), pp. 167—170.

Ex. 1. A simple illustration arises in Ex. 1, § 40, for the equation

$$x(2-x^2)w'' - (x^2+4x+2)\{(1-x)w' + w\} = 0.$$

With the notation of the text, we have

$$f_0(a) = a(a-2),$$

so that

$$\rho_0 = 2, \quad \mu = 1, \quad \rho_1 = 0:$$

we thus have to consider $h_\nu(a)$ for $a = \rho_1 = 0$, $\nu = \rho_0 - \rho_1 = 2$. But

$$g_2(a) = \frac{h_2(a)\, g_0(a)}{f_0(a+1)\, f_0(a+2)},$$

so that, as

$$g_2(a) = \frac{4-a}{a^2+a-2}\, g_0(a),$$

we have

$$h_2(a) = \frac{4-a}{a^2+a-2} f_0(a+1)\, f_0(a+2)$$

$$= (4-a)(a+1)(a+2)\, a.$$

The (one) condition in the present case is that

$$h_2(a) = 0,$$

when $a = 0$: which manifestly is satisfied.

Ex. 2. If the roots of the indicial equation are different from one another, then the integrals which belong to them certainly possess terms free from logarithms. (Fuchs.)

Ex. 3. Let $\rho_0, \rho_1, \ldots, \rho_n$ be the roots of the indicial equation which form a group, the members differing by integers and no two being equal; and assume them ranged in descending order of real parts. Denote $\rho_0 - \rho_n$ by $s-1$; and form the equation satisfied by

$$W = \frac{d^s}{dx^s}(wx^{-\rho_n});$$

then according as the indicial equation for the singularity $x=0$ of the equation in W has no negative roots or has negative roots which are integers, the integrals of the original equation in w are free from logarithms or are affected by logarithms. (Fuchs.)

Ex. 4. Shew that the integrals of the equations

$$\text{(i)} \quad w'' + qw' - \frac{2}{x^2} w = 0,$$

$$\text{(ii)} \quad w'' - \left(\tfrac{1}{4}q^2 + \frac{2}{x^2}\right) w = 0,$$

$$\text{(iii)} \quad w'' + (q - 2\theta) w' + \left(\theta^2 - q\theta - \frac{2}{x^2}\right) w = 0,$$

where q and θ are constants, are free from logarithms.

Ex. 5. Discuss the integrals of the equation

$$2x^2 (2-x) w'' - x (4-x) w' + (3-x) w = 0$$

in the vicinity of the origin. [They are $x^{\frac{1}{2}}$, $(x - \frac{1}{2}x^2)^{\frac{1}{2}}$.]

42. If, instead of requiring (as in § 41) that every integral belonging to an exponent ρ_μ shall be free from logarithms, when ρ_μ is one of a group of roots of the indicial equation of the type indicated in § 36, we consider the possibility that there shall be some one integral free from logarithms, belonging to the exponent and belonging to no earlier exponent in the group as arranged, no such large aggregate of conditions is needed as for the earlier requirement. Thus it is no longer necessary to specify that $\rho_0, \ldots, \rho_{\mu-1}$ shall be simple roots of the indicial equation; nor is it necessary to specify that, even if these roots are simple, the integrals associated with them are of the required form. The conditions that arise will be particularly associated with $\alpha = \rho_\mu$; but they will be affected by modifications arising out of the possible multiplicity of $\rho_0, \ldots, \rho_{\mu-1}$ as roots of the indicial equation.

The detailed results are complicated: a mode of obtaining them will be sufficiently indicated by an investigation of the conditions needed to secure that some integral free from logarithms exists belonging to ρ_i and not to ρ_0, with the notation of §§ 36—38. Suppose that ρ_0 is a root of the indicial equation of multiplicity i; and let $y_1, \ldots, y_i$ denote the set of integrals associated with ρ_0, where the expression of y_{s+1}, for $s = 0, 1, \ldots, i-1$, is given by

$$y_{s+1} = \left[\frac{\partial^s g(x, \alpha)}{\partial \alpha^s}\right]_{\alpha=\rho_0}.$$

If ρ_i is a root of the indicial equation of multiplicity $j-i$, only the first of the set of associated $j-i$ integrals can be free from logarithms: even that this may be the case, conditions will be required. Denoting that first integral by W, we have

$$W = \left[\frac{\partial^i g(x, \alpha)}{\partial \alpha^i}\right]_{\alpha=\rho_i}.$$

Now W certainly belongs to the exponent ρ_i. Its expression, in general, involves logarithms; but there is a possibility of obtaining a modification of its expression, so as to free it from logarithms, if we associate with W a linear combination of $y_2, \ldots, y_i$ with constant coefficients; and the modified integral will still belong to ρ_i but not to ρ_0. Accordingly, consider the combination

$$U = W - \sum_{t=2}^{i} A_t y_t,$$

where the constant coefficients A are at our disposal; this gives

$$\begin{aligned} U &= \left[\frac{\partial^i g(x, \alpha)}{\partial \alpha^i}\right]_{\alpha=\rho_i} - \sum_{t=2}^{i} A_t \left[\frac{\partial^{t-1} g(x, \alpha)}{\partial \alpha^{t-1}}\right]_{\alpha=\rho_0} \\ &= x^{\rho_i} \sum_{\nu=0}^{\infty} \sum_{n=0}^{i} \left\{\frac{i\,!}{n\,!\,(i-n)\,!} (\log x)^n \frac{\partial^{i-n} g_\nu}{\partial {\rho_i}^{i-n}} x^\nu\right\} \\ &\quad - x^{\rho_0} \sum_{t=1}^{i-1} \sum_{\nu=0}^{\infty} \sum_{p=0}^{t} \left\{A_{t+1} \frac{t\,!}{p\,!\,(t-p)\,!} (\log x)^p \frac{\partial^{t-p} g_\nu}{\partial {\rho_0}^{t-p}} x^\nu\right\}. \end{aligned}$$

What we require are the conditions that may, if possible, secure that no logarithms occur in this expression for U.

The least aggregate of conditions that will secure this result is: first,

$$g_\nu(\rho_i) = 0,$$

for all values of ν, which secures the disappearance of $(\log x)^i$; next,

$$i \frac{\partial g_n}{\partial \rho_i} = A_i g_m(\rho_0),$$

for all values of m and n such that $\rho_i + n = \rho_0 + m$, as well as

$$\frac{\partial g_p}{\partial \rho_i} = 0,$$

for $p = 0, 1, \ldots, \rho_0 - \rho_i - 1$, these conditions securing the disappearance of $(\log x)^{i-1}$; next,

$$\frac{i(i-1)}{2} \frac{\partial^2 g_n}{\partial {\rho_i}^2} = A_{i-1} g_m(\rho_0) + (i-1) A_i \frac{\partial g_m}{\partial \rho_0},$$

for all values of m and n such that $\rho_i + n = \rho_0 + m$, as well as

$$\frac{\partial^2 g_p}{\partial \rho_i^2} = 0,$$

for $p = 0, 1, \ldots, \rho_0 - \rho_i - 1$, these conditions securing the disappearance of $(\log x)^{i-2}$; next,

$$\frac{i(i-1)(i-2)}{3!}\frac{\partial^3 g_n}{\partial \rho_i^3} = A_{i-2}\, g_m(\rho_0) + (i-2) A_{i-1} \frac{\partial g_m}{\partial \rho_0} + \frac{(i-1)(i-2)}{2} A_i \frac{\partial^2 g_m}{\partial \rho_0^2},$$

for all values of m and n such that $\rho_i + n = \rho_0 + m$, as well as

$$\frac{\partial^3 g_p}{\partial \rho_i^3} = 0,$$

for $p = 0, 1, \ldots, \rho_0 - \rho_i - 1$; and so on. This aggregate is both necessary and sufficient.

Manifestly any attempt to reduce it to conditions independent of the constants A would be exceedingly laborious, even if possible. The difficulty arises in even greater measure when we deal with the conditions that some integral belonging to ρ_μ, where $\mu > i$, and to no earlier index, should exist free from logarithms.

43. If we assume zero values for all the constants $A_2, \ldots, A_i$ in the preceding investigation, the surviving conditions are certainly sufficient to secure the result that the integral exists, free from logarithms and belonging to its proper exponent: but the conditions cannot be declared necessary.

The aggregate of this set of sufficient conditions is, in the case of ρ_i, that the equation

$$\left[\frac{\partial^\sigma g_n}{\partial \alpha^\sigma}\right]_{\alpha=\rho_i} = 0$$

shall hold for $\sigma = 0, 1, \ldots, i-1$, and for all values of n. As in § 41, it can be proved that all these conditions will be satisfied if the equation

$$h_{\rho_0 - \rho_i}(\alpha) = 0$$

has a simple root equal to ρ_i. Assuming this to be the case, then an integral exists in the form

$$x^{\rho_i} \sum_{\nu=0}^{\infty} \left[\frac{\partial^i g_\nu}{\partial \alpha^i}\right]_{\alpha=\rho_i} x^\nu,$$

which is free from logarithms and belongs to ρ_i (but not to ρ_0) as its proper exponent. If ρ_i is a multiple root of the indicial equation, the remaining integrals belonging to ρ_i as their proper exponent are certainly affected with logarithms.

Corresponding conditions, that are sufficient (but are more than can be declared necessary) to secure the existence of an integral, free from logarithms and belonging to an exponent ρ_μ (but to no earlier exponent in its group), can similarly be found; they are inferred from the investigation in § 41. If the equation

$$h_n(\alpha) = 0,$$

when $n = \rho_{\mu-1} - \rho_\mu$, has a simple root equal to ρ_μ; if the same equation, when $n = \rho_{\mu-2} - \rho_\mu$, has a double root equal to ρ_μ; if the same equation, when $n = \rho_{\mu-3} - \rho_\mu$, has a triple root equal to ρ_μ; and so on, up to the case of $n = \rho_0 - \rho_\mu$, when the equation must have a root equal to ρ_μ of multiplicity μ: then an integral exists, belonging to ρ_μ as its proper exponent (and not to any of the exponents $\rho_0, \rho_1, \ldots, \rho_{\mu-1}$), and free from logarithms. If ρ_μ is a multiple root of the indicial equation, the remaining integrals belonging to ρ_μ as their proper exponent are certainly affected with logarithms.

On the preceding basis, the identification of the integrals, belonging to the group of exponents, with the sub-groups as arranged by Hamburger (§§ 23, 24) can be effected. The aggregate of integrals in the group, which are free from logarithms and belong to their proper exponents, not merely indicate the number of sub-groups in Hamburger's arrangement but constitute the respective first members in the respective sub-groups. The general functional forms of the remaining integrals belonging to any exponent are (save as to a power of a factor $2\pi i$) similar to those which occur in Jürgens' form of the integrals in a sub-group*.

44. In the practical determination of the integrals of specified equations, it sometimes† is convenient to begin with that root

* In this connection, the following memoirs may be consulted: Jürgens, *Crelle*, t. LXXX (1875), pp. 150—168; Schlesinger, *Crelle*, t. CXIV (1895), pp. 159—169, 309—311.

† As to this process, see the remarks by Cayley, *Coll. Math. Papers*, t. VIII, pp. 458—462.

among the group of roots which has the smallest real part, instead of beginning with the root that has the largest real part, as in § 36. When the process about to be discussed is effective, it has the advantage of indicating at once the number of integrals associated with the group which are free from logarithms; but it is not always effective for this purpose, and it does not determine the integrals that are affected with logarithms.

The equations determining the successive coefficients $g_1, g_2, \ldots$ in the expression

$$\sum_{\nu=0}^{\infty} g_\nu x^{\alpha+\nu}$$

in the method of Frobenius are (§ 33)

$$0 = g_n f_0(\alpha+n) + g_{n-1} f_1(\alpha+n-1) + \ldots + g_0 f_n(\alpha),$$

for $n = 1, 2, \ldots$. Let a group of roots of the indicial equation

$$f_0(\alpha) = 0,$$

differing from one another by integers, be denoted by $\rho_0, \rho_1, \ldots, \sigma$, where σ is the root of the group with the smallest real part; and replace α by σ in the foregoing typical equation for the g's. Then, whenever $\sigma + n$ is equal to another root of the group, the equation in its given form ceases to determine g_n, as a unique finite quantity.

It may happen that the equation is satisfied identically; in that case g_n is arbitrary, as well as g_0. It may happen that the equation appears to determine g_n as an infinite quantity: in that case, we modify g_0 as in § 36, and g_n is determinate after the modification.

As often as the former case arises, we have a new arbitrary coefficient; if κ be the number of these coefficients left arbitrary, then κ is the number of different integrals, associated with the group of roots and free from logarithms. These integrals themselves are the quantities multiplying the arbitrary coefficients in the expression

$$\sum_{\nu=0}^{\infty} g_\nu x^{\sigma+\nu}.$$

Ex. 1. As an example in which the process, of dealing first with the root of a group that has the smallest real part, is effective as indicating the

number of integrals free from logarithms, consider the equation

$$(z^4+z^5)\frac{d^4w}{dz^4}-(7z^3+12z^4+4z^5)\frac{d^3w}{dz^3}+(29z^2+57z^3+30z^4+6z^5)\frac{d^2w}{dz^2}$$

$$-(74z+154z^2+93z^3+28z^4+4z^5)\frac{dw}{dz}+(90+194z+125z^2+43z^3+9z^4+z^5)w=0.$$

The indicial equation is easily found to be

$$(\rho-2)(\rho-3)^2(\rho-5)=0,$$

so that there certainly will be an integral belonging to the exponent 5, free from logarithms; there may be a similar integral belonging to the exponent 3, and there will certainly be an integral, belonging to that exponent and affected with logarithms; and there may be an integral belonging to the exponent 2, free from logarithms.

Accordingly, take the value $\rho=2$, and substitute

$$w=c_0z^2+c_1z^3+c_2z^4+c_3z^5+\ldots$$

in the equation; for the immediate purpose, we need not consider powers higher than z^5 in w, because $\rho=5$ is the root of the indicial equation with the highest real part. The equations for determining the successive coefficients are

$$\begin{aligned}
0&=c_0\,.\,0,\\
0&=c_1\,.\,0+c_0\,.\,0,\\
0&=c_2(-2)+c_1(2)+c_0(-1),\\
0&=c_3\,.\,0+c_2(-2)+c_1(2)+c_0(-1);
\end{aligned}$$

from which we see that c_0, c_1, c_3 remain arbitrary. All the other coefficients are expressible in terms of them. Consequently, the equation has three integrals free from logarithms belonging to 2, 3, 5, as their respective proper exponents.

(The equation was constructed so as to have

$$z^5e^z,\quad z^3e^z,\quad z^3e^z\log z+z^4e^z,\quad z^2e^z$$

for a fundamental system; the system is easily derived by writing

$$y=we^{-z},$$

when the equation for y is

$$z^4(1+z)y''''-z^3(7+8z)y'''+z^2(29+36z)y''-z(74+96z)y'+(90+120z)y=0,$$

which can easily be treated by the general method of Frobenius.)

Ex. 2. As an example in which the process is ineffective, consider the equation

$$Dw=(1-z)z^2\frac{d^2w}{dz^2}+(5z-4)z\frac{dw}{dz}+(6-9z)w=0.$$

Taking, as usual,

$$W=\sum_{n=0} c_nz^{n+\rho},$$

we have

$$DW=(\rho-2)(\rho-3)\,c_0 z^\rho,$$

provided

$$(\rho+n-2)(\rho+n-3)\,c_n=(\rho+n-4)^2\,c_{n-1},$$

for values $n=1, 2, \ldots$.

If instead of beginning with the root $\rho=3$ as in the general theory (§§ 35, 36), we try $\rho=2$, the equation for the coefficients c gives

$$n(n-1)\,c_n=(n-2)^2\,c_{n-1},$$

determining c_1 apparently as infinite. To modify this, we take

$$c_0=C(\rho-2);$$

the equation for c_1 then becomes

$$(\rho-1)(\rho-2)\,c_1=(\rho-3)^2(\rho-2)\,C,$$

which is satisfied identically, when $\rho=2$. Thus c_1 remains arbitrary; but $c_0=0$. The integral which would be obtained is, in fact, that which belongs to $\rho=3$; and the process is ineffective. There happens to be no integral belonging to $\rho=2$ (and not to $\rho=3$) free from logarithms.

The actual solution is easily obtained by the general method of Frobenius. We have

$$W=Cz^\rho\left\{(\rho-2)+\frac{(\rho-3)^2}{\rho-1}z+(\rho-2)^2(\rho-3)^2R(z,\rho)\right\},$$

where $R(z,\rho)$ is a holomorphic function of z when ρ is either 2 or 3; and then

$$DW=C(\rho-2)^2(\rho-3)\,z^\rho.$$

For $\rho=3$, we have the integral

$$w_1=[W]_{\rho=3}=Cz^3.$$

For $\rho=2$, we have the two integrals

$$w_2=[W]_{\rho=2}=Cz^3=w_1,$$

and

$$w_3=\left[\frac{\partial W}{\partial\rho}\right]_{\rho=2}=Cz^2+Cz^3\log z-3Cz^3.$$

The integral belonging to the index 3 is

$$z^3,$$

free from logarithms; that which belongs to the index 2 is effectively

$$z^2+z^3\log z,$$

which is affected with a logarithm, so that the index 2 possesses no proper integral free from logarithms.

Discrimination between Real Singularity and Apparent Singularity.

45. The singularity, in the vicinity of which the integrals have been considered, is a singularity of coefficients of the equation

$$\frac{d^m w}{dz^m} + \frac{P_1(z)}{z-a}\frac{d^{m-1}w}{dz^{m-1}} + \ldots + \frac{P_m(z)}{(z-a)^m} w = 0;$$

and the indices to which the integrals belong are the roots of the indicial equation for $z = a$, which is

$$\rho(\rho-1)\ldots(\rho-m+1) + \rho(\rho-1)\ldots(\rho-m+2)P_1(a) + \ldots$$
$$\ldots + P_m(a) = 0.$$

In general, the integrals of the equation in the vicinity of a cease to be holomorphic functions of $z-a$; thus they may involve fractional powers or negative powers of $z-a$, and they may involve powers of $\log(z-a)$. When this is the case, a is called* a *real* singularity. If, on the contrary, every integral of the equation in the vicinity of a is a holomorphic function of $z-a$, then a is called an *apparent* singularity of the differential equation. The conditions that must be satisfied when a singularity of the equation is only apparent, so that it is an ordinary point for each of the integrals, may be obtained as follows.

Let $w_1, w_2, \ldots, w_m$ denote a fundamental system of integrals in the vicinity of the singularity a; and suppose that each member of the system is a holomorphic function of $z-a$ in that vicinity, so that the singularity a is only apparent. Let Δ denote the determinant (§ 10) of this fundamental system, so that

$$\Delta = \begin{vmatrix} \frac{d^{m-1}w_1}{dz^{m-1}}, & \frac{d^{m-2}w_1}{dz^{m-2}}, & \ldots, & \frac{dw_1}{dz}, & w_1 \\ \frac{d^{m-1}w_2}{dz^{m-1}}, & \frac{d^{m-2}w_2}{dz^{m-2}}, & \ldots, & \frac{dw_2}{dz}, & w_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{d^{m-1}w_m}{dz^{m-1}}, & \frac{d^{m-2}w_m}{dz^{m-2}}, & \ldots, & \frac{dw_m}{dz}, & w_m \end{vmatrix};$$

* Weierstrass (see Fuchs, *Crelle*, t. LXVIII (1868), p. 378) calls the singularity *wesentlich* in this case: in the alternative case, he calls it *ausserwesentlich*.

and let Δ_r denote the determinant which results from Δ when the column $\dfrac{d^{m-r} w_s}{dz^{m-r}}$ is replaced by $\dfrac{d^m w_s}{dz^m}$, (for $s = 1, \ldots, m$). Then as every constituent in Δ_r and Δ is a holomorphic function of $z - a$ in the vicinity of a, both Δ_r and Δ are holomorphic functions of $z - a$ in that vicinity; neither of them is infinite there. But as in § 31, we have

$$\frac{P_r(z)}{(z-a)^r} = -\frac{\Delta_r}{\Delta}, \qquad (r = 1, \ldots, m),$$

and some one at least of the quantities $P_r(a)$ is not zero; hence, for that value of r,

$$\frac{\Delta_r(a)}{\Delta(a)}$$

is infinite, and therefore

$$\Delta(a) = 0,$$

or the determinant of a fundamental system vanishes at an apparent singularity. Moreover, as in § 10, we have

$$\frac{1}{\Delta}\frac{d\Delta}{dz} = -\frac{P_1(z)}{z-a} = -\frac{P_1(a)}{z-a} + \frac{dG(z-a)}{dz},$$

where $G(z-a)$ is a holomorphic function of $z - a$; whence

$$\Delta(z) = A\,(z-a)^{-P_1(a)}\, e^{G(z-a)},$$

where A is a constant. Now Δ is not identically zero near a, for the system of integrals is fundamental; hence A is not zero. We have seen that $\Delta(a) = 0$, and $\Delta(z)$ is a holomorphic function of $z - a$; hence $P_1(a)$ *must be a negative integer*, numerically greater than zero. This condition is required, in order to ensure that a is a singularity of the equation.

As each of the integrals is a function, that is holomorphic in the vicinity of a, it follows that the respective indices to which they belong must be positive integers; and therefore the roots of the indicial equation

$$\rho(\rho-1)\ldots(\rho-m+1) + \rho(\rho-1)\ldots(\rho-m+2)P_1(a) + \ldots$$
$$\ldots + \rho P_{m-1}(a) + P_m(a) = 0$$

must be positive integers. (When one of these is zero, then $P_m(a)$ vanishes.) Moreover, no two of these roots may be equal;

for otherwise, the expressions for the integrals that belong to the repeated root would certainly include logarithms, contrary to the current hypothesis. Accordingly, let the roots be $\rho_1, \rho_2, \ldots, \rho_m$, a set of unequal positive integers which we shall assume to be ranged in decreasing order of magnitude: they thus form a single group the members of which differ from one another by integers. The integral belonging to ρ_1 involves no logarithm. In order that every integral belonging to ρ_2 may involve no logarithm, one condition must be satisfied: it is as set out in § 41. In order that every integral belonging to ρ_3 may involve no logarithm, two further conditions must be satisfied; they are as set out in § 41. And so on, for each of the roots in succession until the last: in order that every integral belonging to ρ_m may involve no logarithms, $m-1$ further conditions must be satisfied, being the conditions set out in § 41.

The aggregate of these conditions, and the property that the roots of the indicial equation are unequal positive integers, give the requisite character to the integrals. The condition that $P_1(a)$ is a negative integer makes a a singularity of the differential equation. When all the conditions are satisfied, the singularity is apparent.

In all other cases, the singularity is real.

Ex. 1. Consider whether it is possible that $x=0$ should be only an apparent singularity of the equation

$$Dw = x^2w'' - (4x + \lambda x^2) w' + (4 - \kappa x) w = 0,$$

where κ and λ are constants.

The first condition, that $P_1(a)$ should be equal to a negative integer, is satisfied: in the present instance, it is -4. To discuss the integrals, let

$$w = c_0 x^a + c_1 x^{a+1} + \ldots + c_n x^{a+n} + \ldots,$$

and substitute: then

$$Dw = c_0 (a-4)(a-1) x^a,$$

provided

$$c_n (a+n-4)(a+n-1) = \{\lambda (a+n-1) + \kappa\} c_{n-1},$$

for $n = 1, 2, \ldots$.

The indicial equation, being $(a-4)(a-1) = 0$, has all its roots equal to positive integers; so that another of the conditions is satisfied. The two roots form a group.

The integral, which belongs to the (greater) root 4 as its index, is a holomorphic function of x; it is easily proved to be a constant multiple of (say) u, where

$$u=x^4\left\{1+\frac{4\lambda+\kappa}{1.4}x+\frac{4\lambda+\kappa}{1.4}\cdot\frac{5\lambda+\kappa}{2.5}x^2+\frac{4\lambda+\kappa}{1.4}\cdot\frac{5\lambda+\kappa}{2.5}\cdot\frac{6\lambda+\kappa}{3.6}x^3+\ldots\right\}$$

$$=x^4(1+\gamma_1x+\gamma_2x^2+\gamma_3x^3+\ldots),$$

for brevity.

As regards the other root given by $a=1$, we have to assign the conditions that the integral which belongs to it contains no logarithms. In accordance with the results of § 41, we see that there will be a single condition; expressing it in the notation there used, we write

$$\rho_0=4,\ \ \rho_1=1,\ \ \nu=\rho_0-\rho_1=3,\ \ \mu=1,$$

and we have to find $h_\nu(a)$ for $\nu=3$, $a=\rho_1=1$. Now (§ 38)

$$f_0(a)=(a-4)(a-1),$$

$$g_3(a)=A_3,\ \ g_0(a)=A_0,$$

and

$$g_3(a)=\frac{h_3(a)\,g_0(a)}{f_0(a+1)f_0(a+2)f_0(a+3)},$$

so that

$$h_3(a)=\{\lambda(a+2)+\kappa\}\{\lambda(a+1)+\kappa\}\{\lambda a+\kappa\}.$$

The sole condition is that

$$h_3(1)=0;$$

and therefore we must have

$$\kappa=-\lambda,\ \text{ or }\ -2\lambda,\ \text{ or }\ -3\lambda.$$

If κ has any one of these values, the origin is only an apparent singularity of the equation.

If $\kappa=-\lambda$, the independent integral belonging to the root 1 is

$$v=x.$$

If $\kappa=-2\lambda$, the integral is

$$v=x+\tfrac{1}{2}\lambda x^2.$$

If $\kappa=-3\lambda$, the integral is

$$v=x+\lambda x^2+\tfrac{1}{2}\lambda^2x^3.$$

In all other cases, the origin is a real singularity of the differential equation.

The result, as to the relations between λ and μ, can be verified independently. As w and u are solutions of the differential equation, we have

$$uw''-wu''=\left(\frac{4}{x}+\lambda\right)(uw'-wu'),$$

and therefore

$$uw'-wu'=Kx^4e^{\lambda x},$$

where K is a constant. Hence

$$\frac{d}{dx}\left(\frac{w}{u}\right)=\frac{K}{x^4}\,\frac{e^{\lambda x}}{(1+\gamma_1x+\gamma_2x^2+\gamma_3x^3+\ldots)^2}.$$

If every integral is to be holomorphic in the vicinity of the origin, it is easy to see that, as u belongs to the index 4, the only condition necessary is that the coefficient of $\frac{1}{x}$ on the right-hand side should be zero. Thus

$$\tfrac{1}{6}\lambda^3-\tfrac{1}{2}\lambda^2\,.\,2\gamma_1+\lambda\,(3\gamma_1^{\,2}-2\gamma_2)-2\gamma_3+6\gamma_1\gamma_2-4\gamma_1^{\,3}=0,$$

which, on substitution for γ_1, γ_2, γ_3, and multiplication by -36, gives

$$(\lambda+\kappa)\,(2\lambda+\kappa)\,(3\lambda+\kappa)=0,$$

thus verifying the condition obtained by the general method.

In this example it appears that the integral, which belongs to the smaller root of the indicial equation, is, in each of the three possible instances, a polynomial in x. It must not be assumed that such a result always holds when a singularity is only apparent; this is not the case*.

Ex. 2. Prove that the origin is an apparent singularity for the equation

$$x^2w''-x\,(4+\lambda x^2)\,w'+(6+\mu x^2)\,w=0,$$

where λ and μ are constants; and shew that no integral, holomorphic in the vicinity of the origin, can be a polynomial in x unless μ is a positive integer multiple of λ.

Ex. 3. Prove that $z=0$ and $z=1$ are real singularities for the equation

$$z\,(1-z)\,w''+(1-2z)\,w'-\tfrac{1}{4}w=0\,;$$

and that $z=1$, $z=-1$, are real singularities for

$$(1-z^2)\,w''-2zw'+n\,(n+1)\,w=0,$$

when n is an integer.

Ex. 4. Shew that $z=\infty$ is a real singularity for every equation of the form

$$\frac{d^2w}{dz^2}+wR\,(z)=0,$$

where $R\,(z)$ denotes a rational function of z.

Ex. 5. Shew that, if $z=\infty$ be an apparent singularity for each integral of the equation

$$\frac{d^2w}{dz^2}+P\left(\frac{1}{z}\right)\frac{dw}{dz}+Q\left(\frac{1}{z}\right)w=0$$

where P and Q are holomorphic functions of z^{-1} for large values of $|z|$, then, if

$$zP\left(\frac{1}{z}\right)=\lambda+\text{negative powers of } z,$$

$$z^2Q\left(\frac{1}{z}\right)=\mu+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

* See some remarks by Cayley, in the memoir quoted on p. 113, note.

λ must be a positive integer equal to or greater than 2, and μ must be a positive integer which may be zero. Shew also that, if $\lambda=2$, then μ must be zero.

Are these conditions sufficient to secure that each integral of the equation is a holomorphic function of z^{-1} for large values of $|z|$?

Ex. 6. Verify that every integral of the equation

$$\frac{d^2w}{dz^2}+\left(\frac{3}{z}+\frac{1}{2z^2}\right)\frac{dw}{dz}+\left(\frac{1}{z^3}-\frac{1}{2z^4}\right)w=0$$

is holomorphic for large values of $|z|$.

Note on § 34, *p.* 85.

To establish the uniform convergence of a series $\Sigma g_\nu x^\nu$ for values of a, Osgood shews (*l.c.*, p. 85) that it is sufficient to have quantities M_n, independent of a, such that

$$|g_n x^n| \leqslant M_n,$$

provided the series ΣM_n converges.

Take a circle in the a-plane large enough to enclose all the regions round the roots of $f(\rho)=0$ given by $|a-\rho|=r'-\kappa'$; and let this circle be of radius r_1, so that r_1 is a constant independent of a. With the notation of § 34, take constants C_ν, for values of $\nu \geqslant \epsilon$, such that

$$C_{\nu+1}=C_\nu\left\{\frac{\psi(r_1+\nu)}{(-r_1+\nu)^m-\phi(r_1+\nu)}+\frac{1}{R}\right\},$$

while $C_\epsilon=\Gamma_\epsilon=\gamma_\epsilon$. Then, as

$$\psi(r_1+\nu)>M(a+\nu),\quad (-r_1+\nu)^m-\phi(r_1+\nu)\leqslant|f_0(a+\nu+1)|,$$

we have

$$\gamma_{\nu+1}\leqslant\Gamma_{\nu+1}<C_{\nu+1},$$

for all values of ν. Now, as in § 34 for the ratios of the Γ's, we find

$$\operatorname*{Lim}_{\nu=\infty}\frac{C_{\nu+1}}{C_\nu}=\frac{1}{R};$$

and therefore the series

$$C_\epsilon(R'-\kappa)^\epsilon+C_{\epsilon+1}(R'-\kappa)^{\epsilon+1}+\dots$$

converges, R' being less than R. Accordingly, by taking

$$M_n=C_n(R'-\kappa)^n,$$

the uniform convergence of the series $\Sigma g_\nu x^\nu$ is established.

CHAPTER IV.

Equations having their Integrals regular in the Vicinity of every Singularity (including Infinity).

46. We have seen that, if a linear differential equation is to have all its integrals regular in the vicinity of any singularity a, it is necessary and sufficient that the equation should be of the form

$$\frac{d^m w}{dz^m} = \frac{P_1}{z-a}\frac{d^{m-1}w}{dz^{m-1}} + \frac{P_2}{(z-a)^2}\frac{d^{m-2}w}{dz^{m-2}} + \dots + \frac{P_m}{(z-a)^m}w,$$

in the vicinity of that singularity, the quantities $P_1, P_2, \dots, P_m$ being holomorphic functions of $z-a$ in a region round a that encloses no other singularity of the equation. We can immediately infer the general form of a homogeneous linear differential equation which has all its integrals regular in the vicinity of every singularity of the equation, including $z=\infty$. As Fuchs was the first to give a full discussion* of this class of equations, it is sometimes described by his name; the equations are said† to be *of Fuchsian type* or *of Fuchsian class.*

Let $a_1, a_2, \dots, a_\rho$ denote all the singularities of the differential equation in the finite part of the z-plane, and write

$$\psi = (z-a_1)(z-a_2)\dots(z-a_\rho);$$

then the conditions are satisfied for each of these singularities by the equation

$$\frac{d^m w}{dz^m} = \sum_{\kappa=1}^{m} \frac{Q_\kappa}{\psi^\kappa}\frac{d^{m-\kappa}w}{dz^{m-\kappa}},$$

* See his memoir, *Crelle*, t. LXVI (1866), pp. 139—154.

† Care must be exercised in order to discriminate between *equations of Fuchsian type* and *Fuchsian equations.* The latter arise in connection with automorphic functions and differential equations having algebraic coefficients : see Chap. x.

provided the functions Q_κ are holomorphic functions of z everywhere in the finite part of the plane. To secure that the integrals possess the assigned characteristics for infinitely large values of z, we note that

$$\psi = z^\rho R\left(\frac{1}{z}\right),$$

where R is a polynomial in $\frac{1}{z}$ and is unity when $z = \infty$, and therefore

$$\psi^\kappa = z^{\rho\kappa} R^\kappa\left(\frac{1}{z}\right) = z^{\rho\kappa} R_1\left(\frac{1}{z}\right),$$

where R_1 is of the same polynomial character as R, and is unity when $z = \infty$. Now suppose that, for very large values of z, the determinant $\Delta(z)$ of a fundamental system belongs to the index σ, so that

$$\Delta(z) = z^{-\sigma} T\left(\frac{1}{z}\right),$$

where T is a regular function of $\frac{1}{z}$ which does not vanish when $z = \infty$. Then, with the notation of § 31, we have

$$\Delta_\kappa(z) = z^{-\sigma-\kappa} T_\kappa\left(\frac{1}{z}\right),$$

where T_κ is of the same character as T, save that it may possibly vanish when $z = \infty$: taking account of the latter, we have

$$\Delta_\kappa(z) = z^{-\sigma-\kappa-\epsilon} T_\kappa'\left(\frac{1}{z}\right),$$

where ϵ is an integer $\geqslant 0$. Thus

$$p_\kappa = \frac{\Delta_\kappa}{\Delta} = z^{-\kappa-\epsilon} U\left(\frac{1}{z}\right),$$

where U is a regular function of $\frac{1}{z}$ which does not vanish when $z = \infty$; and therefore

$$\begin{aligned} Q_\kappa &= p_\kappa \psi^\kappa \\ &= z^{(\rho-1)\kappa-\epsilon} R_1\left(\frac{1}{z}\right) U\left(\frac{1}{z}\right), \end{aligned}$$

for very large values of z. But Q_κ is a holomorphic function of z near $z=\infty$; this property, imposed on the preceding expression, shews* that Q_κ is a polynomial in z, of degree not higher than $(\rho-1)\kappa$.

Moreover, it was proved in the last chapter that all the integrals of the equation

$$\frac{d^m w}{dz^m} = \frac{P_1}{z-a}\frac{d^{m-1}w}{dz^{m-1}} + \dots + \frac{P_m}{(z-a)^m}w$$

are regular in the vicinity of $z=a$, when the quantities $P_1, \dots, P_m$ are holomorphic functions of z in that vicinity. Applying this proposition to each of the singularities (including ∞) of the equation

$$\frac{d^m w}{dz^m} = \sum_{\kappa=1}^{m} \frac{Q_\kappa}{\psi^\kappa}\frac{d^{m-\kappa}w}{dz^{m-\kappa}},$$

with the restriction upon $Q_1, \dots, Q_m$ as polynomials in z of the appropriate degrees, we infer that all its integrals are regular in the vicinity of each of the singularities (including ∞).

Combining the results, we have the theorem, due to Fuchs†:—

When the m integrals in the fundamental system of a linear homogeneous equation of order m have $a_1, a_2, \dots, a_\rho$ as the whole of their possible singularities in the finite part of the z-plane; and when all the integrals are regular in the vicinity of each of these singularities, as well as for infinitely large values of z; the equation is of the form

$$\frac{d^m w}{dz^m} = \frac{G_{\rho-1}}{\psi}\frac{d^{m-1}w}{dz^{m-1}} + \frac{G_{2(\rho-1)}}{\psi^2}\frac{d^{m-2}w}{dz^{m-2}} + \dots + \frac{G_{m(\rho-1)}}{\psi^m}w,$$

where ψ denotes $\prod_{\kappa=1}^{\rho}(z-a_\kappa)$, and $G_{\mu(\rho-1)}$, for $\mu=1, 2, \dots, m$, is a polynomial in z of degree not higher than $\mu(\rho-1)$.

Conversely, all the integrals of this differential equation are everywhere regular, whatever be the polynomials G and ψ of proper degree.

Accordingly, this is the most general form of linear equation of order m, which is of Fuchsian type.

* This result may also be obtained by using the transformation $zx=1$ and applying to the equation, transformed by the relations in § 5, the proper conditions for the immediate vicinity of $x=0$.

† *Crelle*, t. LXVI (1866), p. 146.

Ex. 1. Legendre's equation is

$$(1-z^2)\,w''-2zw'+n\,(n+1)\,w=0,$$

say

$$w''=\frac{2z}{1-z^2}\,w'-\frac{n\,(n+1)}{1-z^2}\,w.$$

Its form satisfies all the necessary conditions; hence its integrals are regular in the vicinity of $z=1$, $z=-1$, and are regular also for infinitely large values of z.

Similarly, the hypergeometric equation, which is

$$z\,(1-z)\,w''+\{\gamma-(\alpha+\beta+1)\,z\}\,w'-\alpha\beta w=0,$$

has all its integrals regular in the vicinity of $z=0$, $z=1$, and regular also for infinitely large values of z.

Bessel's equation of order zero is

$$\begin{aligned} w'' &= -\frac{1}{z}\,w'-w \\ &= -\frac{1}{z}\,w'-\frac{z^2}{z^2}\,w; \end{aligned}$$

its integrals are regular in the vicinity of $z=0$; but, on account of the order of the numerator of the coefficient of w in its fractional form, they are not regular for infinitely large values of z.

The same result as the last holds for

$$w''=-\frac{1}{z}\,w'+\frac{n^2-z^2}{z^2}\,w,$$

which is Bessel's equation of order n.

A form of Lamé's equation, which proves useful (see Chap. IX, §§ 148—151), is

$$w''=\{A\wp\,(z)+B\}\,w,$$

where A and B are constants; its integrals are regular in the vicinity of any point in the finite part of the z-plane congruent with $z=0$, and these are all the singularities in the finite part of the plane; but they are not regular for infinitely large values of z.

Ex. 2. The sum of all the exponents associated with all the singularities (including ∞) of the equation of Fuchsian type obtained at the end of the preceding investigation is the integer $\frac{1}{2}\,(\rho-1)\,m\,(m-1)$, a result first given by Fuchs*. The proof is simple.

The polynomial $G_{\rho-1}$ is of order not higher than $\rho-1$: say

$$G_{\rho-1}=Az^{\rho-1}+\ldots.$$

The indicial equation for the singularity a_n is

$$\theta\,(\theta-1)\ldots(\theta-m+1)=\frac{G_{\rho-1}\,(a_n)}{\psi'\,(a_n)}\,\theta\,(\theta-1)\ldots(\theta-m+2)+\ldots,$$

* *Crelle*, t. LXVI, p. 145.

the unexpressed terms on the right-hand side constituting a polynomial in θ of order not higher than $m-2$. Hence the sum of the indices for the singularity a_n is

$$\tfrac{1}{2}m(m-1)+\frac{G_{\rho-1}(a_n)}{\psi'(a_n)};$$

and therefore the sum of the indices for all the singularities $a_1, a_2, \ldots, a_\rho$ in the finite part of the plane

$$\begin{aligned}&=\tfrac{1}{2}\rho m(m-1)+\sum_{n=1}^{\rho}\frac{G_{\rho-1}(a_n)}{\psi'(a_n)}\\&=\tfrac{1}{2}\rho m(m-1)+A,\end{aligned}$$

because $a_1, a_2, \ldots, a_\rho$ are the roots of $\psi=0$.

The indices for ∞ are obtainable by substituting

$$w=z^{-p}\left(1+\frac{\lambda}{z}+\ldots\right);$$

the indicial equation for ∞ is

$$(-1)^m p(p+1)\ldots(p+m-1)=(-1)^{m-1}Ap(p+1)\ldots(p+m-2)+\ldots,$$

so that the sum of the indices for ∞ is

$$-\tfrac{1}{2}m(m-1)-A.$$

The total sum of all the indices is therefore

$$\tfrac{1}{2}(\rho-1)m(m-1).$$

Ex. 3. The general equation of Fuchsian type, which has all its integrals regular in the vicinity of every singularity (including ∞), has been obtained. The limitations upon the form of the type are mainly as to degree, so that generally the construction of the equations, when definite singularities and definite exponents at the singularities are assigned, will leave arbitrary elements in the form. The instances when the equations are made completely determinate by such an assignment are easily found.

Taking the equation as of order m, we have polynomials

$$G_{\rho-1}(z),\quad G_{2\rho-2}(z), \ldots, G_{m\rho-m}(z)$$

which, in their most general form, contain

$$\begin{aligned}&\rho+(2\rho-1)+(3\rho-2)+\ldots(m\rho-m+1)\\&=\tfrac{1}{2}\rho m(m+1)-\tfrac{1}{2}m(m-1)\end{aligned}$$

constants.

The assignment of the positions of the singularities merely determines ψ: it gives no assistance to the determination of the constants in the polynomials G.

Each of the ρ singularities in the finite part of the plane requires m exponents, as does also the point $z=\infty$; so that there are $m(\rho+1)$ constants thus provided. But, by the preceding example, their sum is definite: and thus the total number of independent constants thus provided is

$$m(\rho+1)-1.$$

If therefore the equation is to be made fully determinate by the assignment of these constants, we must have

$$\tfrac{1}{2}\rho m(m+1)-\tfrac{1}{2}m(m-1)=m(\rho+1)-1,$$

and therefore

$$\tfrac{1}{2}\rho m(m-1)=\tfrac{1}{2}(m-1)(m+2).$$

When $m=1$, ρ can have any value; that is, any homogeneous linear equation of the first order, which has its integral regular in the vicinity of each of its singularities and of $z=\infty$, is completely determined by the assignment of singularities and of the exponents for the integral in the vicinity of the singularities.

For such equations of the first order, let $a_1, \ldots, a_\rho$ be the singularities in the finite part of the plane; let $m_1, \ldots, m_\rho$ be the indices to which the integral belongs in their respective vicinities, and let m be the index for $z=\infty$, so that $m+\sum_{r=1}^{\rho} m_r=0$. The equation is

$$\frac{dw}{dz}=w\sum_{r=1}^{\rho}\frac{m_r}{z-a_r}:$$

which gives the index for $z=\infty$ as equal to $-\sum_{r=1}^{\rho} m_r$, being its proper value.

When $m>1$, then

$$\rho=1+\frac{2}{m},$$

so that, as ρ is an integer, m must be 2 and then $\rho=2$. Thus the only homogeneous linear equation of order higher than the first, which is of the Fuchsian type, and is completely determined by the assignment of the singularities and of the exponents to which the integrals at the singularities belong, is an equation of the second order: it has two singularities in the finite part of the plane, and it has $z=\infty$ for a singularity; and the sum of the six indices to which the integrals belong, two at each of the singularities, is $\tfrac{1}{2}(2-1)\,2\,(2-1)$, that is, the sum is unity.

The discussion of the determinate equation of the second order of the foregoing type will be resumed later (§§ 47—50).

Note. If $\rho=0$, so that the equation has no singularities in the finite part of the plane, the coefficients are constants if the equation is to be of Fuchsian type. The only singularity of the integrals is at ∞.

If $\rho=1$, $m>1$, the number of arbitrary constants is less than the number of constants, due from the assignment of the indices at the finite point and at $z=\infty$: the latter cannot then all be assigned at will.

For values of ρ greater than 1 and for values of m greater than 1, the number of arbitrary constants in a linear differential equation, which are left undetermined by the assignment of the singularities and their indices, is

$$=\tfrac{1}{2}\rho m(m+1)-\tfrac{1}{2}m(m-1)-\{m(\rho+1)-1\}$$
$$=\tfrac{1}{2}(m-1)\{m(\rho-1)-2\},$$

which for all the specified values of ρ and m, other than $m=2$ and $\rho=2$ taken simultaneously, is greater than zero.

Ex. 4. Consider the equation, indicated in the Note to Ex. 3, all whose integrals are regular at the only finite singularity, which can be taken at the origin, and regular also at infinity: it is

$$\frac{d^m w}{dz^m} = \frac{f_1}{z}\frac{d^{m-1}w}{dz^{m-1}} + \frac{f_2}{z^2}\frac{d^{m-2}w}{dz^{m-2}} + \ldots + \frac{f_m}{z^m} w_1,$$

where $f_1, f_2, \ldots, f_m$ are constants. The assignment of indices for $z=0$ determines $f_1, \ldots, f_m$, and so determines the indices for $z=\infty$; and similarly the assignment of indices for $z=\infty$ determines those for $z=0$. In fact, the indicial equation for $z=0$ is

$$\rho(\rho-1)\ldots(\rho-m+1) = \sum_{\kappa=1}^{m} \rho(\rho-1)\ldots(\rho-m+\kappa-1) f_\kappa,$$

and the indicial equation for $z=\infty$ is

$$(-1)^m \theta(\theta+1)\ldots(\theta+m-1) = \sum_{\kappa=1}^{m} (-1)^{m-\kappa} \theta(\theta+1)\ldots(\theta+m-\kappa+1) f_\kappa :$$

it is at once evident that the roots can be arranged in pairs, one from each equation, in the form $\rho+\theta=0$.

As regards the integrals, it is easy to verify, in accordance with the general theory, that the integral which belongs to a simple root r of the indicial equation for $z=0$ is a constant multiple of z^r: and that the n integrals, which belong to any n-tuple root s of that equation, are constant multiples of

$$z^s (\log z)^a,$$

for $a=0, 1, \ldots, n-1$.

Ex. 5. Consider the equation

$$Dw = z(1-z)w'' + (1-2z)w' - \tfrac{1}{4}w = 0,$$

which* clearly satisfies the conditions that its integrals should be regular, both in the vicinity of its singularities and for large values of z.

To obtain the integrals in the vicinity of $z=0$, we substitute

$$w = c_0 z^a + c_1 z^{a+1} + \ldots + c_n z^{a+n} + \ldots,$$

and find

$$zDw = c_0 a^2 z^a,$$

provided

$$(a+n)^2 c_n = (a+n-\tfrac{1}{2})^2 c_{n-1};$$

so that, writing

$$\gamma_m = \left\{\frac{(a+\frac{1}{2})(a+\frac{3}{2})\ldots(a+m-\frac{1}{2})}{(a+1)(a+2)\ldots(a+m)}\right\}^2,$$

the value of w is

$$w = c_0 z^a (1 + \gamma_1 z + \gamma_2 z^2 + \ldots).$$

* It is the differential equation of the quarter-period in elliptic functions: for a detailed discussion of the equation, see Tannery, *Ann. de l'Éc. Norm. Sup.*, Sér. 2^{me}, t. VIII (1879), pp. 169—194, and Fuchs, *Crelle*, t. LXXI (1870), pp. 121—136.

The indicial equation is $a^2=0$: accordingly, the two integrals belong to the index 0, and they are given by

$$[w]_{a=0}, \quad \left[\frac{\partial w}{\partial a}\right]_{a=0}.$$

To particularise the integrals, we take $c_0=\frac{1}{2}\pi$; the first of the integrals then becomes

$$K(z)=\tfrac{1}{2}\pi\left\{1+\left(\frac{1}{2}\right)^2 z+\left(\frac{1\,.\,3}{2\,.\,4}\right)^2 z^2+\dots\right\}$$
$$=\tfrac{1}{2}\pi\{1+a_1z+a_2z^2+\dots\},$$

say: and the second of them becomes $L(z)$, where

$$L(z)=K(z)\log z+\tfrac{1}{2}\pi\sum_{m=1}^{\infty}a_m z^m\, 2\left\{\frac{1}{\frac{1}{2}}+\frac{1}{\frac{3}{2}}+\dots+\frac{1}{m-\frac{1}{2}}-\frac{1}{1}-\frac{1}{2}-\dots-\frac{1}{2m}\right\}$$
$$=K(z)\log z+2\pi\sum_{m=1}^{\infty}a_m\beta_m z^m$$
$$=K(z)\log z+I(z),$$

say, where

$$\beta_m=\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots+\frac{1}{2m-1}-\frac{1}{2m}.$$

And now the two integrals in the vicinity of the origin are

$$K(z), \quad L(z).$$

To obtain the integrals in the vicinity of $z=1$, we substitute

$$z=1-x,$$

when the equation takes the form

$$x(1-x)\frac{d^2w}{dx^2}+(1-2x)\frac{dw}{dx}-\tfrac{1}{4}w=0,$$

which is of the same form as in the vicinity of $z=0$. Accordingly, the integrals in the vicinity of $z=1$ are given by

$$K(x), \quad L(x).$$

To obtain the integrals in the vicinity of $z=\infty$, we substitute

$$z=\frac{1}{t},$$

when the equation takes the form

$$t^2(1-t)\frac{d^2w}{dt^2}-t^2\frac{dw}{dt}+\tfrac{1}{4}w=0.$$

The indicial equation for $t=0$ is

$$a(a-1)+\tfrac{1}{4}=0;$$

we take

$$w=t^{\frac{1}{2}}u,$$

and we find the equation for u to be

$$t(1-t)\frac{d^2u}{dt^2}+(1-2t)\frac{du}{dt}-\tfrac{1}{4}u=0,$$

of the same form as in the first and the second cases. Accordingly, the integrals of the original equation in the vicinity of $z=\infty$ are

$$t^{\frac{1}{2}} K(t), \quad t^{\frac{1}{2}} L(t).$$

The integrals are thus regular in the vicinity of the three singularities $0, 1, \infty$. Of these, the integrals $K(z)$, $L(z)$ are significant in the domain $|z|<1$, say in D_0; the integrals $K(x)$, $L(x)$ are significant in the domain $|x|=|z-1|<1$, say in D_1; and the integrals $t^{\frac{1}{2}} K(t)$, $t^{\frac{1}{2}} L(t)$ are significant in the domain $|t|<1$, that is, $|z|>1$, say in D_∞. The series $K(z)$ diverges when $z=1$, so that the integrals cease to be significant for such a value.

The domains D_0 and D_1 have a common portion, so that values of z exist which are defined by

$$|z|<1, \quad |z-1|<1.$$

Within this common portion, the integrals $K(z)$, $L(z)$, $K(x)$, $L(x)$ are significant: so that, as $K(z)$ and $L(z)$ make up a fundamental system, we have

$$K(x)=AK(z)+BL(z), \quad L(x)=A'K(z)+B'L(z),$$

where A, B, A', B' are constants. The values of the constants are determined as follows by Tannery.

The integrals are compared for real values of z which are positive and slightly less than 1, so that, as z then approaches 1, $K(z)$ tends to an infinite value. To obtain this infinite value, we note that, as

$$\tfrac{1}{2}\pi(2n+1) > \left\{\frac{2 \,.\, 4 \ldots 2n}{1 \,.\, 3 \ldots (2n-1)}\right\}^2 > \pi n,$$

by Wallis's theorem, we have

$$\frac{1}{(n+\frac{1}{2})\,\pi} < a_n < \frac{1}{n\pi};$$

and therefore, for real values of z between 0 and 1, we have

$$\tfrac{1}{2}\pi\left\{1+\sum_{n=1}^{\infty}\frac{x^n}{(n+\frac{1}{2})\,\pi}\right\} < K(z) < \tfrac{1}{2}\pi\left\{1+\sum_{n=1}^{\infty}\frac{z^n}{n\pi}\right\}.$$

The difference of the two quantities, between which the value of $K(z)$ lies, is

$$\tfrac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+\frac{1}{2}}\right)z^n,$$

which increases as the real value of z increases and, for $z=1$, is

$$\sum_{n=1}^{\infty}\left(\frac{1}{2n}-\frac{1}{2n+1}\right),$$

that is, $1-\log 2$. Hence we may take

$$\begin{aligned} K(z) &= \epsilon(z)+\tfrac{1}{2}\sum_{n=1}^{\infty}\frac{z^n}{n} \\ &= \epsilon(z)-\tfrac{1}{2}\log(1-z), \end{aligned}$$

where

$$\tfrac{1}{2}\pi > \epsilon(z) > \tfrac{1}{2}\pi - 1 + \log 2\,;$$

and the values of z are real, positive, and less than 1. The result shews that $K(z)$ is logarithmically infinite for $z=1$.

Proceeding similarly with $I(z)$ in the expression for $L(z)$, we have, for real values of z between 0 and 1,

$$\sum_{m=1}^{\infty} \frac{\beta_m}{2m+1} z^m < \tfrac{1}{4} I(z) < \sum_{m=1}^{\infty} \frac{\beta_m}{2m} z^m.$$

The difference of the two quantities, between which the value of $\tfrac{1}{4}I(z)$ lies, is

$$\sum_{m=1}^{\infty} \beta_m \left(\frac{1}{2m} - \frac{1}{2m+1}\right) z^m,$$

which increases as the real value of z increases and, for $z=1$, is

$$\sum_{m=1}^{\infty} \beta_m \left(\frac{1}{2m} - \frac{1}{2m+1}\right).$$

Now

$$\begin{aligned}\beta_m &= \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{2m-1} - \frac{1}{2m}\\ &< \log 2\,;\end{aligned}$$

and therefore the foregoing difference is less than

$$\log 2 \sum_{m=1}^{\infty} \left(\frac{1}{2m} - \frac{1}{2m+1}\right),$$

that is, less than $(1-\log 2)\log 2$. Hence we may take

$$\tfrac{1}{4} I(z) = \tfrac{1}{2} \sum_{m=1}^{\infty} \frac{\beta_m}{m} z^m - \epsilon'(z),$$

where, for real positive values of z that are less than 1,

$$0 < \epsilon'(z) < (1-\log 2)\log 2.$$

The expression can be further modified. We have

$$\sum_{m=1}^{\infty} \frac{\beta_m}{m} z^m < \log 2 \sum_{m=1}^{\infty} \frac{z^m}{m},$$

for the values of z considered. The difference between these two series is

$$\sum_{m=1}^{\infty} \frac{\log 2 - \beta_m}{m} z^m,$$

a quantity which increases as the real value of z increases and, for $z=1$, is

$$\sum_{m=1}^{\infty} \frac{1}{m} (\log 2 - \beta_m).$$

But

$$\begin{aligned}\log 2 - \beta_m &= \frac{1}{2m+1} - \frac{1}{2m+2} + \dots\\ &< \frac{1}{2m+1},\end{aligned}$$

and therefore the difference is

$$< \sum_{m=1}^{\infty} \frac{1}{m(2m+1)} < 2(1-\log 2),$$

on evaluating the series. We may therefore take

$$\sum_{m=1}^{\infty} \frac{\beta_m}{m} z^m = \sum_{m=1}^{\infty} \frac{z^m}{m} \log 2 - \epsilon''(z)$$

$$= -\log(1-z)\log 2 - \epsilon''(z),$$

where

$$0 < \epsilon''(z) < 2(1-\log 2).$$

Therefore, finally, we have

$$\tfrac{1}{4} I(z) = -\tfrac{1}{2}\log(1-z)\log 2 - \epsilon_1(z),$$

where

$$\epsilon_1(z) = \epsilon'(z) + \tfrac{1}{2}\epsilon''(z),$$

so that

$$0 < \epsilon_1(z) < 1 - (\log 2)^2;$$

and the values of z considered are real, positive, and less than 1.

In the region common to D_0 and D_1, we have

$$K(x) = AK(z) + BL(z);$$

and therefore, for real values of z less than (but nearly equal to) 1, that is, for real, positive, small values of x,

$$K(x) = A\epsilon(z) - \tfrac{1}{2}A\log x - 2B\log x\log 2 - 4B\epsilon_1(z) + B\{\epsilon(z) - \tfrac{1}{2}\log x\}\log z.$$

When z tends to the value 1, the term $\log x \log z$ tends to the value 0: moreover, $K(x)$ then tends to the value $\frac{1}{2}\pi$; hence, taking account of the infinite terms on the right-hand side, we have

$$A + 4B\log 2 = 0.$$

Again, when z is real, small, and positive, x is real, positive, and less than (but nearly equal to) 1; hence

$$K(x) = \epsilon(x) - \tfrac{1}{2}\log(1-x) = \epsilon(1-z) - \tfrac{1}{2}\log z,$$

so that

$$\epsilon(1-z) - \tfrac{1}{2}\log z = AK(z) + BK(z)\log z + BI(z),$$

all the terms in which are finite except those involving $\log z$; moreover, when $|z|$ is small,

$$K(z) = \tfrac{1}{2}\pi + zR(z),$$

where R is a holomorphic function of z; thus

$$B = -\frac{1}{\pi}.$$

Consequently,

$$A = \frac{4}{\pi}\log 2;$$

so that A and B are known.

Similarly, for the other equation

$$L(x) = A'K(z) + B'L(z),$$

for values of x and z in the common region, we have, for real, positive values of z less than 1, that is, for real, positive values of x that are small,

$$K(x)\log x + I(x) = A'\{\epsilon(z) - \tfrac{1}{2}\log(1-z)\} - 4B'\{\tfrac{1}{2}\log(1-z)\log 2 + \epsilon_1(z)\};$$

hence, taking account of the logarithmically infinite terms on both sides, we see that

$$A' + 4B'\log 2 = -\pi.$$

Next, taking the same equation for values of z that are small, real, and positive, so that x is real, positive, and less than 1, we have

$$A'K(z) + B'\{K(z)\log z + I(z)\} = K(x)\log x + I(x).$$

When x is nearly unity,

$$K(x) = \epsilon(x) - \tfrac{1}{2}\log(1-x),$$

so that $K(x)\log x$, for x nearly equal to 1, is small: and it vanishes when $x=1$. Also, for those values of x,

$$\begin{aligned} I(x) &= -2\log(1-x)\log 2 - 4\epsilon_1(x) \\ &= -2\log z\log 2 - 4\epsilon_1(x); \end{aligned}$$

whence, equating coefficients, we have

$$\tfrac{1}{2}\pi B' = -2\log 2.$$

Thus

$$B' = -\frac{4}{\pi}\log 2, \quad A' = \frac{16}{\pi}(\log 2)^2 - \pi.$$

Accordingly, when z lies within the portion common to the two domains D_0 and D_1, defined by the relations

$$|z| < 1, \quad |z-1| < 1,$$

we have

$$\left.\begin{aligned} K(x) &= \left(\frac{4}{\pi}\log 2\right)K(z) - \frac{1}{\pi}L(z) \\ L(x) &= \left\{\frac{16}{\pi}(\log 2)^2 - \pi\right\}K(z) - \left(\frac{4}{\pi}\log 2\right)L(z) \end{aligned}\right\},$$

where $x = 1-z$.

These results shew that, for complex values of z such that $|z|=1$, both $K(z)$ and $L(z)$ converge. The first of them is a known result in the theory of elliptic integrals; writing $z=k^2$, $x=k'^2$, $K(z)=K$, $K(x)=K'$, we have

$$K' = \frac{2K}{\pi}\log\frac{4}{k} - 2\sum_{m=1}^{\infty} a_m\beta_m k^{2m},$$

an equation which is specially useful for small values of k. Similarly, for values of k nearly equal to unity, we have

$$K = \frac{2K'}{\pi}\log\frac{4}{k'} - 2\sum_{m=1} a_m\beta_m k'^{2m}.$$

Ex. 6. With the notation of the preceding example shew that, for values of z common to the domains D_1 and D_∞ as defined by

$$|z|>1, \quad |z-1|<1,$$

the integrals $K(x)$, $L(x)$, $t^{\frac{1}{2}}K(t)$, $t^{\frac{1}{2}}L(t)$ are connected by the relations

$$\left.\begin{aligned} t^{\frac{1}{2}}K(t) &= \frac{4\log 2 - i\pi}{\pi} K(x) - \frac{1}{\pi} L(x) \\ t^{\frac{1}{2}}L(t) &= \frac{16(\log 2)^2 - 4\pi i \log 2 - \pi^2}{\pi} K(x) - \frac{4\log 2}{\pi} L(x) \end{aligned}\right\}.$$

(Tannery.)

Ex. 7. Denoting the integrals of the equation in Ex. 5 that are associated with the values $z=0, 1, \infty$ by K, L; K', L'; K'', L''; respectively; denoting also the effect upon a function U of a simple cycle round a point a by $[U]_a$, and of simple cycles round a and b in succession by $[U]_{ab}$, prove that

$$[K]_0 = K, \quad [L]_0 = L + 2\pi i K;$$

$$[K']_0 = \left(1 - \frac{8i}{\pi}\log 2\right) K' + \frac{2i}{\pi} L';$$

$$[K']_{01} = -\left(3 + \frac{8i}{\pi}\log 2\right) K' + \frac{2i}{\pi} L';$$

and express $[L']_0$, $[L']_{01}$ in terms of K', L'. (Tannery.)

Ex. 8. Discuss, in the same manner as in Ex. 5, the integrals of the equations

(i) $z(1-z)w'' - \frac{1}{4}w = 0$;

(ii) $z(1-z)w'' + (1-z)w' + \frac{1}{4}w = 0$;

(iii) $z(1-z)w'' \pm w' - \frac{1}{4}w = 0$.

Riemann's P-function.

47. It has already been proved (Ex. 3, § 46) that the only linear differential equation of any order other than the first, which is made completely determinate by the assignment of its singularities and of the exponents to which the integrals belong in the respective vicinities of those singularities, is an equation of the second order which, if it have ∞ for a singularity, has two other singularities in the finite part of the plane. If the latter be at h, k, then the transformation

$$\frac{z-h}{z-k} = \frac{h}{k}\,\frac{c-b}{c-a}\,\frac{x-a}{x-b}$$

gives a, b, c in the x-plane as the representatives of h, k, ∞ in the z-plane. The transformation manifestly does not affect the order

of the equation, its sole result being to make a, b, c (but not now ∞) singularities; we shall therefore suppose this transformation made. Accordingly, we proceed to consider the properties of the function, which thus determines a differential equation; they depend upon the properties initially assigned, which are taken as follows.

In the vicinity of all values of z, except a, b, c (and not excepting ∞ when a, b, c are finite), the function is a holomorphic function of the variable.

In the vicinity of any point (including the three points a, b, c), there are two distinct branches of the function; and all branches of the function in the vicinity of any point are such that, between any three of them, a linear relation

$$A'P' + A''P'' + A'''P''' = 0$$

exists, having constant coefficients A', A'', A'''. (So far as this condition affects the differential equation, it manifestly determines the order as equal to two.)

As exponents are assigned to the three points, let them be α and α' for a: β and β' for b: γ and γ' for c; these quantities being subject (§ 46, Ex. 2) to the condition

$$\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1.$$

It further is assumed that $\alpha - \alpha'$, $\beta - \beta'$, $\gamma - \gamma'$ are not equal to integers. The branches distinct from one another in the respective vicinities are denoted by P_α and $P_{\alpha'}$; P_β and $P_{\beta'}$; P_γ and $P_{\gamma'}$. From the definition of the exponents to which they belong, the functions $(z-a)^{-\alpha}P_\alpha$ and $(z-a)^{-\alpha'}P_{\alpha'}$ are holomorphic in the domain of a and do not vanish when $z=a$. Similarly for b and c.

After the earlier assumption, it follows that any branch existing in the vicinity of a can be expressed in a form

$$c_\alpha P_\alpha + c_{\alpha'} P_{\alpha'},$$

where c_α and $c_{\alpha'}$ are constants; and likewise for branches in the vicinity of b and c. The assumption made as to $\alpha - \alpha'$, $\beta - \beta'$, $\gamma - \gamma'$ not being integers will, by the results obtained in §§ 35—38, secure the absence of logarithms from the integrals of the differential equation: it manifestly excludes the possibility of either of the branches P_α and $P_{\alpha'}$, P_β and $P_{\beta'}$, P_γ and $P_{\gamma'}$, being absorbed into the other.

Riemann* denotes the function, which is thus defined, by

$$P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{Bmatrix};$$

and the function itself is usually called *Riemann's P-function.* It is clear that α and α' are interchangeable without affecting P; likewise β and β'; likewise γ and γ'. Also, the three vertical columns in the symbol can be interchanged among one another without affecting P; six such interchanges are possible. Again, if P be multiplied† by $(x-a)^\delta (x-b)^{-\delta-\epsilon} (x-c)^\epsilon$, the effect is to give a new function, having a singularity at a with exponents $\alpha+\delta$, $\alpha'+\delta$: a singularity at b with exponents $\beta-\delta-\epsilon$, $\beta'-\delta-\epsilon$; and a singularity at c with exponents $\gamma+\epsilon$, $\gamma'+\epsilon$. Every other point (including ∞) is of the same character as for P. Hence

$$\frac{(x-a)^\delta (x-c)^\epsilon}{(x-b)^{\delta+\epsilon}} P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = P\begin{Bmatrix} a & b & c & \\ \alpha+\delta & \beta-\delta-\epsilon & \gamma+\epsilon & x \\ \alpha'+\delta & \beta'-\delta-\epsilon & \gamma'+\epsilon & \end{Bmatrix},$$

the exponents on the right-hand side still satisfying the condition that the sum of the exponents shall be equal to unity.

A homographic transformation of the independent variable can always be chosen so as to give any three assigned points a', b', c' as the representatives of a, b, c. Accordingly, let such a transformation be adopted as will make a and 0, b and ∞, c and 1, respectively correspond to one another: it manifestly is

$$x' = \frac{x-a}{x-b}\,\frac{c-b}{c-a}.$$

The indices are transferred to the critical points 0, ∞, 1; every other point is ordinary for the new function, as every other point was for the old. For brevity, the transformed function is denoted by

$$P\begin{pmatrix} \alpha & \beta & \gamma & x' \\ \alpha' & \beta' & \gamma' & \end{pmatrix},$$

* *Ges. Werke*, p. 63.

† The sum of the indices in the factor is made zero; otherwise $x=\infty$ would be a singularity for the new function.

where the two-term columns are to be associated with 0, ∞, 1 in order. Also, since

$$1 - x' = \frac{a-b}{a-c}\frac{x-c}{x-b},$$

it follows that, except as to a constant factor,

$$\frac{(x-a)^{\delta}(x-c)^{\epsilon}}{(x-b)^{\delta+\epsilon}} \text{ and } x'^{\delta}(1-x')^{\epsilon}$$

agree; and thus, as regards general character, we have

$$x'^{\delta}(1-x')^{\epsilon} P\begin{pmatrix}\alpha & \beta & \gamma & \\ \alpha' & \beta' & \gamma' & x'\end{pmatrix} = P\begin{pmatrix}\alpha+\delta & \beta-\delta-\epsilon & \gamma+\epsilon & \\ \alpha'+\delta & \beta'-\delta-\epsilon & \gamma'+\epsilon & x'\end{pmatrix}.$$

As $\alpha-\alpha'$, $\beta-\beta'$, $\gamma-\gamma'$ are the same for the P-function on the right-hand side as for the P-function on the left, Riemann denotes all functions of the type represented by the expression on the left by

$$P(\alpha-\alpha',\ \beta-\beta',\ \gamma-\gamma',\ x').$$

In the transformation of the variable, the points a, b, c were made to be congruent with 0, ∞, 1 in the assigned order. A similar result would follow if they had been made congruent with 0, ∞, 1 in any order or, in other words, if 0, ∞, 1 be interchanged among themselves by homographic substitution. As is known, six such substitutions are possible, viz.

$$x'' = x',\ 1-x',\ \frac{1}{x'},\ 1-\frac{1}{x'},\ \frac{x'}{x'-1},\ \frac{1}{1-x'};$$

or, taking account of the association of the exponents with the first arrangement, the table of singularities, exponents, and variables for the six cases is

$$\begin{matrix}0 & \infty & 1 & \\ \alpha & \beta & \gamma & x'; \\ \alpha' & \beta' & \gamma' & \end{matrix} \qquad \begin{matrix}0 & \infty & 1 & \\ \gamma & \beta & \alpha & 1-x'; \\ \gamma' & \beta' & \alpha' & \end{matrix} \qquad \begin{matrix}0 & \infty & 1 & \\ \beta & \alpha & \gamma & \dfrac{1}{x'}; \\ \beta' & \alpha' & \gamma' & \end{matrix}$$

$$\begin{matrix}0 & \infty & 1 & \\ \gamma & \alpha & \beta & 1-\dfrac{1}{x'}; \\ \gamma' & \alpha' & \beta' & \end{matrix} \qquad \begin{matrix}0 & \infty & 1 & \\ \alpha & \gamma & \beta & \dfrac{x'}{x'-1}; \\ \alpha' & \gamma' & \beta' & \end{matrix} \qquad \begin{matrix}0 & \infty & 1 & \\ \beta & \gamma & \alpha & \dfrac{1}{1-x'}; \\ \beta' & \gamma' & \alpha' & \end{matrix}$$

so that P-functions of these arguments with properly permuted exponents can be associated with one another.

48. The significance of the relation

$$\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1,$$

in connection with the function, appears from the following considerations. When the singularities are taken at 0, ∞, 1, the axis of real variables, stretching from $-\infty$ to $+\infty$, divides the plane into two parts in each of which every branch of the function is uniform; or, if the singularities be taken at a, b, c, then a circle through a, b, c divides the plane in the same way. In either case, taking (say) the positive side of the axis or the inside of the circle, the linear relations among the branches of the function give

$$\left.\begin{aligned} P_\alpha &= B_1 P_\beta + B_2 P_{\beta'} \\ P_{\alpha'} &= B_1' P_\beta + B_2' P_{\beta'} \end{aligned}\right\}, \quad \left.\begin{aligned} P_\alpha &= C_1 P_\gamma + C_2 P_{\gamma'} \\ P_{\alpha'} &= C_1' P_\gamma + C_2' P_{\gamma'} \end{aligned}\right\},$$

say

$$P_\alpha, P_{\alpha'} = \begin{pmatrix} B_1, & B_2 \\ B_1', & B_2' \end{pmatrix})(P_\beta, P_{\beta'}) = (b)(P_\beta, P_{\beta'}),$$

$$P_\alpha, P_{\alpha'} = \begin{pmatrix} C_1, & C_2 \\ C_1', & C_2' \end{pmatrix})(P_\gamma, P_{\gamma'}) = (c)(P_\gamma, P_{\gamma'});$$

and with the usual notation of substitutions, let

$$P_\beta, P_{\beta'} = (\overset{-1}{b})(P_\alpha, P_{\alpha'}),$$

$$P_\gamma, P_{\gamma'} = (\overset{-1}{c})(P_\alpha, P_{\alpha'}).$$

Consider the effect upon any two branches, say P_α and $P_{\alpha'}$, of circuits of the variable round the singularities.

When it describes positively a circuit round a alone, they become $e^{2\pi i\alpha}P_\alpha$ and $e^{2\pi i\alpha'}P_{\alpha'}$ respectively, so that, in the above notation,

$$P_\alpha, P_{\alpha'} \text{ become } \begin{pmatrix} e^{2\pi i\alpha}, & 0 \\ 0, & e^{2\pi i\alpha'} \end{pmatrix})(P_\alpha, P_{\alpha'}).$$

When it describes positively a circuit round b alone, then P_β and $P_{\beta'}$ become $e^{2\pi i\beta}P_\beta$ and $e^{2\pi i\beta'}P_{\beta'}$ respectively; and therefore

$$P_\alpha, P_{\alpha'} \text{ become } (b)\begin{pmatrix} e^{2\pi i\beta}, & 0 \\ 0, & e^{2\pi i\beta'} \end{pmatrix})(\overset{-1}{b})(P_\alpha, P_{\alpha'}).$$

Similarly, when it describes positively a circuit round c alone,

$$P_\alpha, P_{\alpha'} \text{ become } (c)\begin{pmatrix} e^{2\pi i\gamma}, & 0 \\ 0, & e^{2\pi i\gamma'} \end{pmatrix})(\overset{-1}{c})(P_\alpha, P_{\alpha'}).$$

Accordingly, when z describes a simple circuit round a, b, c, the initial branches P_α, $P_{\alpha'}$ are transformed into branches

$$(c)\begin{pmatrix} e^{2\pi i\gamma}, & 0 \\ 0, & e^{2\pi i\gamma'} \end{pmatrix}(c)^{-1}(b)\begin{pmatrix} e^{2\pi i\beta}, & 0 \\ 0, & e^{2\pi i\beta'} \end{pmatrix}(b)^{-1}\begin{pmatrix} e^{2\pi i\alpha}, & 0 \\ 0, & e^{2\pi i\alpha'} \end{pmatrix}(P_\alpha, P_{\alpha'}),$$

say

$$(I)(P_\alpha, P_{\alpha'}).$$

Such a circuit encloses all the singularities of the functions; and therefore* each of the functions returns to its initial value at the end of the circuit, so that

$$(I) = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}.$$

The determinant of the right-hand side is unity; hence the determinant of I is unity, and it is the product of the determinants of all the component substitutions. Now as (c) and $(c)^{-1}$ are inverse, the product of their determinants is unity; and likewise, the product of the determinants of (b) and $(b)^{-1}$ is unity. Hence we must have

$$e^{2\pi i(\alpha+\alpha'+\beta+\beta'+\gamma+\gamma')} = 1,$$

an equation which is satisfied in virtue of the relation

$$\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1:$$

the sum of the exponents could be equal to any integer merely so far as the preceding considerations are concerned.

In the present instance, the property, that a function returns to its initial value after the description of a circuit enclosing all its singularities, can be used in the form that the effect of a positive circuit round c is the same as the effect of a negative circuit round a and round b. Applying this to P_α, we have

$$C_1 P_\gamma e^{2\gamma\pi i} + C_2 P_{\gamma'} e^{2\gamma'\pi i} = e^{-2\alpha\pi i}(B_1 P_\beta e^{-2\beta\pi i} + B_2 P_{\beta'} e^{-2\beta'\pi i});$$

and from the expressions for P_α, we have

$$C_1 P_\gamma + C_2 P_{\gamma'} = B_1 P_\beta + B_2 P_{\beta'}.$$

As P_β and $P_{\beta'}$ are linearly independent of one another, it follows that $e^{2\gamma\pi i} - e^{2\gamma'\pi i}$ must not be zero, that is, $\gamma - \gamma'$ must not be an integer. Similarly for $\alpha - \alpha'$ and $\beta - \beta'$.

Ex. Prove, by means of these relations, that

$$\frac{C_1}{C_1'} e^{(\alpha-\alpha')\pi i} = \frac{B_1 \sin(\alpha+\beta+\gamma')\pi}{B_1' \sin(\alpha'+\beta+\gamma')\pi} = \frac{B_2 \sin(\alpha+\beta'+\gamma')\pi}{B_2' \sin(\alpha'+\beta'+\gamma')\pi},$$

$$\frac{C_2}{C_2'} e^{(\alpha-\alpha')\pi i} = \frac{B_1 \sin(\alpha+\beta+\gamma)\pi}{B_1' \sin(\alpha'+\beta+\gamma)\pi} = \frac{B_2 \sin(\alpha+\beta'+\gamma)\pi}{B_2' \sin(\alpha'+\beta'+\gamma)\pi}.$$

(Riemann.)

* *T. F.*, § 90.

Differential Equation determined by Riemann's P-function.

49. As regards the differential equation, associated with these P-functions, and determined by the assignment of the three singularities a, b, c, and their exponents, we know that it must be of the form

$$\frac{d^2w}{dz^2}+\frac{A'z^2+B'z+C'}{(z-a)(z-b)(z-c)}\frac{dw}{dz}+\frac{A''z^4+B''z^3+C''z^2+D''z+E''}{(z-a)^2(z-b)^2(z-c)^2}w=0,$$

which (§ 46) secures that a, b, c, ∞ are points in whose vicinity the integrals are regular. Now the singularities are to be merely the three points a, b, c, so that ∞ must be an ordinary point of the integral. Taking the most general case, when the value of every integral is not necessarily zero for $z=\infty$, we have an integral

$$w=K_0+\frac{K_1}{z}+\frac{K_2}{z^2}+\ldots,$$

where K_0 does not vanish. Substituting, we have

$$\frac{K_0A''}{z^2}+\frac{1}{z^3}[(2-A')K_1+A''K_1+\{B''+2A''(a+b+c)\}K_0]+\ldots=0,$$

the unexpressed terms being lower powers of z; hence

$$K_0A''=0,$$
$$(2-A')K_1+A''K_1+\{B''+2A''(a+b+c)\}K_0=0,$$
$$\vdots$$

that is,

$$A''=0,$$
$$(2-A')K_1+B''K_0=0,$$

and so on. Using the result that $A''=0$, the equation may be written in the form

$$\frac{d^2w}{dz^2}+\left(\frac{A}{z-a}+\frac{B}{z-b}+\frac{C}{z-c}\right)\frac{dw}{dz}$$
$$+\frac{w}{(z-a)(z-b)(z-c)}\left(B''+\frac{\lambda}{z-a}+\frac{\mu}{z-b}+\frac{\nu}{z-c}\right)=0.$$

Forming the indicial equations for the singularities, we have

$$\theta(\theta-1)+A\theta+\frac{\lambda}{(a-b)(a-c)}=0$$

as the indicial equation for a; and therefore, as its roots are to be α and α', it follows that

$$A = 1 - \alpha - \alpha', \quad \lambda = \alpha\alpha'(a-b)(a-c).$$

Similarly

$$B = 1 - \beta - \beta', \quad \mu = \beta\beta'(b-a)(b-c),$$

$$C = 1 - \gamma - \gamma', \quad \nu = \gamma\gamma'(c-a)(c-b).$$

Moreover

$$A' = A + B + C = 2,$$

on account of the value of the sum of the six exponents; the condition

$$(2 - A')K_1 + B''K_0 = 0$$

is thus satisfied by $B'' = 0$. All the quantities are thus determined, and the equation has the form*

$$\frac{d^2w}{dz^2} + \left(\frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c}\right)\frac{dw}{dz}$$

$$+ \left\{\frac{\alpha\alpha'(a-b)(a-c)}{z-a} + \frac{\beta\beta'(b-a)(b-c)}{z-b} + \frac{\gamma\gamma'(c-a)(c-b)}{z-c}\right\}$$

$$\frac{w}{(z-a)(z-b)(z-c)} = 0;$$

from the mode of construction we know that the integrals are regular in the vicinity of the singularities a, b, c, and are holomorphic for large values of z. This is the differential equation, associated with (and determined by) the function

$$P\left\{\begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\}.$$

The branches of the integral in the vicinity of a are P_α, $P_{\alpha'}$; those in the vicinity of b are P_β, $P_{\beta'}$; and those in the vicinity of c are P_γ, $P_{\gamma'}$.

Passing to the form of the function represented by

$$P\begin{pmatrix} \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{pmatrix},$$

where the three singularities are 0, ∞, 1, we deduce the associated differential equation from the preceding case by taking

$$a = 0, \quad b = \infty, \quad c = 1;$$

* First given by Papperitz, *Math. Ann.*, t. xxv (1885), p. 213.

after a slight reduction, the equation is found to be

$$\frac{d^2w}{dz^2} + \frac{1-\alpha-\alpha'-(1+\beta+\beta')z}{z(1-z)}\frac{dw}{dz} + \frac{\alpha\alpha'-(\alpha\alpha'+\beta\beta'-\gamma\gamma')z+\beta\beta'z^2}{z^2(1-z)^2}w = 0.$$

The branches of the integral in the vicinity of the origin are P_α, $P_{\alpha'}$, so that $z^{-\alpha}P_\alpha$, $z^{-\alpha'}P_{\alpha'}$ are holomorphic functions of z, not vanishing when $z=0$; those in the vicinity of $z=1$ are P_γ, $P_{\gamma'}$, so that $(z-1)^{-\gamma}P_\gamma$, $(z-1)^{-\gamma'}P_{\gamma'}$ are holomorphic functions of $z-1$, not vanishing when $z=1$; and those in the vicinity of $z=\infty$ are P_β, $P_{\beta'}$, so that $z^\beta P_\beta$, $z^{\beta'}P_{\beta'}$ are holomorphic functions of $\frac{1}{z}$, not vanishing when $z=\infty$.

Lastly, passing to the form of the functions included in

$$P(\alpha-\alpha',\ \beta-\beta',\ \gamma-\gamma',\ z),$$

we saw that they arise from the association of arbitrary powers of z and $1-z$ with the above function in the form

$$z^\delta(1-z)^\epsilon P\begin{pmatrix}\alpha & \beta & \gamma & \\ \alpha' & \beta' & \gamma' & z\end{pmatrix},$$

and that they lead to a function

$$P\begin{pmatrix}\alpha+\delta, & \beta-\delta-\epsilon, & \gamma+\epsilon & \\ \alpha'+\delta, & \beta'-\delta-\epsilon, & \gamma'+\epsilon & z\end{pmatrix}.$$

Thus we can make any (the same) change on α and α' and, as they are interchangeable, we can select either for the determinate change; accordingly, we take

$$\alpha-\alpha=0,\quad \alpha'-\alpha,\ =1-\nu,$$

say, as the modified exponents. Similarly, we can make any (the same) change on γ and γ': we take

$$\gamma-\gamma=0,\quad \gamma'-\gamma=\nu-\lambda-\mu,$$

say. Then the new values of the exponents for ∞ are

$$\beta+\alpha+\gamma,\ =\lambda,\text{ say},$$

and

$$\begin{aligned}\beta'+\alpha+\gamma &= \alpha+\gamma+1-(\alpha+\alpha'+\beta+\gamma+\gamma')\\ &= \mu,\end{aligned}$$

on reduction: or the exponents are

$$\begin{array}{ll} 0,\ 1-\nu & ,\ \text{for } z=0\ , \\ \lambda,\ \mu & ,\ \text{for } z=\infty\ , \\ 0,\ \nu-\lambda-\mu, & \text{for } z=1\ . \end{array}$$

Their sum clearly is unity: moreover, with the preceding hypotheses, the quantities $1-\nu$, $\mu-\lambda$, $\nu-\lambda-\mu$ are not integers. Specialising the last form of the equation by substituting this set of values for α, α', β, β', γ, γ', we find the equation, after reduction, to be

$$z(1-z)\frac{d^2w}{dz^2}+\{\nu-(\lambda+\mu+1)z\}\frac{dw}{dz}-\lambda\mu w=0,$$

which is the differential equation of Gauss's hypergeometric series with elements λ, μ, ν. Either from the original form of the P-function, or from the resulting form of the equation, the quantities λ and μ are interchangeable.

50. Taking the equation in the more familiar notation

$$z(1-z)\frac{d^2w}{dz^2}+\{\gamma-(\alpha+\beta+1)z\}\frac{dw}{dz}-\alpha\beta w=0,$$

so that the exponents are 0, $1-\gamma$, for $z=0$; α, β, for $z=\infty$; 0, $\gamma-\alpha-\beta$, for $z=1$, we use the preceding method to deduce the well-known set of 24 integrals.

Denoting as usual by $F(\alpha, \beta, \gamma, z)$ the integral which belongs to the exponent zero for the vicinity of $z=0$, we have

$$F(\alpha, \beta, \gamma, z)=1+\frac{\alpha\beta}{1\,.\,\gamma}z+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\,.\,2\,.\,\gamma(\gamma+1)}z^2+\ldots,$$

assigning to the integral the value unity when $z=0$. If

$$z^\delta(1-z)^\epsilon F(\alpha', \beta', \gamma', z)$$

be also an integral, then the exponents for each of the critical points must be the same as above; hence

$$\begin{array}{rll} \delta,\ \delta+1-\gamma' & =0,\ 1-\gamma & ,\ \text{for } z=0\ , \\ \epsilon,\ \epsilon+\gamma'-\alpha'-\beta' & =0,\ \gamma-\alpha-\beta, & \text{for } z=1\ , \\ \alpha'-\delta-\epsilon,\ \beta'-\delta-\epsilon & =\alpha,\ \beta & ,\ \text{for } z=\infty\ . \end{array}$$

Apparently there are eight solutions of these equations; but as α and β can be interchanged, and likewise α' and β', there are only four independent solutions. These are:—

I. $\delta = 0$, $\epsilon = 0$; giving $\alpha' = \alpha$, $\beta' = \beta$, $\gamma' = \gamma$; and the integral is

$$F(\alpha, \beta, \gamma, z);$$

II. $\delta = 1 - \gamma$, $\epsilon = 0$; giving $\alpha' = 1 + \alpha - \gamma$, $\beta' = 1 + \beta - \gamma$, $\gamma' = 2 - \gamma$; and the integral is

$$z^{1-\gamma} F(1 + \alpha - \gamma,\ 1 + \beta - \gamma,\ 2 - \gamma,\ z);$$

III. $\delta = 0$, $\epsilon = \gamma - \alpha - \beta$; giving $\alpha' = \gamma - \alpha$, $\beta' = \gamma - \beta$, $\gamma' = \gamma$; and the integral is

$$(1 - z)^{\gamma - \alpha - \beta} F(\gamma - \alpha,\ \gamma - \beta,\ \gamma,\ z);$$

IV. $\delta = 1 - \gamma$, $\epsilon = \gamma - \alpha - \beta$; giving $\alpha' = 1 - \beta$, $\beta' = 1 - \alpha$, $\gamma' = 2 - \gamma$; on interchanging the first two elements, the integral is

$$z^{1-\gamma} (1 - z)^{\gamma - \alpha - \beta} F(1 - \alpha,\ 1 - \beta,\ 2 - \gamma,\ z).$$

Next, it has been seen (§ 47) that, in the most general case, P-functions can be associated with a given P-function, when the argument of the latter is submitted to any of the six homographic substitutions which interchange 0, 1, ∞ among one another, provided there is the corresponding interchange of exponents. Taking the substitution $z'z = 1$, the new arrangement of exponents is

$$\begin{array}{ll} \alpha,\ \beta & \text{, for } z' = 0, \\ 0,\ \gamma - \alpha - \beta, & \text{for } z' = 1, \\ 0,\ 1 - \gamma & \text{, for } z' = \infty; \end{array}$$

hence, if

$$z'^{\delta} (1 - z')^{\epsilon} F(\alpha', \beta', \gamma', z')$$

is an integral, we must have

$$\begin{array}{rlll} \delta,\ \delta + 1 - \gamma' & = \alpha,\ \beta & , & \text{for } z' = 0, \\ \epsilon,\ \epsilon + \gamma' - \alpha' - \beta' & = 0,\ \gamma - \alpha - \beta, & & \text{for } z' = 1, \\ \alpha' - \delta - \epsilon,\ \beta' - \delta - \epsilon & = 0,\ 1 - \gamma & ; & \text{for } z' = \infty. \end{array}$$

Again there are four independent solutions; they are:—

IX. $\delta = \alpha$, $\epsilon = 0$; giving $\alpha' = \alpha$, $\beta' = 1 + \alpha - \gamma$, $\gamma' = 1 + \alpha - \beta$; and the integral is

$$z^{-\alpha} F\left(\alpha,\ 1 + \alpha - \gamma,\ 1 + \alpha - \beta,\ \frac{1}{z}\right);$$

X. $\delta=\beta$, $\epsilon=0$; giving $\alpha'=\beta$, $\beta'=1+\beta-\gamma$, $\gamma'=1-\alpha+\beta$; and the integral is

$$z^{-\beta}F\left(\beta,\ 1+\beta-\gamma,\ 1-\alpha+\beta,\ \frac{1}{z}\right);$$

XI. $\delta=\beta$, $\epsilon=\gamma-\alpha-\beta$; giving $\alpha'=\gamma-\alpha$, $\beta'=1-\alpha$, $\gamma'=1-\alpha+\beta$; on interchanging α' and β', the integral is

$$z^{-\beta}\left(1-\frac{1}{z}\right)^{\gamma-\alpha-\beta}F\left(1-\alpha,\ \gamma-\alpha,\ 1-\alpha+\beta,\ \frac{1}{z}\right);$$

XII. $\delta=\alpha$, $\epsilon=\gamma-\alpha-\beta$; giving $\alpha'=\gamma-\beta$, $\beta'=1-\beta$, $\gamma'=1+\alpha-\beta$; on interchanging α' and β', the integral is

$$z^{-\alpha}\left(1-\frac{1}{z}\right)^{\gamma-\alpha-\beta}F\left(1-\beta,\ \gamma-\beta,\ 1+\alpha-\beta,\ \frac{1}{z}\right).$$

The remaining four sets, each containing four integrals, and belonging to the substitutions

$$z'=1-z,\quad \frac{1}{1-z},\quad \frac{z}{z-1},\quad \frac{z-1}{z},$$

respectively, can be obtained in a similar manner*. They are :—

V. $F(\alpha,\ \beta,\ \alpha+\beta-\gamma+1,\ \zeta)$;

VI. $(1-\zeta)^{1-\gamma}F(\alpha-\gamma+1,\ \beta-\gamma+1,\ \alpha+\beta-\gamma+1,\ \zeta)$;

VII. $\zeta^{\gamma-\alpha-\beta}F(\gamma-\alpha,\ \gamma-\beta,\ \gamma-\alpha-\beta+1,\ \zeta)$;

VIII. $(1-\zeta)^{1-\gamma}\zeta^{\gamma-\alpha-\beta}F(1-\alpha,\ 1-\beta,\ \gamma-\alpha-\beta+1,\ \zeta)$;

in which set ζ denotes $1-z$:

XIII. $\zeta^{\alpha}F(\alpha,\ \gamma-\beta,\ \alpha-\beta+1,\ \zeta)$;

XIV. $\zeta^{\beta}F(\beta,\ \gamma-\alpha,\ \beta-\alpha+1,\ \zeta)$;

XV. $(1-\zeta)^{-1+\gamma}\zeta^{\alpha}F(\alpha-\gamma+1,\ 1-\beta,\ \alpha-\beta+1,\ \zeta)$;

XVI. $(1-\zeta)^{-1+\gamma}\zeta^{\beta}F(\beta-\gamma+1,\ 1-\alpha,\ \beta-\alpha+1,\ \zeta)$;

in which set ζ denotes $\dfrac{1}{1-z}$:

XVII. $(1-\zeta)^{\alpha}F(\alpha,\ \gamma-\beta,\ \gamma,\ \zeta)$;

XVIII. $(1-\zeta)^{\beta}F(\beta,\ \gamma-\alpha,\ \gamma,\ \zeta)$;

* The complete set of expressions, differently obtained and originally due to Kummer, are given in my *Treatise on Differential Equations*, (2nd ed.), pp. 192—194; the Roman numbers, used above to specify the cases, are in accord with the numbers there used.

$$\text{XIX.}\quad \zeta^{1-\gamma}(1-\zeta)^{\alpha} F(\alpha-\gamma+1,\ 1-\beta,\ 2-\gamma,\ \zeta);$$

$$\text{XX.}\quad \zeta^{1-\gamma}(1-\zeta)^{\beta} F(\beta-\gamma+1,\ 1-\alpha,\ 2-\gamma,\ \zeta);$$

in which set ζ denotes $\dfrac{z}{z-1}$: and

$$\text{XXI.}\quad (1-\zeta)^{\alpha} F(\alpha,\ \alpha-\gamma+1,\ \alpha+\beta-\gamma+1,\ \zeta);$$

$$\text{XXII.}\quad (1-\zeta)^{\beta} F(\beta,\ \beta-\gamma+1,\ \alpha+\beta-\gamma+1,\ \zeta);$$

$$\text{XXIII.}\quad \zeta^{\gamma-\alpha-\beta}(1-\zeta)^{\beta} F(1-\alpha,\ \gamma-\alpha,\ \gamma-\alpha-\beta+1,\ \zeta);$$

$$\text{XXIV.}\quad \zeta^{\gamma-\alpha-\beta}(1-\zeta)^{\alpha} F(1-\beta,\ \gamma-\beta,\ \gamma-\alpha-\beta+1,\ \zeta);$$

in which set ζ denotes $\dfrac{z-1}{z}$.

The preceding investigations have been based upon the assumption, among others, that no one of the quantities

$$1-\gamma,\quad \gamma-a-\beta,\quad a-\beta,$$

is an integer or zero: the determination of the integrals of the differential equation

$$Dw = z(1-z)\,w'' + \{\gamma - (a+\beta+1)\,z\}\,w' - a\beta w = 0,$$

when the assumption is not justified, can be effected by the methods of §§ 36—38.

Consider, in particular, the integrals in the vicinity of $z=0$, when $1-\gamma$ is an integer; there are three cases, according as the integer is zero, positive, or negative. We substitute

$$w = c_0 z^{\theta} + c_1 z^{\theta+1} + \ldots + c_n z^{\theta+n} + \ldots$$

in the equation; and we find

$$zDw = \theta(\theta+\gamma-1)\,c_0 z^{\theta},$$

provided

$$c_n(\theta+n)(\theta+n-1+\gamma) = c_{n-1}(\theta+n-1+a)(\theta+n-1+\beta),$$

so that

$$c_n = \frac{(n-1+a+\theta)\ldots(a+\theta)\,(n-1+\beta+\theta)\ldots(\beta+\theta)}{(n+\theta)\ldots(1+\theta)\,(n-1+\gamma+\theta)\ldots(\gamma+\theta)}\,c_0.$$

(i) Let $1-\gamma=0$, so that the indicial equation is $\theta^2=0$: then the two integrals belong to the index 0, and one of them certainly involves a logarithm; and they are given by

$$[w]_{\theta=0},\quad \left[\frac{dw}{d\theta}\right]_{\theta=0}.$$

The former, when we take $c_0=1$, is

$$F(a,\ \beta,\ 1,\ z),$$

with the usual notation for the hypergeometric function; as the coefficients are required for the other integral, we write

$$F(a,\ \beta,\ 1,\ z) = 1 + \kappa_1 z + \kappa_2 z^2 + \ldots + \kappa_n z^n + \ldots.$$

The second integral, when to it we add

$$\{\psi(\alpha)+\psi(\beta)\}\, F(\alpha, \beta, 1, z),$$

c_0 again being made equal to unity, becomes

$$F(\alpha, \beta, 1, z)\log z + \sum_{n=1}^{\infty} \kappa_n z^n \{\psi(n+\alpha-1)+\psi(n+\beta-1)-2\psi(n)\},$$

where $\psi(m)$ denotes $\dfrac{d}{dm}\{\log \Pi(m)\}$.

(ii) Let $1-\gamma$ be a positive integer, say p, where $p>0$. The indicial equation, being $\theta(\theta-p)=0$, has its roots equal to p, 0. We have

$$c_n = \frac{(n-1+\alpha+\theta)\ldots(\alpha+\theta)\,(n-1+\beta+\theta)\ldots(\beta+\theta)}{(n+\theta)\ldots(1+\theta)\,(n-p+\theta)\ldots(1-p+\theta)}\, c_0.$$

Of the two integrals, that, which belongs to the greater of the two exponents, is equal to

$$z^p F(\alpha+p, \beta+p, 1+p, z),$$

when we take $c_0=0$. The other integral may or may not involve logarithms. If it is not to involve logarithms, then, as in § 41, the numerator of c_p must vanish when $\theta=0$, so that

$$(p-1+\alpha)\ldots\alpha\,(p-1+\beta)\ldots\beta$$

must vanish: in other words, either α or β must be zero or a negative integer not less than γ. When this condition is satisfied, the integral belonging to the index zero is effectively a polynomial in z of degree $-\alpha$ or $-\beta$ as the case may be, and it contains a term independent of z.

When the preceding condition is not satisfied, the integral certainly involves logarithms. In accordance with § 36, we take

$$c_0 = C\theta,$$

so that

$$zDw = C\theta^2(\theta-p)\, z^{\theta};$$

and now

$$w = C\sum_{n=0} \frac{(n-1+\alpha+\theta)\ldots(\alpha+\theta)}{(n+\theta)\ldots(1+\theta)}\;\frac{(n-1+\beta+\theta)\ldots(\beta+\theta)}{(n-p+\theta)\ldots(1-p+\theta)}\,\theta z^{\theta+n}.$$

There are two integrals given by

$$[w]_{\theta=0},\quad \left[\frac{dw}{d\theta}\right]_{\theta=0}.$$

The first is easily seen to be a constant multiple of

$$z^p F(\alpha+p,\ \beta+p,\ 1+p,\ z),$$

thus in effect providing no new integral. The second, after reduction, and making $C=1$, is

$$\frac{(p-1+\alpha)\ldots\alpha\,(p-1+\beta)\ldots\beta}{p!\,(p-1)!\,(-1)^{p-1}}\, z^p F(\alpha+p, \beta+p, 1+p, z)\log z$$

$$+\sum_{n=0}^{p-1} \frac{(n-1+\alpha)\ldots\alpha\,(n-1+\beta)\ldots\beta}{n!\,(n-p)\,(n-1-p)\ldots(1-p)}\, z^n$$

$$+(-1)^{p-1}\sum_{n=p}^{\infty} \frac{(n-1+\alpha)\ldots\alpha\,(n-1+\beta)\ldots\beta}{n!\,(p-1)!\,(n-p)!}\, z^n\Phi_n,$$

where

$$\Phi_n = \psi(n-1+a) + \psi(n-1+\beta) - \psi(n) - \psi(n-p).$$

(iii) Let $1-\gamma$ be a negative integer, say $-q$, where $q>0$. The indicial equation, being $\theta(\theta+q)=0$, has its roots equal to 0, $-q$. We have

$$c_n = \frac{(n-1+a+\theta)\ldots(a+\theta)(n-1+\beta+\theta)\ldots(\beta+\theta)}{(n+\theta)\ldots(1+\theta)(n+q+\theta)\ldots(1+q+\theta)} c_0.$$

The greater of the two exponents is 0; the integral which belongs to it, on making $c_0=1$, becomes

$$F(a, \beta, 1+q, z).$$

The integral which belongs to the exponent $-q$ may, or may not, involve logarithms. If it is not to involve logarithms, then, as before, the numerator of c_q must vanish when $\theta=-q$, so that

$$(a-1)\ldots(a-q)(\beta-1)\ldots(\beta-q)$$

must vanish: hence either a or β must be a positive integer greater than 0 and less than $\gamma\,(=1+q)$. When the condition is satisfied, the integral is a polynomial in z^{-1}, beginning with z^{-q}, and ending with z^{-a} or $z^{-\beta}$, as the case may be.

When the preceding condition is not satisfied, the integral certainly involves logarithms. As before, in accordance with § 36, we take

$$c_0 = (\theta+q) K,$$

so that

$$zDw = K\theta(\theta+q)^2 z^\theta;$$

and now

$$w = K\Sigma \frac{(n-1+a+\theta)\ldots(a+\theta)}{(n+q+\theta)\ldots(1+q+\theta)} \frac{(n-1+\beta+\theta)\ldots(\beta+\theta)}{(n+\theta)\ldots(1+\theta)} (\theta+q) z^{\theta+n}.$$

Two integrals are given by

$$[w]_{\theta=-q}, \quad \left[\frac{dw}{d\theta}\right]_{\theta=-q}.$$

The first is easily seen to be a constant multiple of

$$F(a, \beta, 1+q, z),$$

so that no new integral is thus provided. The second, after reduction, and making $K=1$, is

$$\begin{aligned}&\frac{(a-1)\ldots(a-q)(\beta-1)\ldots(\beta-q)}{q!(q-1)!(-1)^{q-1}} F(a, \beta, 1+q, z)\log z\\ &\quad+\sum_{n=0}^{q-1} \frac{(n-1+a-q)\ldots(a-q)(n-1+\beta-q)\ldots(\beta-q)}{n!(n-q)\ldots(1-q)} z^{n-q}\\ &\quad+(-1)^{q-1}\sum_{n=q}^{\infty} \frac{(n-1+a-q)\ldots(a-q)(n-1+\beta-q)\ldots(\beta-q)}{n!(q-1)!(n-q)!} z^{n-q}\Phi_n,\end{aligned}$$

where

$$\Phi_n = \psi(n-1+a-q) + \psi(n-1+\beta-q) - \psi(n) - \psi(n-q).$$

The integrals are thus obtained in all the cases, when γ is an integer.

Similar treatment can be applied to the integrals of the equation, when $\gamma-\alpha-\beta$ is an integer, positive, zero, or negative, contrary to the original hypothesis as to the exponents for $z=1$; likewise, when $\alpha-\beta$ is an integer, positive, zero, or negative, contrary to the original hypothesis as to the exponents for $z=\infty$. These instances are left as exercises.

Note. There is a great amount of literature dealing with the hypergeometric series, with the linear equation which it satisfies, and with the integrals of that equation. The detailed properties of the series and all the associated series are of great importance: but as they are developed, they soon pass beyond the range of illustrating the general theory of linear differential equations, and become the special properties of the particular function. Accordingly, such properties will not here be discussed: they will be found in Klein's lectures *Ueber die hypergeometrische Function* (Göttingen, 1894), where many references to original authorities will be found.

Equations of the Second Order and Fuchsian Type.

51. No equations of the Fuchsian type, other than those already discussed, are made completely determinate merely by the assignment of the singularities and their exponents. It is expedient to consider one or two instances of equations, which shall indicate how far they contain arbitrary elements after singularities and exponents are assigned.

Suppose that an equation of the second order has ρ singularities in the finite part of the plane and has ∞ for a singularity; the sum of the exponents which belong to these $\rho+1$ singularities is (by Ex. 2, § 46) equal to $\rho-1$. Now let a homographic substitution be applied to the independent variable, and let it be chosen so that all the points, congruent to the $\rho+1$ singularities, lie in the finite part of the plane. Thus ∞ is not a singularity of the transformed equation: there are $\rho+1$, say n, singularities in the finite part of the plane: and the adopted transformation has not affected the exponents, which accordingly are transferred to the respective congruent points. Hence, when an equation of the second order and Fuchsian type has n singularities in the finite part of the plane and when infinity is not a singularity, the sum of the exponents belonging to the n points is equal to $n-2$. For

such an equation of the second order, let the singularities and their exponents be

$$a_1, a_2, \ldots, a_n,$$
$$\alpha_1, \alpha_2, \ldots, \alpha_n,$$
$$\beta_1, \beta_2, \ldots, \beta_n;$$

then

$$\sum_{r=1}^{n} (\alpha_r + \beta_r) = n - 2.$$

Let

$$\psi = \psi(z) = (z - a_1)(z - a_2)\ldots(z - a_n);$$

then, as the equation is of the second order and as all its integrals are regular, it is of the form

$$w'' + \frac{F_1}{\psi} w' + \frac{F_2}{\psi^2} w = 0,$$

where F_1 and F_2 are polynomials in z of orders not higher than $n-1$ and $2n-2$ respectively. Also, let

$$\frac{F_1}{\psi} = \frac{A_1}{z - a_1} + \frac{A_2}{z - a_2} + \ldots + \frac{A_n}{z - a_n};$$

and let

$$F_2 = F_2(z) = A''z^{2n-2} + B''z^{2n-3} + C''z^{2n-4} + \ldots.$$

The indicial equation for the point $z = a_r$ is

$$\theta(\theta - 1) + A_r\theta + \frac{F_2(a_r)}{\{\psi'(a_r)\}^2} = 0;$$

and therefore

$$\alpha_r + \beta_r = 1 - A_r,$$

so that

$$A_r = 1 - \alpha_r - \beta_r.$$

Hence

$$\sum_{r=1}^{n} A_r = n - \Sigma(\alpha_r + \beta_r)$$
$$= 2,$$

and therefore the polynomial F_1 is of the form

$$F_1 = 2z^{n-1} + \text{lower powers of } z.$$

Again, ∞ is to be an ordinary point of an integral; hence, taking the most general case, we must have an integral

$$w = K_0 + \frac{K_1}{z} + \frac{K_2}{z^2} + \ldots,$$

where K_0 is not zero: for otherwise we should have a special limitation that every integral is zero at infinity. Substituting, so as to have the equation identically satisfied, and writing

$$\sum_{r=1}^{n} A_r a_r^{\kappa} = s_{\kappa},$$

(so that $s_0 = 2$), we find, as the necessary conditions,

$$0 = K_0 A'',$$

$$0 = (2 - s_0) K_1 + K_1 A'' + K_0 \left\{ B'' + 2A'' \sum_{r=1}^{n} a_r \right\},$$

$$0 = (6 - 2s_0 + A'') K_2 + K_1 \left(-s_1 + B'' + 2A'' \sum_{r=1}^{n} a_r \right)$$

$$+ K_0 \left\{ A'' \left(3 \sum_{r=1}^{n} a_r^2 + 2 \Sigma a_r a_s \right) + 2B'' \sum_{r=1}^{n} a_r + C'' \right\},$$

and so on. The first gives

$$A'' = 0;$$

then the second gives

$$B'' = 0;$$

both of these equations leaving K_0 and K_1 arbitrary. The third equation then gives

$$2K_2 = s_1 K_1 - C'' K_0,$$

and so on, in succession. The remaining coefficients K are uniquely determinate; they are linear in K_1 and K_0, the various coefficients involving the singularities and their exponents, as well as the coefficients in F_2. *The equation therefore has $z = \infty$ for an ordinary point of its integrals, provided F_2 is of order not higher than $2n - 4$.*

The equation can, in this case, be expressed in a different form. Let

$$\frac{F_2}{\psi} = \frac{1}{\psi} (C'' z^{2n-4} + \dots)$$

$$= P_{n-4} + \frac{\lambda_1}{z - a_1} + \frac{\lambda_2}{z - a_2} + \dots + \frac{\lambda_n}{z - a_n},$$

where P_{n-4} is a polynomial of order $n - 4$. (Of course, if $2n - 4$ is less than n, which is the case when $n = 3$, there is no such polynomial.) As the coefficients in F_2 are not subject to any further conditions in connection with the nature of $z = \infty$ for the

integrals, any values or relations imposed upon $\lambda_1, \lambda_2, \ldots, \lambda_n$ and the coefficients in P_{n-4} must be associated with the singularities. The equation now is

$$w'' + \left(\sum_{r=1}^{n} \frac{1-\alpha_r-\beta_r}{z-a_r}\right) w' + \frac{1}{\psi}\left(P_{n-4} + \sum_{r=1}^{n} \frac{\lambda_r}{z-a_r}\right) w = 0.$$

The indicial equation for $z = a_r$ is

$$\theta(\theta-1) + (1-\alpha_r-\beta_r)\,\theta + \frac{\lambda_r}{\psi'(a_r)} = 0,$$

and its roots must be α_r, β_r: thus

$$\lambda_r = \alpha_r \beta_r \psi'(a_r),$$

and therefore the equation is

$$w'' + \left(\sum_{r=1}^{n} \frac{1-\alpha_r-\beta_r}{z-a_r}\right) w' + \frac{w}{\psi}\left\{P_{n-4} + \sum_{r=1}^{n} \frac{\alpha_r\beta_r\psi'(a_r)}{z-a_r}\right\} = 0.$$

It follows that the only coefficients which remain arbitrary are the $n-3$ coefficients in the polynomial P_{n-4} (where $n \geqslant 4$). When the polynomial P_{n-4} is arbitrarily taken, the foregoing is the most general form of equation of the second order and of Fuchsian type, which has n assigned singularities in the finite part of the plane with assigned exponents, and has ∞ for an ordinary point of its integrals. This is the form adopted by Klein*.

If a new dependent variable y be introduced, defined by the relation

$$w = y\,(z-a_1)^{\rho_1}(z-a_2)^{\rho_2}\ldots(z-a_n)^{\rho_n},$$

then the exponents to which y belongs in the vicinity of a_r are

$$\alpha_r - \rho_r, \quad \beta_r - \rho_r,$$

the difference of which is the same as for w; but $z=\infty$ will have become a singularity, unless

$$\rho_1 + \rho_2 + \ldots + \rho_n \geqslant 0.$$

Now

$$\sum_{r=1}^{n} \{(\alpha_r-\rho_r) + (\beta_r-\rho_r)\} = n - 2 - 2\sum_{r=1}^{n} \rho_r;$$

and therefore

$$\sum_{r=1}^{n} \{1 - (\alpha_r-\rho_r) - (\beta_r-\rho_r)\} = 2 + 2\sum_{r=1}^{n} \rho_r.$$

* *Vorlesungen über lineare Differentialgleichungen der zweiten Ordnung* (Göttingen, 1894), p. 7.

Hence, if $z=\infty$ is not to be a singularity, the quantities $\rho_1, \ldots, \rho_n$ cannot all be chosen so that each of the magnitudes

$$1-(\alpha_r-\rho_r)-(\beta_r-\rho_r)$$

vanishes. Conversely, if the quantities ρ_r be chosen so that each of these magnitudes vanishes, then $z=\infty$ has become a singularity of the equation; having regard to the form of w for large values of z, we see that 0 and 1 are the exponents to which y belongs for large values of z; and the differential equation for y is easily seen to be

$$y''+\frac{y}{\psi}\left[P_{n-3}+\sum_{r=1}^{n}\tfrac{1}{4}\{1-(\alpha_r-\beta_r)^2\}\frac{\psi'(a_r)}{z-a_r}\right]=0,$$

where P_{n-3} is a polynomial of order $n-3$.

This equation, however, has n singularities in the finite part of the plane, and a specially limited singularity at $z=\infty$: we proceed, in the next paragraph, to the more general case.

Note. The indicial equation for $z=\infty$ in the case of the equation for w is

$$\phi(\phi+1)-\phi\sum_{r=1}^{n}(1-\alpha_r-\beta_r)=0,$$

that is,

$$\phi(\phi-1)=0.$$

The root $\phi=0$ gives an integral of the form

$$K_0\left(1-\tfrac{1}{2}\frac{C_0}{z^2}+\ldots\right);$$

and the root $\phi=1$ gives an integral of the form

$$K_1\left(\frac{1}{z}+\tfrac{1}{2}\frac{s_1}{z^2}+\ldots\right);$$

both of which are holomorphic for large values of $|z|$, so that all integrals are holomorphic functions of $\dfrac{1}{z}$ for large values of $|z|$. In this case, ∞ is not a singularity of the integrals: it can be regarded as an apparent singularity of the differential equation, and (if we please) we may consider 0 and -1 as its exponents.

Ex. Shew that the preceding equation can be exhibited in the form

$$w''+\left(\sum_{r=1}^{n}\frac{1-\alpha_r-\beta_r}{z-a_r}\right)w'+\left\{\sum_{r=1}^{n}\frac{\alpha_r\beta_r}{(z-a_r)^2}+\sum_{r=1}^{n}\frac{c_r}{z-a_r}\right\}w=0,$$

where the n constants $c_1, \ldots, c_n$ satisfy the three relations

$$\sum_{r=1}^{n} c_r = 0, \quad \sum_{r=1}^{n} c_r a_r + \sum_{r=1}^{n} \alpha_r \beta_r = 0, \quad \sum_{r=1}^{n} c_r a_r^2 + 2 \sum_{r=1}^{n} \alpha_r \beta_r a_r = 0,$$

and otherwise are arbitrary in the most general case. (Klein.)

52. Now consider the equation of the second order and of Fuchsian type, which has n singularities in the finite part of the plane, say $a_1, a_2, \ldots, a_n$, with exponents α_1 and $\beta_1, \ldots, \alpha_n$ and β_n, respectively, and for which ∞ also is a singularity with exponents α and β: the exponents being subject to the relation

$$\sum_{r=1}^{n} (\alpha_r + \beta_r) = n - 1 - \alpha - \beta.$$

Let ψ denote $(z - a_1)(z - a_2) \ldots (z - a_n)$: then the equation is of the form

$$w'' + \left(\sum_{r=1}^{n} \frac{A_r}{z - a_r} \right) w' + \frac{G}{\psi^2} w = 0,$$

where G is a polynomial of order not higher than $2n - 2$. When G is divided by ψ, we have a polynomial of order $n - 2$ and a fractional part: and so we may write

$$\frac{G}{\psi} = h_{n-2} z^{n-2} + h_{n-3} z^{n-3} + \ldots + h_0 + \sum_{r=1}^{n} \frac{\mu_r}{z - a_r}.$$

The indicial equation for $z = a_r$ now is

$$\theta(\theta - 1) + A_r \theta + \frac{\mu_r}{\psi'(a_r)} = 0,$$

so that

$$A_r = 1 - \alpha_r - \beta_r,$$
$$\mu_r = \alpha_r \beta_r \psi'(a_r):$$

holding for $r = 1, 2, \ldots, n$. The indicial equation for $z = \infty$ is

$$\phi(\phi + 1) - \phi \sum_{r=1}^{n} A_r + h_{n-2} = 0,$$

so that

$$\alpha + \beta = \sum_{r=1}^{n} A_r - 1, \quad h_{n-2} = \alpha\beta:$$

the former being satisfied on account of the relation between the exponents. The equation thus is

$$w'' + \sum_{r=1}^{n} \frac{1 - \alpha_r - \beta_r}{z - a_r} w'$$
$$+ \frac{w}{\psi} \left\{ \alpha\beta z^{n-2} + h_{n-3} z^{n-3} + \ldots + h_0 + \sum_{r=1}^{n} \frac{\alpha_r \beta_r \psi'(a_r)}{z - a_r} \right\} = 0,$$

the coefficients $h_0, h_1, \ldots, h_{n-3}$ being independent of the singularities and their exponents.

When a new dependent variable y is defined by the transformation

$$w = (z-a_1)^{\alpha_1}(z-a_2)^{\alpha_2}\ldots(z-a_n)^{\alpha_n}y,$$

then the exponents of y for a_r are 0 and $\beta_r - \alpha_r$, say 0 and λ_r, this holding for $r = 1, 2, \ldots, n$: and its exponents for ∞ are

$$\alpha + \sum_{r=1}^{n} \alpha_r, \quad \beta + \sum_{r=1}^{n} \alpha_r,$$

$= \sigma, \tau$ say: where

$$\sigma + \tau + \sum_{r=1}^{n} \lambda_r = n - 1.$$

The function y is, in general character, similar to w: it has the same singularities as w, and it is regular in the vicinity of each of them but with altered exponents: and it thus satisfies an equation of the second order and Fuchsian type, which (after the earlier investigation) is

$$y'' + \sum_{r=1}^{n} \frac{1-\lambda_r}{z-a_r} y' + \frac{\sigma\tau z^{n-2} + k_{n-3}z^{n-3} + \ldots + k_0}{(z-a_1)(z-a_2)\ldots(z-a_n)} y = 0,$$

where $k_{n-3}, \ldots, k_0$ are independent of the singularities and their exponents*.

This transformation of an equation

$$w'' + \frac{F_{n-1}}{\psi} w' + \frac{F_{2n-2}}{\psi^2} w = 0$$

to an equation

$$y'' + \frac{G_{n-1}}{\psi} y' + \frac{G_{n-2}}{\psi} y = 0,$$

where F_{n-1}, F_{2n-2}, G_{n-1}, G_{n-2} are polynomials of order indicated by their subscript index, appears to have been given first by Fuchs†. The simplest example of importance occurs for $n = 2$, when the hypergeometric equation is once more obtained.

53. It is well known that, when y is determined by the equation

$$y'' + Py' + Q = 0,$$

* The equation for y can be obtained by the direct substitution of the expression for w in the earlier differential equation for w. When reduction takes place, there are $n-2$ linear homogeneous relations between the constants h and k.

† Heffter, *Einleitung in die Theorie der linearen Differentialgleichungen*, (1894), p. 224.

and a new variable Y is introduced by the relation

$$ye^{\frac{1}{2}\int P dz} = Y,$$

the differential equation for Y is

$$\frac{d^2 Y}{dz^2} + IY = 0,$$

where

$$I = Q - \tfrac{1}{2}\frac{dP}{dz} - \tfrac{1}{4}P^2.$$

In the case of the preceding equation, the relation between y and Y is

$$Y = \{\Pi (z - a_r)^{\frac{1}{2}(1-\lambda_r)}\} y,$$

so that Y is a regular integral in the vicinity of all the singularities and of ∞, the exponents being

$$\tfrac{1}{2}(1 - \lambda_r), \quad \tfrac{1}{2}(1 + \lambda_r), \text{ for } z = a_r, \qquad (r = 1, \ldots, n),$$

and

$$\tfrac{1}{2}(-1 + \sigma - \tau), \quad \tfrac{1}{2}(-1 - \sigma + \tau), \text{ for } z = \infty.$$

From the form of P and Q, it is easy to see that

$$\begin{aligned} I\psi^2 &= \text{polynomial of order } 2n - 2 \\ &= \psi\left[P_{n-2} + \sum_{r=1}^{n} \frac{B_r}{z - a_r}\right], \end{aligned}$$

where P_{n-2} is a polynomial of order $n - 2$, say

$$P_{n-2} = Cz^{n-2} + l_{n-3}z^{n-3} + \ldots + l_0.$$

In order that $\frac{1}{2}(1 - \lambda_r)$, $\frac{1}{2}(1 + \lambda_r)$ may be the exponents of a_r for the equation

$$Y'' + IY = 0,$$

they must be the roots of

$$\theta(\theta - 1) + \frac{B_r}{\psi'(a_r)} = 0:$$

hence

$$B_r = \tfrac{1}{4}(1 - \lambda_r^2)\psi'(a_r).$$

In order that $\frac{1}{2}(-1 + \sigma - \tau)$, $\frac{1}{2}(-1 - \sigma + \tau)$ may be the exponents of ∞ for the same differential equation, they must be the roots of

$$\phi(\phi + 1) + C = 0:$$

hence

$$C = \tfrac{1}{4}\{1 - (\sigma - \tau)^2\}.$$

The remaining constants $l_0, l_1, \ldots, l_{n-3}$ are expressible as homogeneous linear functions of $k_0, k_1, \ldots, k_{n-3}$, so that they are independent of the singularities and the exponents: and thus the equation is

$$\frac{d^2Y}{dz^2}+\frac{Y}{\psi}\left[\tfrac{1}{4}\{1-(\sigma-\tau)^2\}z^{n-2}+l_{n-3}z^{n-3}+\ldots+l_0\right.$$
$$\left.+\sum_{r=1}^{n}\frac{\tfrac{1}{4}(1-\lambda_r^2)\psi'(a_r)}{z-a_r}\right]=0.$$

COROLLARY. For the original equation, ∞ was a singularity of the integrals with exponents σ and τ. If it were only an apparent singularity of the original equation, so that the integrals are regular for large values of $|z|$, then we have the case indicated in the Note, § 51, so that we can take

$$\sigma,\ \tau=0,\ -1.$$

The modified equation now is

$$\frac{d^2Y}{dz^2}+\frac{Y}{\psi}\left\{l_{n-3}z^{n-3}+\ldots+l_0+\sum_{r=1}^{n}\frac{\tfrac{1}{4}(1-\lambda_r^2)\psi'(a_r)}{z-a_r}\right\}=0.$$

For this differential equation and its integrals, the exponents to which the integrals belong in the vicinity of a_r are $\tfrac{1}{2}(1-\lambda_r)$, $\tfrac{1}{2}(1+\lambda_r)$; but ∞ is now a singularity of the integrals, and the exponents for $z=\infty$ are $0, -1$, so that $z=\infty$ is a simple zero of one of the linearly independent integrals of the modified equation.

These forms of the equation, from which the term in $\dfrac{dY}{dz}$ is absent, are the *normal forms* used by Klein.

The simplest example of the class of equations, not made entirely determinate by the assignment of the singularities and their exponents, occurs when there are three singularities in the finite part of the plane and ∞ also is a singularity. By a homographic transformation of the variable, two of the singularities can be made to occur at 0 and 1, and ∞ can be left unaltered; let a denote the remaining singularity. Let the exponents be

$$0,\ 1-\lambda_0 \text{ for } z=0;\quad 0,\ 1-\lambda_1 \text{ for } z=1;\quad 0,\ \lambda \text{ for } z=a;$$
$$\sigma,\ \tau \text{ for } z=\infty;$$

where

$$\sigma+\tau-\lambda_0-\lambda_1+\lambda=0.$$

Then the differential equation is

$$y''+\left(\frac{\lambda_0}{z}+\frac{\lambda_1}{z-1}+\frac{1-\lambda}{z-a}\right)y'+\frac{\sigma\tau(z-q)}{z(z-1)(z-a)}y=0,$$

where q is the (sole) arbitrary constant, left undetermined by the assigned properties. The integral of this equation, which is regular in the vicinity of $z=0$ and belongs to the index 0, is denoted* by

$$F(a, q;\ \sigma, \tau, \lambda_0, \lambda_1;\ z).$$

If $a=1$, $q=1$, the equation degenerates into that of a Gauss's hypergeometric series: likewise if $a=0$, $q=0$.

Ex. 1. Verify that, when $a=\frac{1}{2}$, the group of substitutions

$$z,\quad 1-z,\quad \frac{\frac{1}{2}}{z},\quad \frac{z-\frac{1}{2}}{z},\quad \frac{z-\frac{1}{2}}{z-1},\quad \frac{-\frac{1}{2}}{z-1},\quad \frac{\frac{1}{2}(z-1)}{z-\frac{1}{2}},\quad \frac{\frac{1}{2}z}{z-\frac{1}{2}},$$

interchanges among themselves the four points $0, \frac{1}{2}, 1, \infty$.

Prove that, when $a=-1$ and when $a=2$, there is in each case a corresponding group of eight substitutions interchanging the points $0, 1, a, \infty$ among themselves: and that, when $a=\frac{1}{2}(1+i\sqrt{3})$ and when $a=\frac{1}{2}(1-i\sqrt{3})$, there is in each case a corresponding group of twelve substitutions. Construct these groups. (Heun.)

Ex. 2. Prove that there are eight integrals of Heun's equation of the form

$$z^{\alpha}(z-1)^{\beta}(z-a)^{\gamma} F(a, q;\ \sigma', \tau', \lambda_0', \lambda_1';\ z),$$

which are regular in the vicinity of the origin and have the same exponents as $F(a, q;\ \sigma, \tau, \lambda_0, \lambda_1;\ z)$. Hence construct a set of 64 integrals for the equation when $a=\frac{1}{2}$, which correspond to Kummer's set of 24 integrals for the hypergeometric series.

Indicate the corresponding results when

$$a=-1,\ 2,\ \tfrac{1}{2}(1+i\sqrt{3}),\ \tfrac{1}{2}(1-i\sqrt{3}).$$

Ex. 3. A homogeneous linear differential equation of order n is to have n singularities $a_1, a_2, \ldots, a_n$ in the finite part of the plane and also to have ∞ for a singularity: the integrals are to be regular in the vicinity of each of the singularities, and the exponents of a_r are to be $0, 1, \ldots, n-2, \alpha_r$ (for $r=1, \ldots, n$), while the exponents of ∞ are to be $0, 1, \ldots, n-2, \alpha$, so that

$$\alpha+\sum_{r=1}^{n}\alpha_r=(n-1)^2.$$

Shew that the differential equation is

$$\psi(z)\frac{d^n w}{dz^n}+\sum_{s=1}^{n}E_s(z)\frac{d^{n-s}w}{dz^{n-s}}=0,$$

where $\psi(z)=(z-a_1)(z-a_2)\ldots(z-a_n)$, the coefficient $E_s(z)$ is a polynomial in z of order not greater than s, (for $s=1, \ldots, n$), and

$$E_1(z)=\sum_{r=1}^{n}(\alpha_r-n+1)\frac{\psi(z)}{a_r-z}.$$

(Pochhammer.)

* Heun, *Math. Ann.*, t. XXXIII (1889), pp. 161—179, who has developed some of the properties of these equations, and has applied them, in another memoir (*l.c.*, pp. 180—196), to Lamé's functions.

Equations in Mathematical Physics and Equations of Fuchsian Type.

54. These equations of Fuchsian type include many of the differential equations of the second order that occur in mathematical physics; sometimes such an equation is explicitly of Fuchsian type, sometimes it is a limiting form of an equation of Fuchsian type.

One such example has already been indicated, in Legendre's differential equation (Ex. 1, § 46). Another rises from a transformation of Lamé's differential equation which (§ 148) is of the form

$$\frac{1}{w}\frac{d^2w}{dz^2} + A\wp(z) + B = 0,$$

where A and B are constants*. Writing

$$\wp(z) = x,$$

so that x is a new independent variable, we have

$$\frac{d^2w}{dx^2} + \left(\frac{\frac{1}{2}}{x-e_1} + \frac{\frac{1}{2}}{x-e_2} + \frac{\frac{1}{2}}{x-e_3}\right)\frac{dw}{dx} + \frac{1}{4}\frac{Ax+B}{(x-e_1)(x-e_2)(x-e_3)}w = 0.$$

The singularities of this equation are e_1, e_2, e_3, ∞; the exponents to which the integrals belong in the vicinity of e_1, e_2, e_3 are 0 and $\frac{1}{2}$, in each case; the exponents, to which they belong for large values of x, are the roots of the equation

$$p(p+1) - \tfrac{3}{2}p + \tfrac{1}{4}A = 0.$$

The new equation is of Fuchsian type: and, in this form, it is frequently called *Lamé's equation.*

An equation, similar to Lamé's equation, but having n singularities in the finite part of the plane, each of them with 0 and $\frac{1}{2}$ as their exponents, as well as $z = \infty$ with exponents α and β, such that (§ 52)

$$\alpha + \beta = \tfrac{1}{2}n - 1,$$

is sometimes called *Lamé's generalised equation.* By § 52, it is of the form

$$w'' + w'\sum_{r=1}^{n}\frac{\frac{1}{2}}{z-a_r} + \frac{G_{n-2}}{\prod\limits_{r=1}^{n}(z-a_r)}w = 0,$$

* This is the general form; the value $-n(n+1)$ is assigned (*l.c.*) to A, in order to have those cases of the general form which possess a uniform integral.

where G_{n-2} is a polynomial of order $n-2$, the highest term in which is $\alpha\beta z^{n-2}$.

55. The equation of Fuchsian type which, next after the equation determined by Riemann's P-function, appears to be of most interest is that for which there are five singularities in the finite part of the plane, while $z=\infty$ is an ordinary point. The interest is caused by a theorem*, due to Bôcher, to the effect that *when the five points are made to coalesce in all possible ways, each limiting form of the equation contains, or is equivalent to, one of the linear equations of mathematical physics.*

Let the points be a_1, a_2, a_3, a_4, a_5, with indices α_r and β_r, for $r=1, 2, 3, 4, 5$; then

$$\sum_{r=1}^{5} (\alpha_r+\beta_r)=3,$$

and the equation (p. 153) is

$$w''+w'\sum_{r=1}^{5}\frac{1-\alpha_r-\beta_r}{z-a_r}+\frac{w}{\psi}\left\{P_1+\sum_{r=1}^{5}\frac{\alpha_r\beta_r\psi'(a_r)}{z-a_r}\right\}=0,$$

where $\psi=\prod_{r=1}^{5}(z-a_r)$, and P_1 is a linear polynomial $Ax+B$. The substantially distinct modes of coalescence are:—

(i), a_4 and a_5 into one point;

(ii), a_2 and a_3 into one point, a_4 and a_5 into another;

(iii), a_3, a_4, a_5 into one point;

(iv), a_1 and a_2 into one point, a_3, a_4, a_5 into another;

(v), a_2, a_3, a_4, a_5 into one point;

(vi), all five into one point;

and the various cases will be considered in turn.

Case (i). Let the indices for a_1, a_2, a_3 be made $0, \frac{1}{2}$ for each point; then, as $\psi'(a_4)=0$, $\psi'(a_5)=0$ in the present case, and

$$1-\alpha_4-\beta_4+1-\alpha_5-\beta_5=\tfrac{1}{2},$$

* Ueber die Reihenentwickelungen der Potentialtheorie, *Gött. gekrönte Preisschrift*, (1891), p. 44; and a separate book under the same title, p. 193. See also Klein, *Vorlesungen über lineare Differentialgleichungen der zweiten Ordnung*, (1894), p. 40.

the equation is

$$w'' + w' \sum_{r=1}^{4} \frac{\frac{1}{2}}{z-a_r} + \frac{P_1}{(z-a_1)(z-a_2)(z-a_3)(z-a_4)^2} w = 0.$$

Write

$$z - a_4 = \frac{1}{x}, \quad (a_r - a_4)\, e_r = 1, \quad \text{for } r = 1, 2, 3;$$

the equation becomes

$$\frac{d^2w}{dx^2} + \frac{dw}{dx}\left(\frac{\frac{1}{2}}{x-e_1} + \frac{\frac{1}{2}}{x-e_2} + \frac{\frac{1}{2}}{x-e_3}\right) + \frac{Cx+D}{(x-e_1)(x-e_2)(x-e_3)} w = 0,$$

in effect, the preceding ungeneralised Lamé's equation.

Case (ii). The equation becomes

$$w'' + w'\left\{\frac{1-\alpha_1-\beta_1}{z-a_1} + \frac{1-\alpha'-\beta'}{z-a_2} + \frac{1-\alpha''-\beta''}{z-a_4}\right\}$$
$$+ \frac{w}{(z-a_1)(z-a_2)^2(z-a_4)^2}\left\{P_1 + \frac{\alpha_1\beta_1}{z-a_1}(a_1-a_2)^2(a_1-a_4)^2\right\} = 0,$$

after coalescence of the points, where

$$1 - \alpha' - \beta' = 2 - \alpha_2 - \beta_2 - \alpha_3 - \beta_3,$$
$$1 - \alpha'' - \beta'' = 2 - \alpha_4 - \beta_4 - \alpha_5 - \beta_5,$$

and therefore

$$\alpha_1 + \beta_1 + \alpha' + \beta' + \alpha'' + \beta'' = 1.$$

Writing $\theta = (z-a_1)(z-a_2)(z-a_4)$, we have the coefficient of $\frac{w}{\theta}$ in the form

$$\frac{P_1(z-a_1) + \alpha_1\beta_1(a_1-a_2)^2(a_1-a_4)^2}{(z-a_1)(z-a_2)(z-a_4)}$$
$$= \frac{\alpha_1\beta_1\theta'(a_1)}{z-a_1} + \frac{Q_1}{(z-a_2)(z-a_4)},$$

where Q_1, like P_1, is an arbitrary linear polynomial. Thus Q_1 contains two arbitrary coefficients; these can be determined so that

$$\frac{P}{(z-a_2)(z-a_4)} = \frac{\alpha'\beta'\theta'(a_2)}{z-a_2} + \frac{\alpha''\beta''\theta'(a_4)}{z-a_4};$$

and then the equation becomes

$$w'' + w'\left\{\frac{1-\alpha_1-\beta_1}{z-a_1} + \frac{1-\alpha'-\beta'}{z-a_2} + \frac{1-\alpha''-\beta''}{z-a_4}\right\}$$
$$+ \frac{w}{\theta}\left\{\frac{\alpha_1\beta_1\theta'(a_1)}{z-a_1} + \frac{\alpha'\beta'\theta'(a_2)}{z-a_2} + \frac{\alpha''\beta''\theta'(a_4)}{z-a_4}\right\} = 0.$$

Owing to the form of θ and the relation $\Sigma(\alpha+\beta)=1$, this is the equation of Riemann's P-function (§ 49).

When we write $a_1=1$; $a_2=-1$; $a_4=\infty$;

$$\alpha_1, \beta_1 = 0, 0; \ \alpha', \beta' = 0, 0; \ \alpha'', \beta'' = -n, n+1;$$

the equation becomes

$$w'' + w'\left(\frac{1}{z-1} + \frac{1}{z+1}\right) - \frac{n(n+1)}{(z+1)(z-1)} w = 0,$$

that is,

$$(1-z^2) w'' - 2zw' + n(n+1) w = 0,$$

which is Legendre's equation.

Case (iii). Let $\alpha_1, \beta_1 = 0, \frac{1}{2}$; $\alpha_2, \beta_2 = 0, \frac{1}{2}$; so that

$$1 - \alpha_3 - \beta_3 + 1 - \alpha_4 - \beta_4 + 1 - \alpha_5 - \beta_5 = 1.$$

After the coalescence of the points, the equation is

$$w'' + w'\left(\frac{\frac{1}{2}}{z-a_1} + \frac{\frac{1}{2}}{z-a_2} + \frac{1}{z-a_3}\right) + \frac{P_1}{(z-a_1)(z-a_2)(z-a_3)^3} w = 0,$$

where P_1 is a linear polynomial, say $\{A(z-a_3)+B\}(a_3-a_1)(a_3-a_2)$. Now let

$$z - a_3 = \frac{1}{x};$$

after some easy reduction, the equation becomes

$$\frac{d^2w}{dx^2} + \frac{1}{2}\frac{dw}{dx}\left\{\frac{1}{x+\dfrac{1}{a_3-a_1}} + \frac{1}{x+\dfrac{1}{a_3-a_2}}\right\}$$
$$+ \frac{A+Bx}{\left(x+\dfrac{1}{a_3-a_1}\right)\left(x+\dfrac{1}{a_3-a_2}\right)} w = 0.$$

Let $a_1=\infty$, $a_3-a_2=-1$; the equation is

$$\frac{d^2w}{dx^2} + \frac{1}{2}\frac{2x-1}{x(x-1)}\frac{dw}{dx} + \frac{A+Bx}{x(x-1)} w = 0.$$

Writing $x=\sin^2 t$, we have

$$\frac{d^2w}{dt^2} - \frac{1}{4} w (A + B\sin^2 t) = 0,$$

which is known* as the equation of the elliptic cylinder. This equation will be discussed hereafter (§§ 138—140).

Case (iv). Let $\alpha_1, \beta_1 = 0, \frac{1}{2}$; $\alpha_2, \beta_2 = 0, \frac{1}{2}$; so that, as in the last case,

$$1 - \alpha_3 - \beta_3 + 1 - \alpha_4 - \beta_4 + 1 - \alpha_5 - \beta_5 = 1.$$

After coalescence of the points, the equation is

$$w'' + w'\left(\frac{1}{z - a_1} + \frac{1}{z - a_3}\right) + \frac{P_1}{(z - a_1)^2 (z - a_3)^3} w = 0.$$

Let

$$z - a_3 = \frac{1}{x}, \quad P_1 = \{\alpha (z - a_3) + \beta\} (a_1 - a_3)^2, \quad c (a_1 - a_3) = 1;$$

then the equation becomes

$$\frac{d^2w}{dx^2} + \frac{1}{x - c}\frac{dw}{dx} + \frac{\alpha + \beta x}{(x - c)^2} w = 0;$$

or, taking

$$x - c = y^2,$$

we have

$$\frac{d^2w}{dy^2} + \frac{1}{y}\frac{dw}{dy} + w\left(\frac{4\alpha'}{y^2} + 4\beta\right) = 0,$$

which includes Bessel's equation, sometimes called the equation of the circular cylinder.

Case (v). Let $\alpha_1, \beta_1 = 0, \frac{1}{2}$; then

$$\sum_{r=2}^{5} (1 - \alpha_r - \beta_r) = \tfrac{3}{2},$$

and the equation, after coalescence of the points, becomes

$$w'' + w'\left(\frac{\frac{1}{2}}{z - a_1} + \frac{\frac{3}{2}}{z - a_2}\right) + \frac{P_1}{(z - a_1)(z - a_2)^4} w = 0.$$

Let

$$z - a_2 = \frac{1}{x}, \quad P_1 = \{\alpha (z - a_2) + \beta\} (a_2 - a_1), \quad b (a_1 - a_2) = 1;$$

then the equation is

$$\frac{d^2w}{dx^2} + \frac{dw}{dx}\frac{\frac{1}{2}}{x - b} + \frac{\alpha + \beta x}{x - b} w = 0.$$

* Heine, *Kugelfunctionen*, t. I, p. 404.

Writing

$$x - b = y^2,$$

the equation becomes

$$\frac{d^2w}{dy^2} + 4w(\alpha + \beta b + \beta y^2) = 0,$$

which is the equation* of the parabolic cylinder.

Case (vi). The equation is

$$w'' + \frac{2}{z-a}w' + \frac{P_1}{(z-a)^5}w = 0:$$

when we take

$$z - a = \frac{1}{x}, \quad P_1 = \alpha(z-a) + \beta,$$

the equation becomes

$$\frac{d^2w}{dx^2} + w(\alpha + \beta x) = 0.$$

This corresponds to no particular equation in mathematical physics: it will be recognised as a very special instance of equations most simply integrated by definite integrals†.

Ex. Discuss, in a similar manner, the limiting forms which are obtained when the singularities of

(i) the equation determined by Riemann's P-function,

(ii) Lamé's equation, expressed as an equation of Fuchsian type,

are made to coalesce in the various ways that are possible.

Equations with Integrals that are Polynomials.

56. There is one simple class of integrals which obey the condition of being everywhere regular, so that their differential equations are of the Fuchsian type; it is the class constituted by functions which are algebraic. We shall, however, reserve the discussion of linear differential equations having algebraic integrals until the next chapter; and we proceed to a brief discussion of a more limited question.

* Weber, *Math. Ann.*, t. I, p. 33.

† See Ch. VII of my *Treatise on Differential Equations.*

We have seen that an equation of the second order and of Fuchsian type can be transformed to

$$Dy = \psi y'' + G_{n-1}y' + G_{n-2}y = 0.$$

Its integrals are regular in the vicinity of each of n singularities and of infinity; the question arises whether the coefficients in the polynomials G_{n-1} and G_{n-2} can be chosen so that one integral of the equation at least shall be, not merely free from logarithms or even algebraic, but actually a polynomial in z. This question has been answered by Heine*; the result is that G_{n-1} can be taken arbitrarily, and G_{n-2} has then a limited number of determinations.

If the above equation, in which

$$\psi = (z - a_1)(z - a_2) \ldots (z - a_n),$$
$$G_{n-1} = c_0 z^{n-1} + c_1 z^{n-2} + \ldots + c_{n-2} z + c_{n-1},$$
$$G_{n-2} = k_0 z^{n-2} + k_1 z^{n-3} + \ldots + k_{n-3} z + k_{n-2},$$

is satisfied by a polynomial of order m, say by

$$y = g_0 z^m + g_1 z^{m-1} + \ldots + g_{m-1} z + g_m,$$

then

$$Dy = (*\between x, 1)^{m+n-2} = 0,$$

so that there are $m + n - 1$ relations among constants. The form of these relations shews that $g_1, g_2, \ldots, g_m$ are multiples of g_0: to express these multiples, m of the relations are required, and when the values obtained are substituted in the remainder, we have $n - 1$ relations left, involving the constants c and k. Assuming the points $a_1, a_2, \ldots, a_n$ arbitrarily taken, and the coefficients $c_0, c_1, \ldots, c_{n-1}$ arbitrarily assigned, we shall have these $n - 1$ relations independent of one another, and therefore sufficient for the determination of the $n - 1$ constants $k_0, k_1, \ldots, k_{n-2}$.

The first of these relations is

$$m(m-1) + c_0 m + k_0 = 0,$$

so that k_0 is uniquely determinate. Denoting by

$$[k_1, k_2, \ldots, k_r]_r$$

the generic expression of a function of $k_1, k_2, \ldots, k_r$, which is polynomial in those quantities, and the terms of highest weight in

* Heine, *Kugelfunctionen*, t. I, p. 473.

which are of weight r, when weights $1, 2, \ldots, n-2$ are assigned to $k_1, k_2, \ldots, k_{n-2}$, we have, from the m relations next after the first,

$$g_r = g_0 [k_1, k_2, \ldots, k_r]_r,$$

for $r = 1, 2, \ldots, m$. When these are substituted in the remaining $n-2$ relations, we have

$$[k_1, k_2, \ldots, k_{n-2}]_{m+s} = 0,$$

for $s = 1, 2, \ldots, n-2$. These determine the $n-2$ constants $k_1, k_2, \ldots, k_{n-2}$; the number of determinations may be obtained as follows. Writing

$$k_1 = x_1, \quad k_2 = x_2^2, \quad k_3 = x_3^3, \ldots,$$

the equations become $n-2$ equations to determine $n-2$ quantities $x_1, x_2, \ldots, x_{n-2}$. In these quantities, the equations are of degrees

$$m+1, m+2, \ldots, m+n-2,$$

respectively; and therefore the number of sets of values for $x_1, x_2, \ldots, x_{n-2}$ is

$$(m+1)(m+2)\ldots(m+n-2).$$

But the same value of k_2 is given by two values of x_2, independently of the other constants k; so that the sets of values of $x_1, x_2, \ldots, x_{n-2}$ must range themselves in twos on this account. Similarly, the same value of k_3 is given by three values of x_3, independently of the other constants k; hence the arranged sets of values must further range themselves in threes, on account of k_3. And so on, up to k_{n-2}. Hence, finally, the number of sets of values of $k_1, \ldots, k_{n-2}$ is

$$\frac{(m+1)(m+2)\ldots(m+n-2)}{2 . 3 \ldots n-2},$$

$$= \frac{(m+n-2)!}{m!\,(n-2)!},$$

which therefore is the number of different quantities G_{n-2} permitting the equation

$$\psi y'' + G_{n-1} y' + G_{n-2} y = 0$$

to possess* a polynomial integral of degree m.

* In connection with these equations, a memoir by Humbert, *Journ. de l'École Polytechnique*, t. XXIX (1880), pp. 207—220, may be consulted.

This result is of importance, as being related to those special forms of Lamé's differential equation which possess an integral expressible as a polynomial in an appropriate variable. This polynomial can be taken as one of the regular integrals belonging to each of the singularities; the other regular integral belonging to any singularity is, in general, a transcendental function and, in general, it involves a logarithm in its expression.

Ex. 1. Shew that a linear equation of the third order, having all its integrals regular, can, by appropriate transformation of its dependent variable, be changed to the form

$$\psi^2 y''' + \psi P y'' + Q y' + R y = 0,$$

where

$$\psi = (z - a_1)(z - a_2)\dots(z - a_n),$$

$a_1, a_2, \dots, a_n$ being all the singularities in the finite part of the z-plane, and where P, Q, R are polynomial functions in z of degrees $n-1$, $2n-2$, $2n-3$ respectively.

Shew that, if P and Q be arbitrarily chosen, R can be determined so that one integral of the equation is a polynomial in z; and prove that the number of distinct values of R is

$$\frac{(m+2n-3)!}{m!\,(2n-3)!},$$

where m is the degree of the polynomial integral.

Ex. 2. Determine the conditions to be satisfied if

$$\psi y'' + G_{n-1} y' + G_{n-2} y = 0$$

has two distinct polynomials as integrals, so that every integral is a polynomial.

Ex. 3. Determine how far the constants in the equation

$$\psi^2 y''' + \psi P y'' + Q y' + R y = 0$$

may be assumed arbitrarily if the equation is to possess two polynomial integrals.

Ex. 4. Prove that the equation

$$f(x)\frac{d^2y}{dx^2} + \tfrac{1}{2}f'(x)\frac{dy}{dx} - \tfrac{1}{4}\{n(n+1)x + h\}y = 0$$

where n is an integer, $f(x) = x^3 + ax^2 + bx + c$, and a, b, c are constants, admits of two integrals whose product is a polynomial in x.

Ex. 5. Shew that the only cases, in which the differential equation of the hypergeometric series

$$x(1-x)\frac{d^2y}{dx^2} + \{\gamma - (\alpha+\beta+1)x\}\frac{dy}{dx} - \alpha\beta y = 0$$

possesses two integrals whose product is a polynomial in x of degree n, are as follows. If n is an even integer, then either $\alpha=-\frac{1}{2}n$; or $\beta=-\frac{1}{2}n$; or $\alpha+\beta=-n$, and $\gamma=\frac{1}{2}$, or $-\frac{1}{2}$, or $-\frac{3}{2}$, ..., or $-n+\frac{1}{2}$. If n is an odd integer, then either $\alpha=-\frac{1}{2}n$ and $\gamma=\frac{1}{2}$, or $-\frac{1}{2}$, or $-\frac{3}{2}$, ..., or $-\frac{1}{2}n+1$, or β, or $\beta-1$, ..., or $\beta-\frac{1}{2}(n-1)$; or $\beta=-\frac{1}{2}n$ and $\gamma=\frac{1}{2}$, or $-\frac{1}{2}$, or $-\frac{3}{2}$, ..., or $-\frac{1}{2}n+1$, or α, or $\alpha-1$, ..., or $\alpha-\frac{1}{2}(n-1)$; or $\alpha+\beta=-n$, and $\gamma=\frac{1}{2}$, or $-\frac{1}{2}$, or $-\frac{3}{2}$, ..., or $-n+\frac{1}{2}$. (Markoff.)

Ex. 6. Shew that, if the square root of a polynomial of degree m can be an integral of the equation

$$\frac{d^2y}{dx^2}+\sum_{s=1}^{s=n}\left(\frac{1-\lambda_s-\mu_s}{x-e_s}\right)\frac{dy}{dx}$$
$$+\left\{\frac{(\lambda\mu-\sum\limits_{s=1}^{n}\lambda_s\mu_s)x^{n-2}+a_1x^{n-3}+\ldots+a_{n-2}}{\prod\limits_{s=1}^{n}(x-e_s)}+\Sigma\frac{\lambda_s\mu_s}{(x-e_s)^2}\right\}y=0,$$

where the exponents λ and μ are subject to the usual relation, one of the exponents λ_s, μ_s, say λ_s, must be half of a non-negative integer, this holding for each value of s; also $\frac{1}{2}m-\Sigma\lambda_s$ must be a non-negative integer; and one exponent of the singularity at infinity must be equal to $-\frac{1}{2}m$.

If these conditions are satisfied, how many such equations exist? (van Vleck.)

Ex. 7. If the differential equation

$$\frac{d^2y}{dx^2}+\sum_{r=1}^{n}\frac{a_r}{x-e_r}\frac{dy}{dx}+\frac{\psi(x)}{\prod\limits_{r=1}^{n}(x-e_r)}y=0,$$

where $\psi(x)$ is a polynomial, the constants a are real and positive, and the constants e are real and distinct from one another, be satisfied by a polynomial $\phi(x)$, then all the roots of $\phi(x)$ are real, and no root is less than the least or greater than the greatest of the quantities e. (Stieltjes; Bôcher.)

Equations with Rational Integrals.

57. The investigation in § 56 suggests another question: what are those linear equations, all the integrals of which are rational meromorphic functions of z?

Let a_1, ..., a_m be the singularities in the finite part of the plane; let α_{1r}, α_{2r}, ..., α_{nr} be the roots of the indicial equation for a_r; and let β_1, ..., β_n be the roots of the indicial equation for $z=\infty$. If every integral is to be a rational function of z, all the roots α_{1r}, α_{2r}, ..., α_{nr} must be integers; as no integral is to involve a logarithm, no two of them may be equal. Let the arrangement

of these roots be in decreasing order of the integers. The integral belonging to the index α_{1r} involves no logarithms; in order that the integrals belonging to the indices $\alpha_{2r}, \alpha_{3r}, \ldots, \alpha_{nr}$ respectively may involve no logarithms,

$$1 + 2 + \ldots + (n-1),$$

that is, $\frac{1}{2}n(n-1)$, conditions in all must be satisfied, these conditions being as set out in § 41. Corresponding conditions hold for each of the singularities, and also for $z = \infty$; so that there are

$$\tfrac{1}{2}n(n-1)(m+1)$$

conditions of relation among the constants of the equation, in addition to the necessity that the indicial equation of each singularity shall have unequal integers for its roots.

These conditions are certainly necessary; they are also sufficient to secure that any integral of the equation is a rational function of z. For considering the vicinity of a_r, each integral in that vicinity is of the form

$$(z - a_r)^{\alpha_{nr}} P_m(z - a_r),$$

where α_{nr} is the least of the roots of the indicial equation, and $P_m(z - a_r)$ is holomorphic in the vicinity of a_r, for $m = 1, \ldots, n$; when $m = n$, $P(z - a_r)$ does not vanish, and for all other values of m it does vanish. If then α_{nr} be zero or positive, the point $z = a_r$ is an ordinary point for every integral in the vicinity of a_r; if α_{nr} be negative, then a_r is a pole of some integral, and it may be a pole of several or of all.

As this holds in the vicinity of each of the singularities and of $z = \infty$, it follows that, in the vicinity of every singularity of the equation, including $z = \infty$, every integral is uniform and has that singularity either for an ordinary point or a pole; moreover, every integral is synectic in the vicinity of every other point: hence* the integral is a rational function, which is a polynomial if ∞ be the only pole. Thus the conditions are necessary and sufficient.

It has been seen that the indicial equation for each singularity of the differential equation must have unequal integers for its roots. When these are assigned arbitrarily, subject to the one relation (Ex. 2, § 46) which they are bound to satisfy, they amount

* *T. F.*, § 48.

to $(m+1)n-1$ conditions; so that the total number of necessary conditions is

$$\tfrac{1}{2}n(n-1)(m+1)+(m+1)n-1$$
$$=\tfrac{1}{2}n(n+1)(m+1)-1.$$

If such equations are being constructed, they are necessarily of the form

$$\frac{d^n w}{dz^n}+\frac{G_1}{\psi}\frac{d^{n-1}w}{dz^{n-1}}+\ldots+\frac{G_n}{\psi^n}w=0,$$

where $\psi=(z-a_1)\ldots(z-a_m)$, and G_r is a polynomial of order not greater than $r(m-1)$, for $r=1,\ldots,n$. Hence the total number of disposable constants is

m, from the positions of the singularities,

$+\sum_{r=1}^{n}\{r(m-1)+1\}$, from the constant coefficients in $G_1,\ldots,G_n$,

that is,

$$\tfrac{1}{2}n(n+1)(m-1)+n+m$$

constants in all; and therefore, in order that the equations may exist, we must have

$$\tfrac{1}{2}n(n+1)(m-1)+n+m \geqslant \tfrac{1}{2}n(n+1)(m+1)-1,$$

so that

$$m \geqslant n^2-1.$$

In obtaining this result, an arbitrary assignment of unequal integers as roots of the indicial equations has been made: and it has been assumed that these conditions are independent of the necessary conditions attaching to the coefficients, in order that the integrals of the equation may be free from logarithms. It may, however, happen that a particular assignment does not leave all these conditions independent of one another, so that we might have

$$\tfrac{1}{2}n(n+1)(m-1)+n+m=\tfrac{1}{2}n(n+1)(m+1)-1-\lambda,$$

and therefore

$$m=n^2-1-\lambda,$$

and still have the equation determinate. An instance is furnished by the equation

$$x^2y''-2xy'+2y=0,$$

which, although it has only one singularity in the finite part of the plane, so that $m=1$, $n=2$, has an integral Ax^2+Bx. For the most general case, however, we have

$$m \geqslant n^2-1.$$

Ex. 1. Investigate all the cases in which the differential equation of the hypergeometric series has every integral a rational function of the independent variable.

Ex. 2. When the equation is of the second order, and all the assignments of integer roots are quite general, the smallest value of m is 3. Let the singularities be $a_1, \ldots, a_m$, with exponents $\alpha_1, \beta_1; \alpha_2, \beta_2; \ldots; \alpha_m, \beta_m$; and let the exponents for $z=\infty$ be α, β. Choosing in each case the smaller of the two indices α_r and β_r, let it be α_r, for $r=1, \ldots, m$; then writing

$$\lambda_r=\beta_r-\alpha_r, \quad \alpha+\sum_{r=1}^{m}\alpha_r=\sigma, \quad \beta+\sum_{r=1}^{m}\alpha_r=\tau,$$

we have (§ 52)

$$\sigma+\tau+\sum_{r=1}^{m}\lambda_r=m-1,$$

which is the necessary relation among the exponents. Writing

$$w=(z-a_1)^{\alpha_1}(z-a_2)^{\alpha_2}\ldots(z-a_n)^{\alpha_n}y,$$

so that y also is a rational function of z, our equation in y becomes

$$y''+\sum_{r=1}^{m}\frac{1-\lambda_r}{z-a_r}y'+\frac{\sigma\tau z^{m-2}+k_{m-3}z^{m-3}+\ldots+k_0}{(z-a_1)(z-a_2)\ldots(z-a_m)}y=0,$$

say

$$Dy=y''+\sum_{r=1}^{m}\frac{1-\lambda_r}{z-a_r}y'+\frac{G(z)}{\psi(z)}y=0,$$

and here the integers $\lambda_1, \lambda_2, \ldots, \lambda_m$ are, each of them, equal to or greater than unity.

Substituting, in the vicinity of a_r, the expression

$$y=c_0(z-a_r)^\theta+c_1(z-a_r)^{\theta+1}+\ldots+c_n(z-a_r)^{\theta+n}+\ldots,$$

we have

$$(z-a_r)^2\,Dy=c_0\theta(\theta-\lambda_r)z^\theta,$$

provided

$$c_1(\theta+1)(\theta+1-\lambda_r)+c_0\left\{\sum_s\frac{\lambda_s-1}{a_s-a_r}+\frac{G(a_r)}{\psi'(a_r)}\right\}=0,$$

and

$$\begin{aligned}c_n(\theta+n)(\theta+n-\lambda_r)&+c_{n-1}\frac{G(a_r)}{\psi'(a_r)}\\ &+\sum_s\frac{\lambda_s-1}{a_s-a_r}\left\{c_{n-1}(\theta+n-1)+c_{n-2}\frac{\theta+n-2}{a_s-a_r}+\ldots+c_0\frac{\theta}{(a_s-a_r)^{n-1}}\right\}\\ &-\sum_s\frac{G(a_s)}{\psi'(a_s)}\frac{1}{a_s-a_r}\left\{c_{n-2}+\frac{c_{n-3}}{a_s-a_r}+\ldots+\frac{c_0}{(a_s-a_r)^{n-2}}\right\}=0,\end{aligned}$$

and the summation for s is for $s=1, \ldots, m$ except $s=r$. As λ_r is a positive integer, and thus is the greater root of the modified indicial equation, there is

one regular integral belonging to the exponent λ_r, which is a constant multiple of

$$(z-a_r)^{\lambda_r}\{1+\gamma_1(z-a_r)+\gamma_2(z-a_r)^2+\ldots\},$$

$= Y$, say, where $\gamma_\nu = c_\nu \div c_0$, when $\theta=\lambda_r$.

When we write

$$f(\theta)=\theta(\theta-\lambda_r),$$

and solve the equations for $c_1, c_2, \ldots$, we find

$$c_n=\frac{h_n(\theta)}{f(\theta+1)\ldots f(\theta+n)}c_0.$$

We know (§ 41) that there is a single condition to be satisfied in order that the integral belonging to the exponent 0 may be free from logarithms; as $f(\theta+n)$ vanishes to the first order for $\theta=0$ when $n=\lambda_r$, the condition is

$$h_{\lambda_r}(0)=0.$$

There is a corresponding condition for each of the singularities and for $z=\infty$; so that we have $m+1$ conditions, which involve the arbitrary constants $k_0, \ldots, k_{m-3}$, and the positions of the singularities, as well as the assigned integers $\lambda_1, \ldots, \lambda_m, \sigma, \tau$. Keeping the latter arbitrary, we see that there must be at least three singularities in the finite part of the plane: when there are only three, we obtain a limited number of determinations of the equation; if there are $3+p$, then p elements are left arbitrary among an otherwise limited number of determinations of the equation*.

As the equation is of the second order, it is possible to proceed otherwise. Assuming that the integral Y which belongs to the exponent λ_r of the singularity a_r is known, and denoting by Z the integral which belongs to the exponent 0 of the same singularity, we have

$$YZ''-Y''Z+(YZ'-Y'Z)\sum_{r=1}^{m}\frac{1-\lambda_r}{z-a_r}=0,$$

so that

$$YZ'-Y'Z=A\prod_{r=1}^{m}(z-a_r)^{\lambda_r-1},$$

and therefore

$$\frac{d}{dz}\left(\frac{Z}{Y}\right)=A\frac{1}{Y^2}\prod_{r=1}^{m}(z-a_r)^{\lambda_r-1}.$$

When the right-hand side is expanded in powers of $z-a_r$, the first term involves $(z-a_r)^{-1-\lambda_r}$, that is, the index is negative. If Z is to be free from logarithms, the term in $\dfrac{1}{z-a_r}$ in this expansion must have its coefficient equal to zero—a condition which must be the equivalent of

$$h_{\lambda_r}(0)=0.$$

* The hypergeometric case indicated in the preceding example is given by

$$\lambda_3=\lambda_n=\ldots=\lambda_m=1,\quad G=\sigma\tau(z-a_3)\ldots(z-a_m),$$

which will be found to satisfy the conditions for $a_3, \ldots, a_m$ given in the text.

CHAPTER V.

Linear Equations of the Second and the Third Orders possessing Algebraic Integrals.

58. The general form of equation, having all its integrals regular in the vicinity of each of the singularities (including ∞), has been obtained; in the vicinity of a singularity a, each such integral is of the form

$$(z-a)^{\mu}[\phi_0 + \phi_1 \log(z-a) + \phi_2 \{\log(z-a)\}^2 + \ldots + \phi_\kappa \{\log(z-a)\}^\kappa],$$

where each of the functions $\phi_0, \phi_1, \ldots, \phi_\kappa$ is holomorphic at and near a. In general, each of the functions ϕ is a transcendental function in the domain of a: they are polynomials only when special relations among the coefficients are satisfied.

When attention is paid to the aggregate of the integrals so obtained, it is to be noted that the branches of a function defined by means of an algebraic equation belong to this class. If algebraic functions are to be integrals of the differential equation, they constitute a special class; special relations among coefficients of the differential equation must then be satisfied, and, it may be, special restrictions must be imposed upon its form. Accordingly, we proceed to consider those linear equations whose integrals are algebraic functions, that is, functions of z defined by an algebraic equation between w and z. It has already been proved (§ 17) that each root of such an algebraic equation of any degree in w satisfies a homogeneous linear differential equation, the coefficients of which are rational functions of z. If the algebraic equation were resoluble into a number of other algebraic equations, necessarily of lower degree, each such component equation would lead to its own differential equation of correspondingly lower order; accordingly, we shall assume that the algebraic equation is irre-

soluble and proceed to consider linear differential equations whose integrals are the roots of an algebraic equation. In the most general case, the degree of the algebraic equation is equal to the order of the differential equation: in particular cases (§ 17, Note 1) it can be greater than the order: and as we seek algebraic integrals, it may be expected that these particular cases will occur.

The investigation can be connected with an equivalent problem that arises in a different range of ideas. It has been proved that, given a fundamental system $w_1, w_2, \ldots, w_m$ of integrals of a linear equation of order m, the effect upon the system, caused by the description of a closed path enclosing one or more of the singularities, is to replace the system by another of the form

$$\left.\begin{array}{l} w_1' = \alpha_{11} w_1 + \alpha_{12} w_2 + \ldots + \alpha_{1m} w_m \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ w_m' = \alpha_{m1} w_1 + \alpha_{m2} w_2 + \ldots + \alpha_{mm} w_m \end{array}\right\},$$

say

$$(w_1', \ldots, w_m') = S\,(w_1, \ldots, w_m),$$

where S denotes a linear substitution. By making the independent variable describe an unlimited number of contours any number of times, we may obtain an unlimited number of linear substitutions; and so each integral could, in that case, be made to have an unlimited number of values. If, however, the fundamental system is equivalent to the m roots of an algebraic equation, then each of the integrals can acquire only a limited number of values at a point which are distinct from one another: that is, there can be only a limited number of substitutions in the aggregate. When therefore we know all the groups of linear substitutions in m variables which are of finite order, only those linear differential equations which possess such groups need be considered. Accordingly, if we proceed by this method, it is necessary to construct the finite groups of linear substitutions.

Further, it is clear that the investigation can be associated with the theory of invariantive forms; for the relations between $w_1', \ldots, w_m'$ and $w_1, \ldots, w_m$ constitute a linear transformation of the type under which these invariantive forms persist. Indeed, it was by this association with binary, ternary, and quaternary forms that the earliest results, relating to linear equations of the orders two, three, and four, were obtained. Some brief indications of this method will be given later (§§ 69—72).

KLEIN'S METHOD FOR EQUATIONS OF THE SECOND ORDER.

59. The determination of linear equations of the second order, whose integrals are everywhere algebraic, is effected by Klein*, by a special method that associates it with the finite groups of linear substitutions of two homogeneous variables.

Let w_1 and w_2 denote a fundamental system of integrals for the differential equation; and let

$$W_1 = \alpha w_1 + \beta w_2, \quad W_2 = \gamma w_1 + \delta w_2,$$

be any one of the linear substitutions, representing the change made upon the fundamental system by the description of a closed path. Then taking

$$s = \frac{w_1}{w_2},$$

the quotient of two algebraic integrals, so that s itself is an algebraic function, we have

$$S = \frac{W_1}{W_2} = \frac{\alpha s + \beta}{\gamma s + \delta};$$

thus s is subject to a homographic substitution. Accordingly, the determination of the finite groups of linear substitutions in the present case is effectively the determination of the finite groups of homographic substitutions.

Let any such group containing N substitutions be represented by

$$\psi_0(s), \ \psi_1(s), \ \ldots, \ \psi_{N-1}(s),$$

and let $\psi_0(s) = s$, the identical substitution: every possible combination of these substitutions can be expressed as some one of the members of the group. Take a couple of arbitrary constants a and b, subject solely to the negative restrictions that a is not equal to $\psi_r(b)$ and b is not equal to $\psi_s(a)$, for any of the values $0, 1, \ldots, N-1$ of r and of s; and form the equation

$$\frac{\psi_0(s) - a}{\psi_0(s) - b} \cdot \frac{\psi_1(s) - a}{\psi_1(s) - b} \cdot \ldots \cdot \frac{\psi_{N-1}(s) - a}{\psi_{N-1}(s) - b} = X,$$

* *Math. Ann.*, t. XI (1877), pp. 115—118, *ib.*, t. XII (1877), pp. 167—179; *Vorlesungen über das Ikosaeder*, (Leipzig, Teubner, 1884), pp. 115—123.

which is an algebraic equation of degree N in s. It is unaltered when s is submitted to any of the substitutions of the group; for such a substitution only effects a permutation of the various N fractions on the left-hand side among one another. Hence, if any root s be known, all the N roots can be derived from it by submitting it to the N substitutions of the group in turn.

For quite general values of X, the N roots of the equation are distinct; but it can happen that, for particular values of X, a repeated root arises, of multiplicity ν. From the nature of the equation in relation to the group of substitutions, it follows that each distinct root is of multiplicity ν, so that there are $N \div \nu$ distinct roots. To consider the effect of this property of the equation, let the latter be changed so that the numerator and denominator are multiplied by the denominators of $\psi_1(s), \ldots, \psi_{N-1}(s)$. It thus can be expressed in the form

$$\frac{G(s, a)}{G(s, b)} = X,$$

where $G(s, a)$ is a polynomial in s of degree N, the coefficients being functions of a, and $G(s, b)$ is a similar polynomial, its coefficients being the same functions of b. Let X_1 be a value of X, such that $s = \sigma_1$ is a root of multiplicity ν_1 when $X = X_1$; then the equation

$$\frac{G(s, a)}{G(s, b)} - \frac{G(\sigma_1, a)}{G(\sigma_1, b)} = X - X_1$$

has $\dfrac{N}{\nu_1}$ roots each of multiplicity ν_1 when $X = X_1$. But each such root is a root of multiplicity $\nu_1 - 1$ of the equation

$$\frac{d}{ds}\left\{\frac{G(s, a)}{G(s, b)} - \frac{G(\sigma_1, a)}{G(\sigma_1, b)}\right\} = 0,$$

that is, of the equation

$$\Delta(s) = G(s, b)\frac{dG(s, a)}{ds} - G(s, a)\frac{dG(s, b)}{ds} = 0;$$

as there are $\dfrac{N}{\nu_1}$ such roots, it follows that these repeated roots account for

$$\frac{N}{\nu_1}(\nu_1 - 1)$$

of the roots of this derived equation. Moreover, we then have

$$\frac{\Phi_1^{\nu_1}}{G(s, b)\, G(\sigma_1, b)} = X - X_1,$$

where Φ_1 is a polynomial in s of degree $\dfrac{N}{\nu_1}$.

Let X_2 be another value of X, such that $s = \sigma_2$ is a root of the equation of multiplicity ν_2 when $X = X_2$. A precisely similar argument shews that each distinct root of the equation is of multiplicity ν_2; that there are $N \div \nu_2$ distinct roots; that each such root is of multiplicity $\nu_2 - 1$ for the equation $\Delta(s) = 0$; that these roots account for

$$\frac{N}{\nu_2}(\nu_2 - 1)$$

of the roots of the derived equation; and that we have

$$\frac{\Phi_2^{\nu_2}}{G(s, b)\, G(\sigma_2, b)} = X - X_2,$$

where Φ_2 is a polynomial in s of degree $\dfrac{N}{\nu_2}$.

Proceeding in this way with the various values of X that lead to multiple roots of the initial equation, we shall exhaust all the roots of the equation $\Delta(s) = 0$. The degree of $\Delta(s)$ is $2N - 2$; for if

$$G(s, a) = s^N f_0(a) + s^{N-1} f_1(a) + \ldots,$$

then

$$G(s, b) = s^N f_0(b) + s^{N-1} f_1(b) + \ldots;$$

and therefore

$$\Delta(s) = s^{2N-2}\{f_0(a) f_1(b) - f_0(b) f_1(a)\} + \ldots.$$

But taking account of the roots of $\Delta(s) = 0$, as associated with the multiple roots of the original equation for the respective values of X, we see that its degree is

$$\frac{N}{\nu_1}(\nu_1 - 1) + \frac{N}{\nu_2}(\nu_2 - 1) + \ldots;$$

and therefore

$$\frac{N}{\nu_1}(\nu_1 - 1) + \frac{N}{\nu_2}(\nu_2 - 1) + \ldots = 2N - 2,$$

whence

$$\left(1 - \frac{1}{\nu_1}\right) + \left(1 - \frac{1}{\nu_2}\right) + \ldots = 2 - \frac{2}{N}.$$

Each of the integers ν is equal to or greater than 2, so that each of the quantities $1-\dfrac{1}{\nu}$ is equal to or greater than $\frac{1}{2}$. Hence the smallest number of different integers ν is two; if there were only one, the left-hand side would be < 1, while the right-hand side is > 1. The largest number of different integers ν is three; if there were four or more, the left-hand side would be equal to or greater than 2, while the right-hand side is less than 2.

In the first place, let there be only two integers, ν_1 and ν_2; then

$$\frac{1}{\nu_1}+\frac{1}{\nu_2}=\frac{2}{N}.$$

From the nature of the case, $\nu_1 \leqslant N$, $\nu_2 \leqslant N$, so that

$$\frac{1}{\nu_1} \geqslant \frac{1}{N}, \quad \frac{1}{\nu_2} \geqslant \frac{1}{N};$$

hence the only possible solution is

$$\nu_1 = N, \quad \nu_2 = N, \qquad \text{(I)},$$

and N is an undetermined integer.

In the next place, let there be three integers, ν_1, ν_2, ν_3: then

$$\frac{1}{\nu_1}+\frac{1}{\nu_2}+\frac{1}{\nu_3}=1+\frac{2}{N}.$$

At least one of the integers ν must be 2: for if each of these integers were $\geqslant 3$, the left-hand side would be $\leqslant 1$, while the right-hand side is > 1, as N is a finite integer.

Taking $\nu_1 = 2$, we have

$$\frac{1}{\nu_2}+\frac{1}{\nu_3}=\tfrac{1}{2}+\frac{2}{N}.$$

Another of the integers ν may be 2. Let it be ν_2; then $N = 2\nu_3$, and we have the solution

$$\nu_1 = 2, \quad \nu_2 = 2, \quad \nu_3 = n, \quad N = 2n, \qquad \text{(II)},$$

where n is an undetermined integer.

If neither of the integers ν_2 and ν_3 be 2, one of them must be 3; for if each of them were $\geqslant 4$, then $\dfrac{1}{\nu_2}+\dfrac{1}{\nu_3} \leqslant \frac{1}{2}$, and so

could certainly not be equal to $\frac{1}{2}+\frac{2}{N}$. Taking $\nu_2=3$, we have

$$\frac{1}{\nu_3}=\tfrac{1}{6}+\frac{2}{N},$$

so that $\nu_3<6$: thus possible values of ν_3 are 3, 4, 5. The solutions are

$$\nu_1=2,\quad \nu_2=3,\quad \nu_3=3,\quad N=12,\quad \text{.............(III)},$$

$$\nu_1=2,\quad \nu_2=3,\quad \nu_3=4,\quad N=24,\quad \text{.............(IV)},$$

$$\nu_1=2,\quad \nu_2=3,\quad \nu_3=5,\quad N=60,\quad \text{...............(V)}.$$

60. The finite groups are thus known; the corresponding equations in s are required. The solutions will be taken in order.

I. Instead of X, we take a quantity Z, defined by the relation

$$Z=\frac{X-X_1}{X-X_2},$$

so that $Z=0$ gives $X=X_1$, that is, gives $s=s_1$, a root repeated N times, and $Z=\infty$ gives $X=X_2$, that is, gives $s=s_2$, a root repeated N times. We have

$$X-X_1=\frac{(s-s_1)^N}{G(s, b)\,G(s_1, b)},$$

$$X-X_2=\frac{(s-s_2)^N}{G(s, b)\,G(s_2, b)};$$

and therefore

$$\left(\frac{s-s_1}{s-s_2}\right)^N=Z,$$

absorbing the constant $G(s_1, b)\div G(s_2, b)$ into the variable Z.

II, III, IV, V. These cases are of the same general form. Instead of X, we take a quantity Z, defined by the relation

$$Z=\frac{X-X_2}{X-X_3}\,\frac{X_1-X_3}{X_1-X_2};$$

then $Z=0$ gives $X=X_2$, $Z=1$ gives $X=X_1$, $Z=\infty$ gives $X=X_3$, and thus

$$Z : Z-1 : 1$$

$$=(X-X_2)(X_1-X_3) : (X-X_1)(X_2-X_3) : (X-X_3)(X_1-X_2).$$

But

$$X - X_1 = \frac{\Phi_1^{\nu_1}}{G(s, b)\, G(\sigma_1, b)},$$

$$X - X_2 = \frac{\Phi_2^{\nu_2}}{G(s, b)\, G(\sigma_2, b)},$$

$$X - X_3 = \frac{\Phi_3^{\nu_3}}{G(s, b)\, G(\sigma_3, b)};$$

and therefore

$$Z : Z - 1 : 1 = A\Phi_2^{\nu_2}(s) : B\Phi_1^{\nu_1}(s) : \Phi_3^{\nu_3}(s),$$

where A and B are constants which, if we please, may be absorbed into the functions Φ_2 and Φ_1 respectively.

Now these groups are the groups that occur in connection with the polyhedral functions*: and the polyhedral functions can be associated with the conformal representation†, upon a half-plane, of a triangle, bounded by three circular arcs and having angles equal to $\frac{\pi}{\nu_1}$, $\frac{\pi}{\nu_2}$, $\frac{\pi}{\nu_3}$. The analytical results connected with these investigations can be at once applied to the present problem. Denoting derivatives of Z with regard to s by Z', Z'', Z''', ..., we have (*T. F.*, § 275)

$$-\frac{1}{Z'^2}\left[\frac{Z'''}{Z'} - \frac{3}{2}\left(\frac{Z''}{Z'}\right)^2\right] = \frac{1}{2}\frac{1 - \frac{1}{\nu_2^2}}{Z^2} + \frac{1}{2}\frac{1 - \frac{1}{\nu_1^2}}{(Z-1)^2} + \frac{1}{2}\frac{\frac{1}{\nu_1^2} + \frac{1}{\nu_2^2} - \frac{1}{\nu_3^2} - 1}{Z(Z-1)},$$

or, taking account of the properties‡ of the Schwarzian derivative, we have

$$\{s, Z\} = \frac{\frac{1}{2}\left(1 - \frac{1}{\nu_2^2}\right)}{Z^2} + \frac{\frac{1}{2}\left(1 - \frac{1}{\nu_1^2}\right)}{(Z-1)^2} + \frac{\frac{1}{2}\left(\frac{1}{\nu_1^2} + \frac{1}{\nu_2^2} - \frac{1}{\nu_3^2} - 1\right)}{Z(Z-1)}.$$

The forms of the functions for the various cases II, III, IV, V are:—

for II,

$$Z : Z - 1 : 1 = \{\tfrac{1}{2}(s^n - 1)\}^2 : \{\tfrac{1}{2}(s^n + 1)\}^2 : -s^n;$$

* *T. F.*, §§ 276—279, 300—302.

† *T. F.*, §§ 274, 275.

‡ See Ex. 3, § 62, of my *Treatise on Differential Equations*.

for III,

$$Z : Z-1 : 1$$
$$= (s^4 + 2s^2\sqrt{3} - 1)^3 : 12\sqrt{3}\, s^2 (s^4+1)^2 : (s^4 - 2s^2\sqrt{3} - 1)^3;$$

for IV,

$$Z : Z-1 : 1$$
$$= (s^8 + 14s^4 + 1)^3 : (s^{12} - 33s^8 - 33s^4 + 1)^2 : 108s^4(s^4-1)^2;$$

and for V,

$$\begin{aligned} Z : Z-1 : 1 = {} & (s^{20} - 228s^{15} + 494s^{10} + 228s^5 + 1)^3 \\ & : \{s^{30} + 1 + 522s^5(s^{20}-1) - 10005s^{10}(s^{10}+1)\}^2 \\ & : -1728s^5(s^{10} + 11s^5 - 1)^5. \end{aligned}$$

These results* can be obtained by purely algebraic processes, from the properties of finite groups proved by Gordan†.

61. These results can be applied at once to the determination of linear equations of the second order

$$\frac{d^2w}{dz^2} + p\frac{dw}{dz} + qw = 0,$$

all the integrals of which are algebraic. Denoting the quotient of two integrals w_1 and w_2 by s, we have§

$$w_1 = s'^{-\frac{1}{2}} s e^{-\frac{1}{2}\int p dz}, \quad w_2 = s'^{-\frac{1}{2}} e^{-\frac{1}{2}\int p dz}, \quad w_2 s = w_1,$$

$$\{s, z\} = 2q - \tfrac{1}{2}p^2 - \frac{dp}{dz} = 2I,$$

say. As all integrals are to be algebraic, it follows that s and $s'^{-\frac{1}{2}}$ are algebraic; *accordingly, $\int p\,dz$ must be the logarithm of an algebraic function*, which is a first condition. Further, in the equations under consideration, both p and q (and therefore also $2I$) are rational functions of z; and therefore

$$\{s, z\} = \text{rational function of } z,$$

* They are slightly changed from the forms in § 302, § 278 (*l.c.*); the change is made, so as to associate the indices ν_2, ν_1, ν_3 with the values $Z=0$, $Z=1$, $Z=\infty$ respectively.

† *Math. Ann.*, t. XII (1877), pp. 23—46. See also Cayley's memoir, "On the Schwarzian derivative and the polyhedral functions," *Coll. Math. Papers*, t. XI, pp. 148—216.

§ See my *Treatise on Differential Equations*, §§ 61, 62.

and the quantity s is subject to the transformation of the finite group. Now we have seen that

$$\{s, Z\} = \frac{\frac{1}{2}\left(1 - \frac{1}{\nu_2^2}\right)}{Z^2} + \frac{\frac{1}{2}\left(1 - \frac{1}{\nu_1^2}\right)}{(Z-1)^2} + \frac{\frac{1}{2}\left(\frac{1}{\nu_1^2} + \frac{1}{\nu_2^2} - \frac{1}{\nu_3^2} - 1\right)}{Z(Z-1)},$$

in cases II, III, IV, V; and for case I, it is easy to verify directly that

$$\{s, Z\} = \tfrac{1}{2}\,\frac{1 - \frac{1}{N^2}}{Z^2}.$$

From the properties of the Schwarzian derivative, we have

$$\{s, z\} = \{s, Z\}\left(\frac{dZ}{dz}\right)^2 + \{Z, z\};$$

hence, taking account of the particular form of $\{s, Z\}$ which is actually known, and of the generic form of $\{s, z\}$ which is required, we see that, in order to satisfy the conditions, we must have

$$Z = R(z),$$

where R is a rational function of z. Conversely, the conditions will be satisfied if Z is any rational function of z. Accordingly, *the differential equation of the second order must have the coefficient of w' in the form*

$$\frac{1}{u}\frac{du}{dz},$$

where u is an algebraic function of z; and its invariant $I(z)$, which is $q - \frac{1}{4}p^2 - \frac{1}{2}\frac{dp}{dz}$, must be of the form

$$\tfrac{1}{4}\left[\frac{1 - \frac{1}{\nu_2^2}}{Z^2} + \frac{1 - \frac{1}{\nu_1^2}}{(Z-1)^2} + \frac{\frac{1}{\nu_1^2} + \frac{1}{\nu_2^2} - \frac{1}{\nu_3^2} - 1}{Z(Z-1)}\right]\left(\frac{dZ}{dz}\right)^2 + \tfrac{1}{2}\{Z, z\},$$

or

$$\tfrac{1}{4}\,\frac{1 - \frac{1}{N^2}}{Z^2}\left(\frac{dZ}{dz}\right)^2 + \tfrac{1}{2}\{Z, z\},$$

where Z is any rational function of z; the integers ν_1, ν_2, ν_3 in the first form are the integers of the finite groups in cases II, III, IV,

V; and N in the second form is an integer. When these conditions are satisfied, the integrals are given by

$$w_1 = s'^{-\frac{1}{2}} u^{-\frac{1}{2}} s, \quad w_2 = s'^{-\frac{1}{2}} u^{-\frac{1}{2}},$$

where, for the first form, s is determined in terms of Z, the rational function of z, by the equations at the end of § 60; and for the second form,

$$s^N = Z.$$

Construction of an Integral, when it is Algebraic.

62. The preceding investigation is adequate for the general construction of linear equations of the second order which are integrable algebraically; there still remains the question of determining whether any particular given equation satisfies the test.

When the equation is of the form

$$\frac{d^2w}{dz^2} + p\frac{dw}{dz} + qw = 0,$$

inspection of the form of p at once determines whether it satisfies the condition which governs it specially. Assuming this condition to be satisfied, we construct the invariant $I(z)$ of the equation, where

$$I(z) = q - \tfrac{1}{4}p^2 - \tfrac{1}{2}\frac{dp}{dz};$$

and then, if the original equation is algebraically integrable, we must also have

$$I(z) = \tfrac{1}{4}\left[\frac{1-\dfrac{1}{\nu_2^2}}{Z^2} + \frac{1-\dfrac{1}{\nu_1^2}}{(Z-1)^2} + \frac{\dfrac{1}{\nu_1^2}+\dfrac{1}{\nu_2^2}-\dfrac{1}{\nu_3^2}-1}{Z(Z-1)}\right]\left(\frac{dZ}{dz}\right)^2 + \tfrac{1}{2}\{Z, z\},$$

or else

$$I(z) = \tfrac{1}{4}\frac{1-\dfrac{1}{N^2}}{Z^2}\left(\frac{dZ}{dz}\right)^2 + \tfrac{1}{2}\{Z, z\},$$

where Z is a rational function of z, and the integers ν_1, ν_2, ν_3 belong to one of four definite systems.

It may happen that the identification is easy, because Z has some simple value; the simplest of all is, of course, given by

$Z = z$. When the identification is not thus obvious, it is desirable to have a method of constructing the rational function Z if it exists; when it has been constructed, the further identification is only a matter of comparing coefficients. Should this identification be completely effected, then the integration of the equation is given by the results of § 60.

Such a method is given by Klein*, who uses for the purpose a comparison of those terms on the two sides, which are connected with the poles and have the highest negative index. A rational function is determinate save as to a constant factor, when its zeros, its poles in the finite part of the plane, and their respective multiplicities, all are known; and this constant factor is determinate, when the value of the rational function is known for any other value of the variable. Accordingly, let a denote a zero of Z of multiplicity α, and so for all the zeros; let c denote a pole of Z (and therefore also of $Z-1$) of multiplicity γ, and so for all the poles; and let b denote a zero of $Z-1$ of multiplicity β, and so for all its zeros: then

$$Z = \frac{\Pi (z-a)^\alpha}{\Pi (b-a)^\alpha} \cdot \frac{\Pi (b-c)^\gamma}{\Pi (z-c)^\gamma},$$

where the multiplicity β of b is not used directly in the expression.

Consider now the right-hand side of the expression for $I(z)$. In the vicinity of a, we have

$$Z = (z-a)^\alpha U,$$

where U is a regular function of $z-a$, not vanishing when $z=a$; so that

$$\frac{1}{Z}\frac{dZ}{dz} = \frac{\alpha}{z-a} + R(z-a),$$

and

$$\{Z, z\} = \tfrac{1}{2}\frac{1-\alpha^2}{(z-a)^2} + \dots,$$

the unexpressed terms in $\{Z, z\}$ having exponents greater than -2.

In the vicinity of c, we have

$$Z = (z-c)^{-\gamma} V, \quad Z - 1 = (z-c)^{-\gamma} V_1,$$

* *Math. Ann.*, t. XII (1877), pp. 173—176: the exposition given in the text does not follow his exactly, as he transforms the equation so as to secure that $z=\infty$ is an ordinary point.

where V and V_1 are regular functions of $z-c$, not vanishing when $z=c$; thus

$$\frac{1}{Z}\frac{dZ}{dz}=\frac{-\gamma}{z-c}+S(z-c),$$

$$\frac{1}{Z-1}\frac{dZ}{dz}=\frac{-\gamma}{z-c}+S_1(z-c),$$

$$\{Z, z\}=\tfrac{1}{2}\frac{1-\gamma^2}{(z-c)^2}+\dots,$$

the unexpressed terms in $\{Z, z\}$ having exponents greater than -2.

In the vicinity of b, we have

$$Z-1=(z-b)^{\beta}W,$$

where W is a regular function of $z-b$, not vanishing when $z=b$; so that

$$\frac{1}{Z-1}\frac{dZ}{dz}=\frac{\beta}{z-b}+T(z-b),$$

$$\{Z, z\}=\tfrac{1}{2}\frac{1-\beta^2}{(z-b)^2}+\dots,$$

the unexpressed terms in $\{Z, z\}$ having exponents greater than -2.

We thus have taken account of all the highest terms with negative indices which arise through zeros or poles of Z and $Z-1$. On account of the form of $\{Z, z\}$, which is

$$\frac{Z'''}{Z'}-\tfrac{3}{2}\left(\frac{Z''}{Z'}\right)^2,$$

it is necessary to take account of the poles and the zeros of Z'. As Z is rational, all its poles are poles of Z' and the latter has no others; so that, on this score, no new terms arise. A repeated zero of Z is a zero of Z', and all these have been taken into account; likewise for a repeated zero of $Z-1$. Hence we need only consider those roots of Z', which are not repeated roots of Z or of $Z-1$; let such an one be t, of multiplicity τ, so that

$$Z'=(z-t)^{\tau}Q(z-t),$$

where Q is a regular function of $z-t$, not vanishing when $z=t$; then

$$\{Z, z\}=-\frac{\tau+\frac{1}{2}\tau^2}{(z-t)^2}+\dots,$$

the unexpressed terms in $\{Z, z\}$ having exponents greater than -2.

Gathering together the terms with the largest negative index, we have, for Cases II, III, IV, V,

$$I(z) = \Sigma \frac{\frac{1}{4}\left(1 - \frac{\alpha^2}{\nu_2^2}\right)}{(z-a)^2} + \Sigma \frac{\frac{1}{4}\left(1 - \frac{\beta^2}{\nu_1^2}\right)}{(z-b)^2} + \Sigma \frac{\frac{1}{4}\left(1 - \frac{\gamma^2}{\nu_3^2}\right)}{(z-c)^2} - \Sigma \frac{\frac{1}{2}\tau + \frac{1}{4}\tau^2}{(z-t)^2} + \ldots,$$

where the unexpressed terms have integer exponents greater than -2; and in this expression the significance of a, b, c, for the construction of Z, must be borne in mind. Actual comparison with the form of $I(z)$ then gives indications as to which set of values of ν_1, ν_2, ν_3 must be chosen, and determines the values of α, β, γ. The construction of Z is then possible and, Z being known, the complete identification of the right-hand side with the known value of $I(z)$ is merely a matter of numerical calculation.

For Case I, we have

$$I(z) = \Sigma \frac{\frac{1}{4}\left(1 - \frac{\alpha^2}{N^2}\right)}{(z-a)^2} + \Sigma \frac{\frac{1}{4}\left(1 - \frac{\gamma^2}{N^2}\right)}{(z-c)^2} - \Sigma \frac{\frac{1}{2}\tau + \frac{1}{4}\tau^2}{(z-t)^2};$$

and the method of proceeding is the same as before.

In particular instances, it may happen that no terms of the type

$$-\frac{\frac{1}{2}\tau + \frac{1}{4}\tau^2}{(z-t)^2}$$

occur: Z' then contains no roots other than the repeated roots of Z and $Z-1$. An example is given by

$$Z = -\frac{(z-1)^2}{4z}.$$

Further, it may happen that $\alpha = \nu_2$, or $\beta = \nu_1$, or $\gamma = \nu_3$: so that the corresponding value of z, viz. a, b, or c, is then not a singularity of the differential equation. And, in particular, if $z = \infty$ is not a singularity of the differential equation and therefore also not a singularity of the integral, then, if the equation be integrable algebraically, the numerator of the rational function Z is a polynomial in z of the same degree as the denominator*.

* This form of equation is discussed by Klein in the memoir already quoted (note, p. 185): reference should be made to it for further developments.

Ex. 1. The equation

$$\frac{d^2w}{dz^2}+\tfrac{3}{16}\frac{z^2-z+1}{z^2(z-1)^2}w=0$$

is integrable algebraically. For

$$I(z)=\tfrac{3}{16}\frac{z^2-z+1}{z^2(z-1)^2}$$

$$=\frac{\frac{3}{16}}{(z-1)^2}+\frac{\frac{3}{16}}{z^2}-\frac{\frac{3}{16}}{z(z-1)},$$

so that

$$Z=z;$$

$$\tfrac{3}{16}=\tfrac{1}{4}\left(1-\frac{1}{\nu_2{}^2}\right),\text{ whence }\nu_2=2;$$

$$\tfrac{3}{16}=\tfrac{1}{4}\left(1-\frac{1}{\nu_1{}^2}\right),\text{ whence }\nu_1=2;$$

$$-\tfrac{3}{16}=\tfrac{1}{4}\left(\frac{1}{\nu_1{}^2}+\frac{1}{\nu_2{}^2}-\frac{1}{\nu_3{}^2}-1\right),\text{ whence }\nu_3=2.$$

We thus have an instance of case II, when $n=2$. All the conditions are satisfied: and thus (§ 60) the integrals of the equation are given by

$$\frac{(s^2-1)^2}{-4s^2}=z,$$

$$w_1=s\left(\frac{ds}{dz}\right)^{-\frac{1}{2}},\qquad w_2=\left(\frac{ds}{dz}\right)^{-\frac{1}{2}}.$$

Ex. 2. Construct a linear differential equation of the second order in its normal form, such that the quotient s of two of its solutions is given by

$$\frac{(s^8+14s^4+1)^3}{108s^4(s^4-1)^2}=-\frac{(z-1)^2}{4z}.$$

Ex. 3. Consider the equation

$$\frac{1}{w}\frac{d^2w}{dz^2}+\frac{2z^4-8z^3-15z^2-8z+2}{9z^2(z^2-1)^2}+\tfrac{3}{8}\frac{(z-1)^2}{z(z^2+1)^2}=0.$$

We have

$$I(z)=\frac{2z^4-8z^3-15z^2-8z+2}{9z^2(z^2-1)^2}+\tfrac{3}{8}\frac{(z-1)^2}{z(z^2+1)^2}$$

$$=\frac{\frac{2}{9}}{z^2}+\frac{\frac{5}{36}}{(z+1)^2}-\frac{\frac{3}{4}}{(z-1)^2}+\frac{\frac{3}{16}}{(z-i)^2}+\frac{\frac{3}{16}}{(z+i)^2}+\ldots,$$

the terms indicated constituting all the infinities of $I(z)$ of the second order.

First, it is clear that there is only one root of Z' other than repeated roots of Z and $Z-1$; it is characterised by

$$t=1,\quad \tau=1.$$

As regards the remaining terms, the numbers ν_1, ν_2, ν_3 must be 2 or 3; so that we either have an instance of case II with $n=3$, or we have an instance of case III.

If it were possibly an instance of case II with $n=3$, then we must have

$$\tfrac{1}{4}\left(1-\frac{\beta^2}{\nu_1^2}\right)=\tfrac{3}{16}, \text{ so that } \nu_1=2,\ \beta=1,\ b=i,$$

$$\tfrac{1}{4}\left(1-\frac{\alpha^2}{\nu_2^2}\right)=\tfrac{3}{16}, \ \ldots\ldots\ldots\ \nu_2=2,\ \alpha=1,\ a=-i,$$

$$\tfrac{1}{4}\left(1-\frac{\gamma^2}{\nu_3^2}\right)=\tfrac{2}{9}, \ \ldots\ldots\ldots\ \nu_3=3,\ \gamma=1,\ c=0,$$

$$\tfrac{1}{4}\left(1-\frac{\gamma'^2}{\nu_3^2}\right)=\tfrac{5}{36}, \ \ldots\ldots\ldots\ \nu_3=3,\ \gamma'=2,\ c=-1;$$

and therefore

$$Z=A\frac{z+i}{z(z+1)^2},$$

with the condition that $Z=1$ when $z=b=i$, so that $A=i$. But then

$$Z'=-\frac{i}{z^2(z+1)^3}(2z^2+3iz+i),$$

shewing that Z' does not possess a root $z=t=1$; hence the example is not an instance of case II.

If therefore the equation is algebraically integrable, it must be an instance of case III. We must have therefore

$$\nu_1=2,\quad \nu_2=3,\quad \nu_3=3;$$

so that

$$\tfrac{1}{4}\left(1-\frac{\beta^2}{\nu_1^2}\right)=\tfrac{3}{16}, \text{ whence } \beta=1,\ b=i,$$

$$\tfrac{1}{4}\left(1-\frac{\beta'^2}{\nu_1^2}\right)=\tfrac{3}{16}, \ \ldots\ldots\ldots\ \beta'=1,\ b'=-i;$$

and then, either

$$\tfrac{1}{4}\left(1-\frac{\alpha^2}{\nu_2^2}\right)=\tfrac{2}{9},\quad \tfrac{1}{4}\left(1-\frac{\gamma^2}{\nu_3^2}\right)=\tfrac{5}{36},$$

giving

$$\alpha=1,\quad a=0,\quad \gamma=2,\quad c=-1;$$

or else

$$\tfrac{1}{4}\left(1-\frac{\alpha^2}{\nu_2^2}\right)=\tfrac{5}{36},\quad \tfrac{1}{4}\left(1-\frac{\gamma^2}{\nu_3^2}\right)=\tfrac{2}{9},$$

giving

$$\alpha=2,\quad a=-1,\quad \gamma=1,\quad c=0.$$

Taking the former, we have

$$Z=A\frac{z}{(z+1)^2},$$

from the poles and zeros of Z; as $Z=1$, when $z=i$, we have $A=2$, so that

$$Z=\frac{2z}{(z+1)^2},\quad Z-1=-\frac{z^2+1}{(z+1)^2},$$

so that $Z-1$ has the roots $z=i$, $z=-i$; but

$$Z'=\frac{2}{(z+1)^3},$$

shewing that Z' does not possess the root 1; and thus the first assignment of values is not possible.

Taking the latter, we have

$$Z = A\frac{(z+1)^2}{z},$$

from the poles and zeros of Z; as $Z=1$ when $z=i$, we have $A=\frac{1}{2}$, and then

$$Z=\frac{(z+1)^2}{2z}, \quad Z-1=\frac{z^2+1}{2z},$$

$$Z'=\frac{z^2-1}{2z^2},$$

so that $Z-1$ has $z=i$, $z=-i$ for roots, and Z' has $z=1$ for a root.

The preliminary conditions are thus satisfied; it is easy to verify that this value of Z gives the complete value of $I(z)$. Hence, after the results of § 60, the integral of the differential equation is given by the equations

$$\left(\frac{s^4+2s^2\sqrt{3}-1}{s^4-2s^2\sqrt{3}-1}\right)^3=\frac{(z+1)^2}{2z},$$

$$w_1=s\left(\frac{ds}{dz}\right)^{-\frac{1}{2}}, \quad w_2=\left(\frac{ds}{dz}\right)^{-\frac{1}{2}}:$$

so that the differential equation is algebraically integrable.

Ex. 4. Shew that the equations

$$z(1-z)\frac{d^2w}{dz^2}+(\tfrac{2}{3}-\tfrac{7}{6}z)\frac{dw}{dz}+\tfrac{1}{48}w=0,$$

$$z(1-z)\frac{d^2w}{dz^2}+(\tfrac{2}{3}-z)\frac{dw}{dz}+\tfrac{1}{36}w=0,$$

$$z(1-z)\frac{d^2w}{dz^2}+(\tfrac{5}{4}-\tfrac{19}{12}z)\frac{dw}{dz}-\tfrac{5}{72}w=0,$$

are integrable algebraically: and obtain their integrals.

Ex. 5. Taking the equation, which has three singularities in the finite part of the plane and for which infinity is an ordinary point, in the form given in § 49, so that, by § 53,

$$I(z)=\frac{1}{\psi}\sum_{r=1}^{3}\frac{\frac{1}{4}(1-\lambda_r^2)}{(z-a_r)^2}\psi'(a_r),$$

where $\psi=(z-a_1)(z-a_2)(z-a_3)$, and $\lambda_1=\frac{1}{2}(a-a')$, $\lambda_2=\frac{1}{2}(\beta-\beta')$, $\lambda_3=\frac{1}{2}(\gamma-\gamma')$; discuss the possibilities of algebraic integrability for the values

$$\lambda_1=\tfrac{1}{3}, \quad \lambda_2=\tfrac{2}{5}, \quad \lambda_3=\tfrac{1}{2}.$$

In particular, shew that, if $a_2=-1$, $a_3=0$, then

$$a_1=-\tfrac{189}{64}. \qquad \text{(Klein.)}$$

EQUATIONS OF THE THIRD ORDER WITH ALGEBRAIC INTEGRALS.

63. When we pass to the consideration of linear equations of order higher than the second which are algebraically integrable, the discussion can be initiated in the same way as for equations of the second order; but the detailed development proves to be exceedingly laborious, and it has not been fully completed for each case. Only a sketch will here be given.

Dealing in particular with the linear equation of the third order, we take it in the form

$$w''' + 3pw'' + 3qw' + rw = 0,$$

where p, q, r are rational functions of z, subject to the limitations imposed by the regularity of the integrals in the vicinity of each singularity (∞ included). If w_1, w_2, w_3 denote three linearly independent integrals, we have (§ 9)

$$\begin{vmatrix} w_1'', & w_2'', & w_3'' \\ w_1', & w_2', & w_3' \\ w_1\;, & w_2\;, & w_3 \end{vmatrix} = Ae^{-3\int p\,dz},$$

so that, as w_1, w_2, w_3 are algebraic functions of z, it follows that p, a rational function of z, must be of the form

$$p = \frac{1}{u}\frac{du}{dz},$$

where u is an algebraic function of z. This is a first condition: it is the same as for the equation of the second order (§ 61): and it is easily obtained as a universal condition attaching to any linear equation which is algebraically integrable.

Now substitute for w by the relation

$$we^{\int p\,dz} = y,$$

and let y_1, y_2, y_3 denote the three integrals corresponding to w_1, w_2, w_3; owing to the character of p and the functional character of the integrals w, the integrals y are also algebraic functions of z. Thus the equation in y, being

$$y''' + 3Qy' + R = 0,$$

where

$$\left.\begin{aligned} Q &= q - p^2 - p' \\ R &= r - 3pq + 2p^3 - p'' \end{aligned}\right\},$$

is to be algebraically integrable. Denoting by s and t the quotients of two integrals by a third, we have

$$s = \frac{w_2}{w_1} = \frac{y_2}{y_1}, \quad t = \frac{w_3}{w_1} = \frac{y_3}{y_1}.$$

The quantities s and t are algebraic functions of z for equations of the class under consideration.

The effect upon a fundamental system, when the independent variable describes a circuit enclosing one or more of the singularities, is represented by relations of the form

$$\left.\begin{aligned} Y_1 &= a\ y_1 + b\ y_2 + c\ y_3 \\ Y_2 &= a' y_1 + b' y_2 + c' y_3 \\ Y_3 &= a'' y_1 + b'' y_2 + c'' y_3 \end{aligned}\right\}.$$

If S and T denote the corresponding integral-quotients, then

$$S = \frac{a' + b's + c't}{a + bs + ct}, \quad T = \frac{a'' + b''s + c''t}{a + bs + ct}.$$

Now if the equation is integrable algebraically, there can exist only a limited number of different sets of values of the integrals; so that the number of sets Y_1, Y_2, Y_3 is finite, and the number of simultaneous values of S and T is finite. If then we know all the homogeneous linear groups in three variables, or (what is the same thing) all the lineo-linear groups in two variables, which are finite, then each such finite group determines its set of values of Y_1, Y_2, Y_3 and the set of values of S and T, and so it determines a linear equation the integrals of which are algebraic: and conversely, each such linear equation is characterised by a finite group.

64. In order to utilise the method for the present purpose on the lines adopted for the equation of the second order, it is necessary to deduce from the differential equation certain differential invariants involving s and t, these invariants being expressed in terms of Q and R. This can be done in two ways. It is clear that, as s implicitly contains five arbitrary constants, it satisfies a differential equation of order five; and that, as t is of the same functional form as s, it satisfies the same differential equation.

On the other hand, as s and t combined contain eight arbitrary constants implicitly, it may be expected that the two differential equations, which they satisfy and which will involve both of them, will be each of the fourth order or will be equivalent to two of the fourth order. The single equation is, for some purposes, the more important in the formal theory of the linear equation, which will be left undiscussed; for the present purpose, the two equations prove to be the more important. Accordingly, we substitute

$$sy_1 \text{ for } y_2, \quad \text{and} \quad ty_1 \text{ for } y_3,$$

in turn in the equation

$$y''' + 3Qy' + Ry = 0;$$

whence, remembering that y_1 is an integral of this equation, we have

$$\left.\begin{aligned} 3s'y_1'' + 3s''y_1' + (3Qs' + s''')\,y_1 = 0 \\ 3t'y_1'' + 3t''y_1' + (3Qt' + t''')\,y_1 = 0 \end{aligned}\right\}.$$

Differentiating each of these once, and substituting for y_1''' from the linear equation which it satisfies, we have

$$\left.\begin{aligned} 6s''y_1'' + (4s''' - 6Qs')\,y_1' + \{s'''' + 3Qs'' + 3\,(Q' - R)\,s'\}\,y_1 = 0 \\ 6t''y_1'' + (4t''' - 6Qt')\,y_1' + \{t'''' + 3Qt'' + 3\,(Q' - R)\,t'\}\,y_1 = 0 \end{aligned}\right\};$$

so that there are four equations, linear and homogeneous in the quantities y_1'', y_1', y_1. When the ratios of $y_1'' : y_1' : y_1$ are eliminated from the first pair and the first of the second pair, we have

$$\begin{vmatrix} s'''', & 4s''', & 6s'' \\ s''', & 3s'', & 3s' \\ t''', & 3t'', & 3t' \end{vmatrix} - 3Q \begin{vmatrix} s'', & 2s', & 0 \\ s''', & 3s'', & 3s' \\ t''', & 3t'', & 3t' \end{vmatrix} - 3\,(R - Q') \begin{vmatrix} s', & 0, & 0 \\ s''', & 3s'', & 3s' \\ t''', & 3t'', & 3t' \end{vmatrix} = 0;$$

and when the same ratios are likewise eliminated from the first pair and the second of the second pair, we have

$$\begin{vmatrix} t'''', & 4t''', & 6t'' \\ s''', & 3s'', & 3s' \\ t''', & 3t'', & 3t' \end{vmatrix} - 3Q \begin{vmatrix} t'', & 2t', & 0 \\ s''', & 3s'', & 3s' \\ t''', & 3t'', & 3t' \end{vmatrix} - 3\,(R - Q') \begin{vmatrix} t', & 0, & 0 \\ s''', & 3s'', & 3s' \\ t''', & 3t'', & 3t' \end{vmatrix} = 0.$$

These, in fact, are the two equations, each of the fourth order, satisfied by s and t.

Suppose now that two solutions (other than the trivial solutions, $s =$ constant, $t =$ constant) are known, say

$$s = \sigma, \quad t = \tau.$$

Solving the first pair of the foregoing equations for $y_1' : y_1$, we have

$$3(\sigma''\tau' - \sigma'\tau'')\, y_1' + (\sigma'''\tau' - \sigma'\tau''')\, y_1 = 0,$$

and therefore

$$y_1 = (\sigma''\tau' - \sigma'\tau'')^{-\frac{1}{3}},$$

neglecting an arbitrary constant arising as a factor on the right-hand side. Hence a fundamental system of integrals of the original equation is

$$(\sigma''\tau' - \sigma'\tau'')^{-\frac{1}{3}}, \quad \sigma(\sigma''\tau' - \sigma'\tau'')^{-\frac{1}{3}}, \quad \tau(\sigma''\tau' - \sigma'\tau'')^{-\frac{1}{3}};$$

or the original equation can be integrated if two particular solutions of the equations in s and t are known.

65. Moreover, from the source of the two equations which serve to determine s and t, it is to be expected that, when the above two (being any two) particular solutions $s = \sigma$, $t = \tau$, are known, the complete primitive of the two equations is

$$s = \frac{a' + b'\sigma + c'\tau}{a + b\sigma + c\tau}, \quad t = \frac{a'' + b''\sigma + c''\tau}{a + b\sigma + c\tau},$$

where the constants $a, b, c, a', b', c', a'', b'', c''$ are arbitrary so far as those two equations are concerned. This result can be stated in a different form. The two equations in question can be written

$$As'''' + 4Bs''' + 6Cs'' - 3Q(As'' + 2Bs') - 3(R - Q')As' = 0,$$
$$At'''' + 4Bt''' + 6Ct'' - 3Q(At'' + 2Bt') - 3(R - Q')At' = 0,$$

where A, B, C are the three determinants in

$$\left\| \begin{matrix} s''', & 3s'', & 3s' \\ t''', & 3t'', & 3t' \end{matrix} \right\|.$$

Now let

$$\begin{aligned} u_1 &= s''\ t' - s't'' , & & \\ u_2 &= s'''\ t' - s't''' , & & \\ u_3 &= s''''t' - s't'''' , & v_2 &= s'''t'' - s''t''', \\ u_4 &= s'''''t' - s't''''' , & v_3 &= s''''t'' - s''t'''', \end{aligned}$$

so that

$$A = 9u_1, \quad B = -3u_2, \quad C = 3v_2;$$

then solving the preceding equations for Q and for $R - Q'$ in turn, we find

$$\left.\begin{aligned} 3Q &= \frac{u_3 + 2v_2}{u_1} - \tfrac{4}{3}\left(\frac{u_2}{u_1}\right)^2 = I(s, t, z) \\ \text{and} \qquad & \\ -27(R - Q') &= 9\,\frac{v_3}{u_1} - 6\,\frac{u_2(u_3 + 4v_2)}{u_1^{\,2}} + 8\left(\frac{u_2}{u_1}\right)^3 = J(s, t, z) \end{aligned}\right\},$$

say. The latter equations may be regarded as the equivalent of the two equations, which have been solved; and therefore we may expect that

$$I\left(\frac{a' + b's + c't}{a + bs + ct}, \ \frac{a'' + b''s + c''t}{a + bs + ct}, \ z\right) = I(s, t, z),$$

$$J\left(\frac{a' + b's + c't}{a + bs + ct}, \ \frac{a'' + b''s + c''t}{a + bs + ct}, \ z\right) = J(s, t, z);$$

the actual verification, which is comparatively simple, is left as an exercise. Clearly these are generalisations of the property of the Schwarzian derivative, represented by

$$\left\{\frac{as + b}{cs + d}, z\right\} = \{s, z\}.$$

The two invariant functions I and J were first indicated* by Painlevé; they subsequently were simplified to a form, which is the equivalent of the above, by Boulanger†.

The invariance of the functions I and J, as indicated, exists for lineo-linear transformation of s and t. There is also an invariance for any transformation of the independent variable z; for we easily find the equations

$$I(s, t, z) = I(s, t, Z)\,Z'^2 + 2\,\{Z, z\},$$

$$J(s, t, z) = J(s, t, Z)\,Z'^3 - 9\,I(s, t, Z)\,Z'Z'' - 9\,\frac{d}{dz}\{Z, z\},$$

where Z is any function of z. Also

$$\begin{aligned} I'(s, t, z) &= \frac{d}{dz}\{I(s, t, z)\} \\ &= I'(s, t, Z)\,Z'^3 + 2I(s, t, Z)\,Z'Z'' + 2\,\frac{d}{dz}\{Z, z\}, \end{aligned}$$

* *Comptes Rendus*, t. CIV (1887), p. 1830.

† See his Thèse, *Contribution à l'étude des équations différentielles linéaires et homogènes intégrables algébriquement*, (Paris, Gauthier-Villars, 1897).

and therefore

$$J(s, t, z) + \tfrac{9}{2} I'(s, t, z) = [J(s, t, Z) + \tfrac{9}{2} I'(s, t, Z)] Z'^3,$$

or

$$J(s, t, z) + \tfrac{9}{2} I'(s, t, z)$$

is an invariant for any change of the independent variable z. Dropping a numerical constant, this is the function

$$R - \tfrac{3}{2} \frac{dQ}{dz},$$

which is the known Laguerre invariant in the formal theory; that is*, if the equation

$$y''' + 3Qy' + Ry = 0$$

be transformed, by the relation

$$y = Y \left(\frac{dZ}{dz}\right)^{-1},$$

to the form

$$\frac{d^3 Y}{dZ^3} + 3Q_1 \frac{dY}{dZ} + R_1 Y = 0,$$

then

$$R - \tfrac{3}{2} \frac{dQ}{dz} = \left(R_1 - \tfrac{3}{2} \frac{dQ_1}{dZ}\right) \left(\frac{dZ}{dz}\right)^3.$$

As the transformation

$$y = Y \left(\frac{dZ}{dz}\right)^{-1}$$

leaves the quotient of two integrals transformed only as by a lineo-linear substitution, it follows that the preceding function, say

$$L(s, t, z) = J(s, t, z) + \tfrac{9}{2} I'(s, t, z),$$

is unchanged by lineo-linear transformations effected on s, t; also, except as to a factor Z'^3, it is unchanged by transformation effected on the independent variable. Now

$$u_3' = u_4 + v_3, \quad v_2' = v_3,$$

$$u_2' = u_3 + v_2, \quad u_1' = u_2,$$

so that we have

$$L(s, t, z) = \frac{9u_4 + 45v_3}{2u_1} - \frac{45u_2}{2u_1^2}(u_3 + 2v_2) + 20\left(\frac{u_2}{u_1}\right)^3,$$

* See a paper by the author, *Phil. Trans.* (1888), pp. 383, 390. Laguerre's invariant was first announced in two notes, *Comptes Rendus*, t. LXXXVIII (1879), pp. 116—119, 224—227.

which is the full expression of Laguerre's invariant in terms of the derivatives of s and t.

66. The next stage is to associate these invariants with the algebraic equations in two variables, which admit of one or other of the finite groups. These groups have been obtained by Jordan* and Valentiner†; and references to other writers are given by Boulanger‡. A method of using the results is outlined by Painlevé§ as follows.

Let $\phi(s, t)$, $\psi(s, t)$ denote two irreducible invariant functions of a finite group of order N; the functions are given by Klein|| for the group of order 168, and by Boulanger (*l.c.*) for the group of order 216. As these functions are invariable for each substitution of the group, and as s, t are algebraic functions of z, it follows that ϕ and ψ are rational functions of z, say

$$\phi(s, t) = \Phi(z), \quad \psi(s, t) = \Psi(z).$$

Conversely, taking Φ and Ψ to be arbitrary rational functions of z, these two equations give rise to N sets of simultaneous values of s and t as algebraic functions of z; and if any one set of values be represented by σ, τ, all the others are obtained on transforming σ and τ by all the $N-1$ substitutions of the group other than the identical substitution. These two equations are used to obtain the first four derivatives of s and t with regard to z; and with these derivatives, the two invariants

$$I(s, t, z), \quad J(s, t, z)$$

are constructed. The functions so formed involve derivatives of Φ and Ψ; and the coefficients of these quantities are rational in the derivatives of $\phi(s, t)$ and $\psi(s, t)$. As I and J are invariantive for the group, the coefficients specified are rational functions of s and t, which must be invariantive for the group and are therefore rationally expressible in terms of ϕ and ψ, that is, in terms of Φ

* *Crelle*, t. LXXXIV (1878), pp. 89—215; *Atti della R. Accad. di Napoli*, t. VIII (1879), No. 11.

† *Kjøb. Vidensk. Selsk. Skr.*, 6 R., t. V (1889), pp. 64—235.

‡ In the *Thèse*, already cited on p. 195, note.

§ *Comptes Rendus*, t. CIV (1887), pp. 1829—1832, *ib.* t. CV (1887), pp. 58—61.

|| *Math. Ann.*, t. XV (1879), pp. 265—267.

and Ψ. Thus $I(s, t, z)$ and $J(s, t, z)$ would be expressed as rational functions of z. Accordingly, taking

$$3Q = I(s, t, z),$$
$$R = \tfrac{1}{3} I'(s, t, z) - \tfrac{1}{27} J(s, t, z),$$

we have the differential equation

$$y''' + 3Qy' + Ry = 0.$$

The earlier investigations shewed that its integrals are expressible in terms of s, t, and their derivatives; and we thus have a method of constructing all the linear differential equations of the third order which are integrable algebraically. There is a double arbitrary element for each group, viz. the arbitrary forms of the rational functions Φ and Ψ; and there is a limited number of groups.

67. While this outline is simple enough in general description, the application to particular cases requires extremely elaborate calculations. These have been effected by Boulanger for the group of order 216; they do not appear to have been yet effected for any one of the other groups. As, however, the enumeration of the finite groups in two quantities s and t is complete, the subject offers an interesting, if a laborious, field of investigation.

In the absence of the complete table of equations, for all the finite groups and for two arbitrarily assumed functions Φ and Ψ, it is not possible to use a method, analogous to that of § 62, to determine whether a given equation of the third order is algebraically integrable or not: it is not even possible to recognise to which of the groups it would belong if it were algebraically integrable. Indications of two general methods of procedure have been given by Painlevé and have been developed to some extent by Boulanger; but the methods, while general in description, suffer from the same kind of difficulty as the method indicated for the construction of the equations, for the calculations are exceedingly laborious. We have seen that, if two particular values of s and t, say σ and τ, are known, then an integral of the differential equation is given by

$$y = (\sigma''\tau' - \sigma'\tau'')^{-\frac{1}{3}}.$$

Hence, if we take

$$u = \frac{y'}{y},$$

we have

$$u = -\tfrac{1}{3}\,\frac{\sigma'''\tau' - \sigma'\tau'''}{\sigma''\tau' - \sigma'\tau''},$$

so that the number of values, which u can acquire, is equal to N or to a submultiple of N, where N is the order of the associated group: let the number of values be n. Now if y is algebraic, every zero of y and every infinity of y are of a finite order, which is commensurable in every instance; and therefore all the infinities of u are simple poles with commensurable residues. Substituting for u in the equation

$$y''' + 3Qy' + Ry = 0,$$

we find

$$u'' + 3uu' + u^3 + 3Qu + R = 0,$$

a non-linear equation of the second order satisfied by u. This equation renders it possible to test the character of the poles and the residues of u. If these are of the appropriate type, then the equation is satisfied by a relation of the form

$$A_0 u^n + A_1 u^{n-1} + \ldots + A_{n-1} u + A_n = 0,$$

where $A_0, A_1, \ldots, A_n$ are polynomials in z, and A_0 is the product of the factors corresponding to the poles of u. Then there is the further test that this algebraic function u must be such that

$$e^{\int u dz}$$

is algebraic. Manifestly, the calculations will generally be too elaborate to make the method effective in practice.

Equations of the Fourth Order.

68. As pointed out* by Painlevé, the processes just indicated can formally be applied to linear equations of any order: but of course, if any advance towards final conditions is to be made, it is necessary to know all the finite lineo-linear groups of transformations in a number of variables less by one than the order of the

* *Comptes Rendus*, t. CV (1887), p. 59.

equation. Towards this enumeration of groups in three variables, which are associated with the linear equation of the fourth order, Jordan* has constructed a characteristic numerical equation which, when completely resolved, would indicate the order and the composition of each such group: but the resolution is exceedingly long and, owing to the number of cases that must be considered, it has not been completed. In these circumstances, no detailed results of a final critical character can be obtained for an equation of the fourth order or of any higher order: the only results obtainable are of a general character, and arise through the association of groups in general with linear equations.

The equation of the fourth order, which may be written

$$w'''' + 4pw''' + 6qw'' + 4rw' + sw = 0,$$

can be transformed by

$$we^{\int p\,dz} = y$$

into

$$y'''' + 6Qy'' + 4Ry' + Sy = 0.$$

We denote a system of four integrals by y_1, y_2, y_3, y_4, and we introduce three quotients s, t, u, such that

$$y_2 = y_1 s, \quad y_3 = y_1 t, \quad y_4 = y_1 u;$$

then s, t, u are simultaneous solutions of three equations of the fifth order in the derivatives. If σ, τ, υ are a special set of solutions, then

$$y_1 = \begin{vmatrix} \sigma''', & \sigma'', & \sigma' \\ \tau''', & \tau'', & \tau' \\ \upsilon''', & \upsilon'', & \upsilon' \end{vmatrix}^{-\frac{1}{4}},$$

and

$$y_2 = y_1\sigma, \quad y_3 = y_1\tau, \quad y_4 = y_1\upsilon.$$

The complete primitive of the three equations is of the form

$$\frac{s}{a' + b'\sigma + c'\tau + d'\upsilon} = \frac{t}{a'' + b''\sigma + c''\tau + d''\upsilon} = \frac{u}{a''' + b'''\sigma + c'''\tau + d'''\upsilon}$$

$$= \frac{1}{a + b\sigma + c\tau + d\upsilon}.$$

* *Atti della R. Accad. di Napoli*, t. VIII (1879), No. 11, p. 25; instead of dealing with lineo-linear transformations in three variables, Jordan deals with homogeneous linear substitutions in four variables.

There are three functions of the derivatives of s, t, u, with regard to z, which are invariantive for substitutions such as the preceding relations expressing s, t, u, in terms of σ, τ, v; and they are equal to

$$Q,\quad R-\frac{dQ}{dz},\quad S-\frac{dR}{dz}-3Q^2.$$

If the determinants

$$\Sigma\pm(s'''t''u'),\quad \Sigma\pm(s^{\text{iv}}t''u'),\quad \Sigma\pm(s^{\text{iv}}t'''u'),\quad \Sigma\pm(s^{\text{v}}t''u')$$

be denoted by ρ, ρ_1, ρ_2, ρ_3 respectively, then

$$Q=\frac{3\rho_3+5\rho_2}{12\rho}-\tfrac{5}{16}\left(\frac{\rho_1}{\rho}\right)^2=I_1(s, t, u, z),$$

say; if, in addition, the determinants

$$\Sigma\pm(s^{\text{iv}}t'''u''),\quad \Sigma\pm(s^{\text{v}}t'''u')$$

be denoted by ρ_4 and ρ_5 respectively, then

$$-\left(R-\frac{dQ}{dz}\right)=\frac{2\rho_5+5\rho_4}{12\rho}-\frac{\rho_1(6\rho_3+25\rho_2)}{72\rho^2}+\tfrac{5}{48}\left(\frac{\rho_1}{\rho}\right)^3=I_2(s, t, u, z),$$

say; and if the determinant $\Sigma\pm(s^{\text{v}}s'''s'')$ be denoted by ρ_6, then

$$\begin{aligned}S-\frac{dR}{dz}-3Q^2&=\frac{\rho_6}{4\rho}-\frac{1}{48\rho^2}(6\rho_1\rho_5+25\rho_1\rho_4+6\rho_2\rho_3+10\rho_2^2)\\&\qquad\qquad+\frac{3}{32\rho^3}\rho_1^2(\rho_3+5\rho_2)-\tfrac{15}{128}\left(\frac{\rho_1}{\rho}\right)^2\\&=I_3(s, t, u, z),\end{aligned}$$

say. The three quantities $I_1(s, t, u, z)$, $I_2(s, t, u, z)$, $I_3(s, t, u, z)$ are unchanged when lineo-linear substitutions are effected on s, t, u; and the combinations

$$\begin{aligned}&I_2-\tfrac{1}{2}I_1',\\&I_3+2I_2'-\tfrac{4}{5}I_1''-\tfrac{6}{25}I_1^2,\end{aligned}$$

are also unchanged, except as to a power of Z', when z is replaced by Z, any function of z.

The proofs of these various statements are left as exercises.

Equations, having Algebraic Integrals, associated with Homogeneous Forms.

69. It has already (§ 58) been stated that the discussion of the equations, which have algebraic integrals, has been associated with the theory of homogeneous forms: the association can be seen to occur as follows.

Using the preceding notation of §§ 63—66 for the quantities connected with any linear equation of the third order, we denote by s and t the quotients of any two by the third out of any three linearly independent integrals of the equation

$$\frac{d^3y}{dz^3} + 3Q\frac{dy}{dz} + Ry = 0.$$

If, then, all the integrals of this equation are algebraic, both s and t are algebraic functions of z; they may therefore be regarded as determined, in the most general case, by a couple of distinct algebraic equations, say

$$f_1(s, t, z) = 0, \quad f_2(s, t, z) = 0,$$

or by

$$g_1(s, z) = 0, \qquad g_2(t, z) = 0.$$

Eliminating z between the pair of equations in whichever form they are taken, we obtain a relation of the type

$$F_0(s, t) = 0,$$

where F_0 is a non-homogeneous polynomial in s and t, because it is the eliminant of two polynomials. Replacing s and t by $y_2 \div y_1$ and $y_3 \div y_1$ respectively, and multiplying by the proper power of y_1 to free the equation from fractions, we have

$$F(y_1, y_2, y_3) = 0,$$

where F is a homogeneous polynomial in its arguments or, in other phrase, is a ternary form in y_1, y_2, y_3.

Further, the above form of equation is obtained from

$$\frac{d^3w}{dz^3} + 3p\frac{d^2w}{dz^2} + 3q\frac{dw}{dz} + rw = 0,$$

by the transformation

$$we^{\int p\,dz} = y;$$

and therefore

$$F(w_1 e^{\int p\,dz}, w_2 e^{\int p\,dz}, w_3 e^{\int p\,dz}) = 0,$$

that is,

$$F(w_1, w_2, w_3) = 0,$$

on rejecting the factor $e^{m\int p\,dz}$, which occurs because F is a ternary form (say) of order m. Hence it follows that *when the integrals of a linear equation of the third order are algebraic functions, a homogeneous relation of finite order exists among any three linearly independent integrals.*

Moreover, when any other set of fundamental integrals Y_1, Y_2, Y_3 is taken, we know that

$$\left.\begin{aligned} y_1 &= a_1 Y_1 + a_2 Y_2 + a_3 Y_3 \\ y_2 &= b_1 Y_1 + b_2 Y_2 + b_3 Y_3 \\ y_3 &= c_1 Y_1 + c_2 Y_2 + c_3 Y_3 \end{aligned}\right\},$$

where the coefficients a, b, c are constants. The variables in the homogeneous ternary form are therefore subject to linear transformation; and thus the theory of ternariants can be associated with those homogeneous linear equations of the third order, which have their integrals algebraic. The various cases will arise according to the order of the form F; this order is always greater than unity, because the integrals considered are linearly independent.

If, still further, we choose to combine the geometry of the ternary form with the form in its association with the equation, then the preceding algebraic relation $F = 0$ is the equation of an algebraic plane curve referred to homogeneous coordinates: the curve is usually called the *integral curve.*

We may proceed similarly with an equation of the fourth order

$$\frac{d^4w}{dz^4} + 4p\frac{d^3w}{dz^3} + 6q\frac{d^2w}{dz^2} + 4r\frac{dw}{dz} + sw = 0,$$

when all its integrals are algebraic. If we choose, we may transform it by the relation

$$we^{\int p\,dz} = y;$$

the quantity $e^{\int p\,dz}$ must be algebraic, because

$$\begin{vmatrix} w_1, & w_2, & w_3, & w_4 \\ w_1', & w_2', & w_3', & w_4' \\ w_1'', & w_2'', & w_3'', & w_4'' \\ w_1''', & w_2''', & w_3''', & w_4''' \end{vmatrix} = Ce^{4\int p\,dz},$$

where C is a non-vanishing constant; and the equation in y, which is of the form

$$\frac{d^4y}{dz^4} + 6P_2\frac{d^2y}{dz^2} + 4P_3\frac{dy}{dz} + P_4 y = 0,$$

has all its integrals algebraic. Taking any four linearly independent solutions y_1, y_2, y_3, y_4, and writing

$$\rho y_1 = y_2, \quad \sigma y_1 = y_3, \quad \tau y_1 = y_4,$$

then as ρ, σ, τ are algebraic functions of z, they must be given by three equations of the form

$$f_1(\rho, \sigma, \tau, z) = 0, \quad f_2(\rho, \sigma, \tau, z) = 0, \quad f_3(\rho, \sigma, \tau, z) = 0,$$

or of simpler equivalent forms, which are completely algebraic in character. Eliminating z between the first and second, and also between the first and third, and taking the eliminants in a form free from irrational quantities if these occur, we have two equations

$$F_0(\rho, \sigma, \tau) = 0, \quad G_0(\rho, \sigma, \tau) = 0,$$

two non-homogeneous polynomials in ρ, σ, τ. Replacing these quantities by their values in terms of y_1, y_2, y_3, y_4, and multiplying each equation by the power of y_1, appropriate to free it from fractions, we find

$$\left.\begin{aligned} F(y_1, y_2, y_3, y_4) = 0 \\ G(y_1, y_2, y_3, y_4) = 0 \end{aligned}\right\},$$

where F and G are homogeneous polynomials in their arguments or, in other phrase, are quaternary forms in y_1, y_2, y_3, y_4. As in the case of the cubic, these equations imply the further equations

$$\left.\begin{aligned} F(w_1, w_2, w_3, w_4) = 0 \\ G(w_1, w_2, w_3, w_4) = 0 \end{aligned}\right\};$$

so that, *when the integrals of a homogeneous linear equation of the fourth order are algebraic functions, two homogeneous relations of finite order exist among any four linearly independent integrals.*

Again, when the variables y_1, y_2, y_3, y_4 are replaced by any other set of fundamental integrals Y_1, Y_2, Y_3, Y_4, the two sets of variables are connected by homogeneous linear relations: and thus the theory of quaternariants can be associated with those homogeneous linear equations of the fourth order which have their integrals algebraic. The various cases will arise according to the orders of the forms F and G; these orders are always greater than unity, because the integrals y_1, y_2, y_3, y_4 are linearly independent.

We may also combine the geometry of quaternary forms with the forms themselves as associated with the equation. In that case, each of the equations $F=0$, $G=0$ is the equation of a non-planar surface in three dimensions referred to homogeneous coordinates: the two equations combined determine a skew curve, which accordingly is the *integral curve*.

Similarly, in the case of equations of the fifth order, of which all the integrals are algebraic, we have three homogeneous non-linear relations among any fundamental set of integrals; and there are corresponding associations with the theory of homogeneous forms in five variables and the allied geometry. And so also for linear equations of higher orders.

Note 1. There cannot be two homogeneous relations among a set of three linearly independent integrals of an equation of the third order: for they would determine a limited number of sets of constant values for the ratios $y_1 : y_2 : y_3$, contrary to the postulate of linear independence.

Similarly, there cannot be three homogeneous relations among a set of four linearly independent integrals of an equation of the fourth order: for their existence would imply a corresponding contradiction of the same postulate. And so for other equations of higher orders.

It might however happen that, for an equation of the fourth order, only a single homogeneous relation exists among four linearly independent integrals; that, for an equation of the fifth order, the number of homogeneous relations among a fundamental set of integrals is less than three; and so on. If the relations thus given in each of the respective cases are the maximum number of homogeneous relations that can exist, we can infer that not all

the integrals of the respective equations are algebraic: and a question arises as to the significance of the respective relations.

Note 2. The converse of the general argument must not be assumed valid: that is to say, the existence of a homogeneous relation between the members of a fundamental system of integrals of an equation of the third order is not sufficient to ensure the property that all the integrals are algebraic. Thus we know that a number of transcendental functions of a variable can be connected by algebraic relations: and such instances are not the only possible exceptions.

70. The preceding method of associating the theory of forms with linear equations does not apply directly when the equation is of the second order: for a homogeneous relation between two integrals would imply one or other of a limited number of constant values for the ratio of the integrals, which accordingly could not be linearly independent. This deficiency, however, is rendered relatively unimportant, because Klein's method explained in §§ 59—62 for the equation of the second order gives the complete solution of the question propounded as to the cases when all its integrals are algebraic. The results there given can be (and have been) obtained by processes directly connected with the theory of binary forms. After the preceding exposition, the analysis is mainly of formal interest, and adds little to the knowledge of the solutions regarded as functions of the independent variable.

It will be sufficiently illustrated* by one or two examples.

Ex. 1. We take the differential equation in the form

$$\frac{d^2y}{dx^2} - Iy = 0,$$

and consider the value of a homogeneous polynomial function of two integrals y_1 and y_2, linearly independent of one another. Let this polynomial be of order n, and write

$$\begin{aligned} f(y_1, y_2) = (a_0, a_1, \ldots, a_n \between y_1, y_2)^n &= y_2^n (a_0, a_1, \ldots, a_n \between s, 1)^n \\ &= y_2^n u, \end{aligned}$$

* For fuller discussion and details, see Fuchs, *Crelle*, t. LXXXI (1876), pp. 97—142, *ib.*, t. LXXXV (1878), pp. 1—25; Brioschi, *Math. Ann.*, t. XI (1877), pp. 401—411; Forsyth, *Quart. Journ.*, t. XXIII (1889), pp. 45—78.

A memoir by Pepin, "Méthode pour obtenir les intégrales algébriques des équations différentielles linéaires du second ordre," *Rom. Acc. P. d. N. L.*, t. XXXIV (1882), pp. 243—389, may also be consulted with advantage.

say, where s is the quotient $y_1 \div y_2$. When substitution is made for y_1 and y_2 in terms of x, let the value of f be $\phi(x)$, so that

$$f(y_1, y_2) = \phi(x).$$

Now if $H(y_1, y_2) = H(f)$ be the Hessian of f, and if $H(u)$ be the Hessian of u, so that

$$H(f) = (a_0 a_2 - a_1^2, \ldots \between y_1, y_2)^{2n-4},$$
$$H(u) = (a_0 a_2 - a_1^2, \ldots \between s, 1)^{2n-4},$$

we have

$$H(f) = y_2^{2n-4} H(u),$$
$$H(f) = n^{-2}(n-1)^{-2}\left\{\frac{\partial^2 f}{\partial y_1^2}\frac{\partial^2 f}{\partial y_2^2} - \left(\frac{\partial^2 f}{\partial y_1 \partial y_2}\right)^2\right\},$$
$$H(u) = n^{-2}(n-1)^{-1}\left\{nu\frac{d^2 u}{ds^2} - (n-1)\left(\frac{du}{ds}\right)^2\right\}.$$

We have also

$$y_2 \frac{dy_1}{dx} - y_1 \frac{dy_2}{dx} = \text{constant} = C,$$

say, so that

$$y_2^2 \frac{ds}{dx} = C.$$

Now

$$y_2^n u = \phi(x);$$

hence

$$\frac{n}{y_2}\frac{dy_2}{dx} + \frac{C}{y_2^2}\frac{1}{u}\frac{du}{ds} = \frac{1}{\phi}\frac{d\phi}{dx}.$$

Differentiating, and substituting for the second derivative of y_2, we have

$$nI - \frac{n}{y_2^2}\left(\frac{dy_2}{dx}\right)^2 - \frac{2C}{y_2^3}\frac{dy_2}{dx}\frac{1}{u}\frac{du}{ds} + \frac{C^2}{y_2^4}\frac{d^2(\log u)}{ds^2} = \frac{d^2(\log\phi)}{dx^2}.$$

Multiply by n, and add the squares of the sides of the preceding equation : then

$$n^2 I + \frac{C^2}{y_2^4}\left\{n\frac{d^2(\log u)}{ds^2} + \frac{1}{u^2}\left(\frac{du}{ds}\right)^2\right\} = \frac{1}{\phi^2}\left(\frac{d\phi}{dx}\right)^2 + n\frac{d^2(\log\phi)}{dx^2}.$$

The coefficient of $C^2 y_2^{-4}$ on the left-hand side is

$$= \frac{n}{u}\frac{d^2 u}{ds^2} - (n-1)\frac{1}{u^2}\left(\frac{du}{ds}\right)^2$$
$$= \frac{n^2(n-1)}{u^2} H(u)$$
$$= y_2^4 n^2 (n-1)\frac{H(y_1, y_2)}{\phi^2};$$

so that

$$n^2(n-1)C^2 H(y_1, y_2) = \left\{\frac{n}{\phi}\frac{d^2\phi}{dx^2} - \frac{n-1}{\phi^2}\left(\frac{d\phi}{dx}\right)^2 - n^2 I\right\}\phi^2,$$

thus expressing the Hessian in terms of functions of x : let this be written

$$H(y_1, y_2) = \phi^2 \chi.$$

If now $\Phi(y_1, y_2)$ denote the cubicovariant of f, so that

$$\Phi(y_1, y_2) = \frac{1}{n(n-2)}\left\{\frac{\partial f}{\partial y_1}\frac{\partial H}{\partial y_2} - \frac{\partial f}{\partial y_2}\frac{\partial H}{\partial y_1}\right\}$$
$$= (a_0{}^2 a_3 - 3a_0a_1a_2 + 2a_1{}^3, \ldots \between y_1, y_2)^{3n-6},$$

then, proceeding in a similar way, we find

$$\Phi(y_1, y_2) = -\frac{\phi^3}{n(n-2)C}\left\{n\frac{d\chi}{dx} + 4\frac{\chi}{\phi}\frac{d\phi}{dx}\right\}.$$

And so for other covariants.

As a special case*, let it be required to find the value of ϕ, if when the binary form is the quadratic

$$a_0y_1{}^2 + 2a_1y_1y_2 + a_2y_2{}^2,$$

$\phi(x)$ is a root of some rational function of x. In this instance,

$$H(y_1, y_2) = a_0a_2 - a_1{}^2,$$

a constant; hence $\phi(x)$ is either a rational function, or is the square root of a rational function. The integration is immediate; for

$$y_2{}^2(a_0s^2 + 2a_1s + a_2) = \phi(x),$$
$$y_2{}^2\,ds \qquad\qquad = C\,dx,$$

whence

$$\frac{ds}{a_0s^2 + 2a_1s + a_2} = \frac{C\,dx}{\phi(x)}.$$

The value of s is thus known: and the consequent values of y_1 and y_2 are immediately given†.

Ex. 2. Shew that, if the integrals of the equation

$$\frac{d^2y}{dx^2} + Iy = 0$$

are such that

$$(a_0, a_1, a_2, a_3 \between y_1, y_2)^3 = \phi(x),$$

and ϕ is a root of some rational function of x, then ϕ^4 must be rational; and obtain the relation between I and $\phi(x)$.

Ex. 3. The integrals of the equation

$$\frac{d^2y}{dx^2} + Iy = 0$$

are such that

$$(a_0, a_1, a_2, a_3, a_4 \between y_1, y_2)^4 = \phi(x),$$

and $\phi(x)$ is a root of some rational function of x; shew that, unless $\phi(x)$ is actually rational, the quadrinvariant of the binary quartic must vanish. In either case, find the relation between I and $\phi(x)$. (Brioschi.)

* Fuchs, *Crelle*, t. LXXXI (1876), p. 116.

† See my *Treatise on Differential Equations*, § 62.

Ex. 4. Find the value of I in the equation

$$\frac{d^2y}{dx^2}+Iy=0,$$

when, in the relation

$$ay_1^n+by_2^n=\phi(x)$$

connecting two integrals, the function ϕ is supposed known.

Ex. 5. Shew that, if two integrals of the equation

$$\frac{d^2y}{dx^2}=P\frac{dy}{dx}+Qy$$

are connected by a relation

$$Ay_1^2+By_1y_2+Cy_2^2+D=0,$$

where A, B, C, D are constants, then

$$\frac{dQ}{dx}-2PQ=0.$$

Assuming the condition satisfied, integrate the equation. (Appell.)

Ex. 6. Two integrals of the equation

$$\frac{d^2y}{dx^2}=P\frac{dy}{dx}+Qy$$

are connected by a relation of the form

$$Ay_1^3+By_1^2y_2+Cy_1y_2^2+Dy_2^3+E=0,$$

where A, B, C, D, E are constants: prove that

$$\frac{d^2Q}{dx^2}-5P\frac{dQ}{dx}-2Q\frac{dP}{dx}-3Q^2+6P^2Q=0.$$

Shew that the quantity on the left-hand side of this conditional equation is invariantive for change of the independent variable; and hence, assuming the condition satisfied, shew that the equation can be transformed so as to become a particular case of Lamé's equation (Chap. IX). (Appell.)

Equations of the Third Order and Ternariants.

71. Returning now to the differential equation of the third order in the form

$$\frac{d^3y}{dz^3}+3Q\frac{dy}{dz}+Ry=0,$$

and supposing that all its integrals are algebraic, we proceed to consider the equation

$$F(y_1, y_2, y_3)=0,$$

where F is a homogeneous polynomial in any three linearly independent integrals. For this purpose, it will be convenient to have an equivalent simpler form of the equation which is given by a known transformation*, viz. we have

$$\frac{d^3u}{dt^3} + Iu = 0,$$

where

$$y\frac{dt}{dz} = u, \quad I\left(\frac{dt}{dz}\right)^3 = R - \tfrac{3}{2}\frac{dQ}{dz}, \quad \{t, z\} = \tfrac{3}{2}Q.$$

If we take

$$\frac{dt}{dz} = \theta^{-2},$$

the last of these relations may be replaced by the equation

$$\frac{d^2\theta}{dz^2} + \tfrac{3}{4}Q\theta = 0.$$

The equation among any three integrals is

$$F(u_1, u_2, u_3) = 0.$$

Consider the simplest case; it arises when $n = 2$, so that F is then a quadratic polynomial involving six terms. Writing

$$a_u = a_1u_1 + a_2u_2 + a_3u_3,$$

where a_1, a_2, a_3 are umbral symbols, the equation can be symbolically represented by

$$a_u^2 = 0.$$

We have

$$a_u a_{u'} = 0,$$

$$a_u a_{u''} + a_{u'}^2 = 0,$$

where u' is du/dt, and so for u''. Differentiating again, and replacing u''' by $-Iu$, we have

$$-Ia_u^2 + 3a_{u'}a_{u''} = 0,$$

that is,

$$a_{u'}a_{u''} = 0,$$

on using the original equation. Similarly, on differentiating this result,

$$-Ia_u a_{u'} + a_{u''}^2 = 0,$$

that is,

$$a_{u''}^2 = 0,$$

* See a paper by the author, *Phil. Trans.*, (1888), p. 441.

on using the first derivative of the original equation. Differentiating once more, we have

$$I a_u a_{u''} = 0,$$

so that either $I = 0$ or $a_u a_{u''} = 0$.

If I is not zero, then we must have

$$a_u a_{u''} = 0,$$

and therefore, by the second derivative of the original equation,

$$a_{u'}^2 = 0.$$

Hence, on the present hypothesis, we have

$$a_u^2 = 0, \quad a_u a_{u'} = 0, \quad a_{u'}^2 = 0, \quad a_u a_{u''} = 0, \quad a_{u'} a_{u''} = 0, \quad a_{u''}^2 = 0.$$

Now each of these equations is linear and homogeneous in the six real coefficients that occur in a_u^2; eliminating these coefficients, we obtain, as equal to zero, a determinant which is the fourth power of

$$\begin{vmatrix} u_1, & u_2, & u_3 \\ u_1', & u_2', & u_3' \\ u_1'', & u_2'', & u_3'' \end{vmatrix},$$

and the latter ought therefore to vanish. But because u_1, u_2, u_3 are linearly independent, this determinant (being the determinant of a fundamental system) does not vanish—it is a non-zero constant in the present case. Accordingly, the hypothesis that I is not zero is invalid.

Hence $I = 0$; and therefore, on returning to the original equation, we have

$$R - \tfrac{3}{2}\frac{dQ}{dz} = 0.$$

Writing

$$3Q = 4P, \quad R = 2\frac{dP}{dz},$$

our original equation becomes

$$\frac{d^3y}{dz^3} + 4P\frac{dy}{dz} + 2\frac{dP}{dz}y = 0.$$

Any three linearly independent integrals are connected by a quadratic relation

$$F(y_1, y_2, y_3) = 0.$$

To obtain the integrals, we note that one value of u is a constant, say unity; thus

$$y = \frac{dz}{dt} = \theta^2,$$

where

$$\frac{d^2\theta}{dz^2} + P\theta = 0.$$

Thus three integrals of the original equation are θ_1^2, $\theta_1\theta_2$, θ_2^2, where θ_1 and θ_2 are two linearly independent integrals of the latter equation of the second order.

It may be noted that three independent integrals of the u-equation are 1, t, t^2; so that

$$y_1 \frac{dt}{dz} = 1, \quad y_2 \frac{dt}{dz} = t, \quad y_3 \frac{dt}{dz} = t^2,$$

and therefore

$$y_2^2 - y_1 y_3 = 0,$$

thus verifying the existence of the quadratic relation obtained in a canonical form.

Assuming θ known, we have

$$\frac{dt}{dz} = \frac{1}{\theta^2},$$

so that

$$t = \int \frac{dz}{\theta^2};$$

and thus three integrals of the original equation are

$$\theta^2, \quad \theta^2 \int \frac{dz}{\theta^2}, \quad \theta^2 \left\{ \int \frac{dz}{\theta^2} \right\}^2.$$

The comparison of these integrals with θ_1^2, $\theta_1\theta_2$, θ_2^2 is immediate; for it is a well-known theorem that, if θ_1 is a solution of an equation

$$\frac{d^2\theta}{dz^2} + P\theta = 0,$$

then another solution, which is linearly independent of θ_1, is given by

$$\theta_1 \int \frac{dz}{\theta_1^2}.$$

Denoting this by θ_2, the above three integrals are at once seen to be θ_1^2, $\theta_1\theta_2$, θ_2^2.

Ex. 1. Prove that, if u be a solution of the equation

$$\frac{d^3y}{dx^3}+P\frac{dy}{dx}+\tfrac{1}{2}\frac{dP}{dx}y=0,$$

the primitive can be expressed in the form

$$y=Au+Bu\exp\left(a\int\frac{dx}{u}\right)+Cu\exp\left(-a\int\frac{dx}{u}\right),$$

where A, B, C are arbitrary constants, and a is a determinate constant. What is the primitive when a vanishes? (Math. Trip. Part I, 1895.)

Ex. 2. Prove that, if three linearly independent integrals of the equation

$$\frac{d^3y}{dx^3}=Iy$$

be connected by a relation $F(y_1, y_2, y_3)=0$, where F is a homogeneous polynomial of the third degree, then I must satisfy the equation

$$\begin{aligned}(56I'^2-48II'')\,I''''+54II'''^2-144I'I''I'''+18^2.\,7I^3I'''+\tfrac{1}{7}24^2I''^3\\ -7.\,36^2I^2I'I''+84^2II'^3+\frac{2.\,3^5.\,7^2}{25}I^4=0.\end{aligned}$$

Ex. 3. Prove that, if both the fundamental invariants* of an equation of the fourth order vanish, so that it can be taken in the form

$$y''''+10Py''+10P'y'+(3P''+9P^2)\,y=0,$$

then four linearly independent integrals are given by θ_1^3, $\theta_1^2\theta_2$, $\theta_1\theta_2^2$, θ_2^3, where θ_1 and θ_2 are linearly independent integrals of

$$\frac{d^2\theta}{dx^2}+P\theta=0.$$

Shew also that, if the relations

$$\left\|\begin{matrix} y_1, & y_2, & y_3 \\ y_2, & y_3, & y_4 \end{matrix}\right\|=0$$

* These arise in the same manner as for the cubic. If the equation

$$\frac{d^4y}{dx^4}+6P_2\frac{d^2y}{dx^2}+4P_3\frac{dy}{dx}+P_4y=0$$

be transformed by the relations

$$\frac{dz}{dx}=\theta^{-2},\quad \frac{d^2\theta}{dx^2}+\tfrac{3}{5}P_2\theta=0,\quad y=u\theta^3,$$

into

$$\frac{d^4u}{dz^4}+4Q_3\frac{du}{dz}+Q_4u=0,$$

then

$$Q_3=\theta^6\left(P_3-\tfrac{3}{2}\frac{dP_2}{dx}\right),\quad Q_4-2\frac{dQ_3}{dz}=\theta^8\left(P_4-2\frac{dP_3}{dx}+\tfrac{6}{5}\frac{d^2P_2}{dx^2}-\tfrac{81}{25}P_2^2\right);$$

and the fundamental invariants are Q_3, $Q_4-2\dfrac{dQ_3}{dz}$. See my memoir quoted p. 210, note.

subsist among four linearly independent integrals of an equation of the fourth order, (so that the integral curve is a twisted cubic), the equation must be of the above form.

Ex. 4. Construct the equation of the fourth order having $\theta_1\phi_1$, $\theta_1\phi_2$, $\theta_2\phi_1$, $\theta_2\phi_2$ for a set of linearly independent integrals, where θ_1 and θ_2, ϕ_1 and ϕ_2, are linearly independent integrals of the respective equations

$$\frac{d^2\theta}{dx^2}+P\theta=0, \quad \frac{d^2\phi}{dx^2}+Q\phi=0.$$

Hence infer the form of a quartic equation when a single homogeneous quadratic relation subsists among a fundamental system of integrals.

Ex. 5. Shew that the equation

$$y''''+ry'''+4sy''+(6s'+4rs)\,y'+2\,(s''+rs')\,y=0$$

is satisfied by $y=\theta^2$, where θ is an integral of

$$\theta''+s\theta=0\,;$$

and hence integrate the equation. (Fano.)

Ex. 6. Shew that, if five linearly independent integrals of an equation of the fifth order are connected by the relations

$$\left\| \begin{matrix} y_1, & y_2, & y_3, & y_4 \\ y_2, & y_3, & y_4, & y_5 \end{matrix} \right\| = 0,$$

the equation can be taken in the form

$$\frac{d^5y}{dx^5}+20s\frac{d^3y}{dx^3}+30\frac{ds}{dx}\frac{d^2y}{dx^2}+\left(18\frac{d^2s}{dx^2}+64s^2\right)\frac{dy}{dx}+\left(4\frac{d^3s}{dx^3}+64s\frac{ds}{dx}\right)y=0\,;$$

and thence integrate the equation as far as possible. (Fano.)

72. Consider now the more general case when three linearly independent integrals of the equation

$$\frac{d^3y}{dz^3}+3Q\frac{dy}{dz}+Ry=0$$

are connected by an irresoluble relation

$$F(y_1, y_2, y_3)=0,$$

where F is a homogeneous polynomial of order greater than two: the question is as to the character of the integrals of the equation. For the discussion, it is assumed that the differential equation has its integrals regular and free from logarithms: it thus is of Fuchsian type.

Let K denote any non-evanescent covariant of the quantic F; such a covariant is the Hessian, which would vanish only if F

contained a linear factor. Let z describe any contour, which encloses any one of the singularities, and return to its initial value; the effect upon the fundamental system of integrals y_1, y_2, y_3 is to change them into another fundamental system Y_1, Y_2, Y_3, the two systems being connected by relations

$$Y_r = \alpha_r y_1 + \beta_r y_2 + \gamma_r y_3, \qquad (r = 1, 2, 3).$$

The determinant of the coefficients α, β, γ (say Δ) is different from zero in every such case; in the present case, owing to the absence of the term in $\dfrac{d^2y}{dz^2}$ from the equation, we have (§ 14)

$$\Delta = 1,$$

by Poincaré's theorem.

Now the preceding relations constitute a linear transformation of the variables in the foregoing homogeneous forms; hence if μ be the index of K, and $\bar{K}$ denote the same function of Y_1, Y_2, Y_3 as K is of y_1, y_2, y_3, we have

$$\begin{aligned} \bar{K} &= \Delta^\mu K \\ &= K, \end{aligned}$$

for μ is necessarily an integer. It thus appears that the value of K is unaltered by the description of the contour.

This holds for each of the singularities, as well as for $z = \infty$; hence K, when expressed as a function of z, is a uniform function. To obtain the form of K in the vicinity of any singularity a, we take account of the fact that the equation is of Fuchsian type: hence in the vicinity we have, for any integral y,

$$(z-a)^{-\rho} y = \text{holomorphic function of } z - a,$$

where $|\rho|$ is a finite quantity. Now K is of finite order in the variables y_1, y_2, y_3; accordingly substituting for them, and remembering that K is a uniform function of z, we have

$$(z-a)^{-\sigma} K = \text{holomorphic function of } z - a,$$

where σ is an integer, positive or negative. This holds for each of the singularities, the number of which is limited when Q and R are rational functions of z; it holds also for $z = \infty$. Hence K is not merely a uniform function, but it is a rational function, of z.

It therefore follows that every covariant of the quantic F is a rational function of z, exceptions of course arising in the case when the covariant in an invariant, so that it is a mere constant.

Take then any two covariants, say the Hessian H, and any other, say K: we have

$$F=0, \quad H=\phi, \quad K=\psi,$$

where ϕ and ψ are rational functions of z. These are three algebraical equations to determine y_1, y_2, y_3 in terms of z; and therefore *the differential equation is integrable algebraically*, a theorem first announced* by Fuchs.

A case of exception arises, when the Hessian is a constant: the quantic F is then of the second order so that the case has already been discussed; the integration of the original equation depends upon the integrals of a linear equation of the second order.

As an illustration, consider the equation

$$y'''+3Qy'+Ry=0,$$

when a fundamental set of integrals is connected by a homogeneous cubic relation. We assume that the equation is of Fuchsian type.

Taking the cubic in the canonical form, we have

$$F=y_1^3+y_2^3+y_3^3+6ly_1y_2y_3=0,$$

l being a constant. The Hessian is a rational function, say $\phi(1+8l^3)$; so that

$$H=l^2(y_1^3+y_2^3+y_3^3)-(1+2l^3)\,y_1y_2y_3=\phi\,(1+8l^3),$$

and therefore

$$y_1y_2y_3=-\phi,$$
$$y_1^3+y_2^3+y_3^3=6l\phi.$$

Taking the other symmetric covariant† of the cubic, which also is a rational function, we have

$$\Psi=(1+8l^3)^2\{y_1^6+y_2^6+y_3^6-10\,(y_2^3y_3^3+y_3^3y_1^3+y_1^3y_2^3)\},$$

and Ψ is equal to a rational function; so that, taking account of the above value of $y_1^3+y_2^3+y_3^3$, we can write

$$y_1^3y_2^3+y_2^3y_3^3+y_3^3y_1^3=\psi.$$

Thus y_1^3, y_2^3, y_3^3 are the roots of

$$\eta^3-6l\phi\eta^2+\psi\eta+\phi^3=0,$$

* *Acta Math.*, t. I (1882), p. 330.

† Cayley, *Coll. Math. Papers*, t. XI, p. 345.

an irreducible cubic. So far as the coefficients are concerned, they are known to be rational functions of z; the denominator of each such function is known, because its factors arise through the singularities of the equation and the multiplicity of any factor can be determined through the associated indicial equation; and the degree of the numerator has an upper limit, determined by the behaviour of the integrals for large values of z. Hence ϕ and ψ can be regarded as known, save as to a polynomial numerator in each case.

We have

$$\left.\begin{aligned} \eta^3 &= 6l\phi\eta^2 - \psi\eta - \phi^3 \\ \eta' &= A\eta^2 + B\eta + C \\ \eta'' &= A_1\eta^2 + B_1\eta + C_1 \\ \eta''' &= A_2\eta^2 + B_2\eta + C_2 \end{aligned}\right\},$$

the last three being obtained, after differentiation, by repeated use of the cubic equation for η, and the quantities $A, B, C, \ldots$ being functions of ϕ, ψ and their derivatives. Now writing $y=\eta^{\frac{1}{3}}$ in the differential equation, we find

$$\tfrac{1}{3}\eta^2\eta''' - \tfrac{2}{3}\eta\eta'\eta'' + \tfrac{10}{27}\eta'^3 + Q\eta^2\eta' + R\eta^3 = 0.$$

When the above values are substituted and the result is reduced by means of the cubic equation, so that no power of η higher than the second occurs, we have an equation of the form

$$\Upsilon_1\eta^2 + \Upsilon_2\eta + \Upsilon_3 = 0,$$

where Υ_1, Υ_2, Υ_3 involve ϕ, ψ and their derivatives, and are linear in Q, R. As the cubic is irreducible, so that this equation holds for each root, we have

$$\Upsilon_1 = 0, \quad \Upsilon_2 = 0, \quad \Upsilon_3 = 0,$$

three equations to determine ϕ and ψ. There consequently exists a relation among the remaining quantities, viz. Q and R: and this must be equivalent to the condition (§ 71, Ex. 2), which must be satisfied in order that the equation $F=0$ may exist.

Similar results hold for the cubic equation, when the homogeneous relation between the integrals is of order greater than three; and corresponding results hold for linear differential equations of higher orders. In fact, *if a general homogeneous relation of finite order higher than the second subsists among a fundamental system of integrals of a linear differential equation of order n, then the equation is integrable algebraically:* the proof follows the lines of the preceding proof exactly.

This range of investigations will not, however, be pursued further, as it becomes mainly formal in character, depending upon

the theory of covariants and upon the application of the theory of groups to linear differential equations. An excellent account of what has been achieved, together with many references, is given in a memoir* by Fano who has made many contributions to the subject; a memoir† by Brioschi contains some investigations connected with ternariants; and other detailed references are given in Schlesinger's treatise‡, which contains an ample discussion of the subject.

* *Math. Ann.*, t. LIII (1900), pp. 493—590.

† *Ann. di Mat.*, 2a Ser., t. XIII (1885), pp. 1—21.

‡ *Theorie der linearen Differentialgleichungen*, II, 1 (1897), pp. viii—xi. The discussion is to be found in chapters 2—6 of the tenth section of the treatise.

CHAPTER VI.

Equations having only some of their Integrals regular near a Singularity.

73. It has been seen that, if all the integrals of an equation are to be regular in the vicinity of each singularity, the coefficients in the equation must be rational functions of z of appropriate form and degree.

It may, however, happen that the coefficients are rational functions of z but are not of the appropriate form and degree: in that case, it is not the fact that all the integrals are regular, and it may even be the fact that none of the integrals are regular. This deviation from regularity need not occur at each singularity of the equation: a fundamental system may be entirely regular in the vicinity of one (or more than one) of the singularities, and may not possess its entirely regular character in the vicinity of some other. The conditions necessary and sufficient to secure that all the integrals are regular in the vicinity of a singularity a have already (Ch. III) been obtained. If these conditions are not satisfied, then the composition of the fundamental system in the vicinity of the singularity a is no longer of an entirely regular character: we desire to know the deviations from regularity.

It may also happen that not all the coefficients are rational functions of z; in that case, if uniform, they are transcendental functions and possess at least one essential singularity, say c. Further, owing either to a possibly excessive degree of the numerator in a rational meromorphic coefficient or to a possibility that $z = \infty$ is an essential singularity of some one or more of the coefficients, it can happen that the conditions for regularity of integrals near $z = \infty$ are not satisfied. The fundamental system

is then not entirely regular near c or for large values of $|z|$, in the respective cases indicated, and it may even be devoid of any regular element; the same question as to its composition arises as in the corresponding hypothesis for the singularity a.

Accordingly, for our present purpose we assume that the coefficients in the differential equation are everywhere uniform: that (unless as otherwise stated) they may have any number of poles, and that they may have one or more essential singularities. When a is a pole of one (or more than one) of the coefficients, and is not an essential singularity of any of them, we have one of the cases just indicated; when ∞ is a pole of coefficients, not being an essential singularity of any one of them, we have another. We write

$$z - a = x, \quad z = \frac{1}{x},$$

in these respective cases; and then our differential equation takes the form

$$\frac{d^m w}{dx^m} + p_1 \frac{d^{m-1} w}{dx^{m-1}} + p_2 \frac{d^{m-2} w}{dx^{m-2}} + \dots + p_{m-1} \frac{dw}{dx} + p_m w = 0,$$

where the point $x=0$ is a pole of some (and it may be of all) the coefficients. If all the integrals were regular in the vicinity of $x=0$, then $x^r p_r$ for $r = 1, 2, \dots, m$ would be a uniform function of x that does not become infinite when $x=0$. As some of the integrals are to be not regular in the vicinity of $x=0$, the multiplicity of the origin as a pole of p_r must be greater than r, for some value or values of r. Let

$$p_r = x^{-\varpi_r} P_r(x), \qquad (r = 1, \dots, m),$$

where ϖ_r is a positive integer (which may be zero for particular coefficients), and $P_r(x)$ is a uniform function of x which does not become infinite when $x=0$: also it will be assumed that, unless p_r vanishes identically, ϖ_r has been chosen so that $P_r(0)$ does not vanish, so that ϖ_r measures the multiplicity of the pole of p_r at the origin. Then one or more than one of the quantities

$$\varpi_r - r \qquad (r = 1, \dots, m)$$

is a positive integer greater than zero.

As in § 23, let

$$L = \frac{1}{2\pi i} \log x;$$

and suppose that

$$\phi_\lambda + \phi_{\lambda-1} L + \phi_{\lambda-2} L^2 + \ldots + \phi_1 L^{\lambda-1} + \phi_0 L^\lambda$$

is an integral of the equation, regular in the vicinity of $x = 0$ and belonging to an exponent μ; then it is known (§§ 25—28) that ϕ_0 is a regular integral also belonging to the exponent μ, so that

$$\phi_0 = x^\mu \Phi_0,$$

where Φ_0 is a uniform function of x which does not vanish when $x = 0$. As this expression, when substituted for w, should make the equation satisfied identically, the aggregate coefficient of the lowest power of x must vanish (as, of course, must all the other aggregate coefficients). The lowest power of x in the respective terms has for its index

$$\mu - m,\ \mu - \varpi_1 - (m-1),\ \mu - \varpi_2 - (m-2),\ \ldots,\ \mu - \varpi_{m-1} - 1,\ \mu - \varpi_m:$$

and for any other integral, belonging to an exponent σ, the corresponding numbers would be

$$\sigma - m,\ \sigma - \varpi_1 - (m-1),\ \sigma - \varpi_2 - (m-2),\ \ldots,\ \sigma - \varpi_{m-1} - 1,\ \sigma - \varpi_m.$$

Let

$$\varpi_s + (m - s) = \Pi_s, \qquad (s = 0, 1, \ldots, m),$$

and consider the set of integers

$$\Pi_0,\ \Pi_1,\ \ldots,\ \Pi_m.$$

Of these, let the greatest be chosen. It may occur several times in the set; when this is the case, let the first occurrence be at Π_n, as we pass in the order of increasing subscripts, so that

$$\Pi_r < \Pi_n\ , \text{ for } r = 0, 1, \ldots, n-1,$$

$$\Pi_n \geqslant \Pi_{n+r}, \qquad r = 0, 1, \ldots, m-n.$$

Then n is called* the *characteristic index* of the equation: when $n = 0$, all the integrals are regular.

The lowest power of x after substitution of the expression for the regular integral has $\mu - \Pi_n$ for its index; it arises through $p_n \dfrac{d^{m-n}w}{dz^n}$ and later terms in the differential equation; as the coefficient of this lowest power must vanish, the exponent μ must

* Thomé, *Crelle*, t. LXXV (1873), p. 267.

satisfy an algebraic equation of degree $m - n$. Similarly for an exponent σ to which any other regular integral belongs; it also is a root of the same algebraic equation; and each such exponent satisfies that same algebraic equation of degree $m - n$, which accordingly is called the *indicial equation*. But it must not be assumed (and, in fact, it is not necessarily the case when $n > 0$) that the number of regular integrals is equal to the degree of the indicial equation. It is clear that, in all cases where $n > 0$, the degree of the indicial equation is less than m.

74. Suppose now that the given differential equation of order m has a number s of regular integrals, which are linearly independent of one another, where $s < m$: (the case $s = m$ has already been discussed): and that there do not exist more than s linearly independent integrals. After the earlier discussion of fundamental systems, it is clear that any regular integral of the equation is expressible as a homogeneous linear combination of the s integrals, with constant coefficients; also that, if every regular integral of the equation is expressible as such a combination of s (and not fewer than s) such integrals, the number of regular integrals linearly independent of one another is s.

Further, a linear relation among the integrals of the equation, involving a number of regular integrals and only a single one that is not of the regular type, cannot exist; for the single non-regular integral would involve an unlimited number of negative powers of x, while each of the others occurring in the linear relation involves only a limited number of such negative powers.

A linear relation might exist among the integrals of the equation, involving a number of regular integrals and two integrals that are not of the regular type. We then regard the relation as shewing that the deviation from regularity is the same for the two integrals: and in constituting the fundamental system for the equation, we could use the relation as enabling us to reject one of the non-regular integrals, because it is linearly expressible in terms of integrals already retained. So also for a linear relation with constant coefficients between regular integrals and more than two integrals of a non-regular type.

Again, suppose that our differential equation of order m has an aggregate of n integrals, regular in the vicinity of $x = 0$ and

linearly independent of one another; and let it be fo 0, $K=0$, is groups of integrals of the type nection

$$\psi_{\lambda+1,1} + \lambda\psi_\lambda L + \frac{\lambda(\lambda-1)}{2\,!}\psi_{\lambda-1,1}L^2 + \ldots + \lambda\psi_{2,1}L^{\lambda-1} + \psi_{1,1}L^\lambda,$$

for $\lambda = 0, 1, 2, \ldots, \kappa$, where

$$\Sigma(\kappa+1) = n.$$

Then, after §§ 25—28, we know that these n linearly independent integrals constitute a fundamental system for a linear differential equation of order n, the coefficients of which are functions of x, uniform in the vicinity of $x=0$; let it be

$$\frac{d^n y}{dx^n} + r_1\frac{d^{n-1}y}{dx^{n-1}} + r_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + r_{n-1}\frac{dy}{dx} + r_n y = 0.$$

Now this equation, being of order n, cannot have more than n linearly independent integrals: and its fundamental system in the vicinity of $x=0$ is composed of the n regular integrals of the original equation. Hence, by § 31, we must have

$$r_\mu = x^{-\mu}R_\mu(x), \qquad (\mu = 1, 2, \ldots, n),$$

where $R_\mu(x)$ is a holomorphic function of x in the vicinity of $x=0$, such that $R_\mu(0)$ is not infinite. Accordingly, *the aggregate of the n linearly independent regular integrals of the original equation are the n integrals in a fundamental system of a linear equation of order n of the foregoing type.*

Reducibility of Equations.

75. If therefore some (but not all) of the integrals of the given equation of order m are of the regular type, it has integrals in common with an equation of lower order. On the analogy of rational algebraic equations, which possess roots satisfying an algebraic equation of the same rational form and of lower degree, the differential equation is said to be *reducible.*

Consider two equations

$$\left.\begin{aligned} M(y) = P_0\frac{d^m y}{dx^m} + P_1\frac{d^{m-1}y}{dx^{m-1}} + \ldots + P_{m-1}\frac{dy}{dx} + P_m y = 0 \\ N(y) = Q_0\frac{d^n y}{dx^n} + Q_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots + Q_{n-1}\frac{dy}{dx} + Q_n y = 0 \end{aligned}\right\},$$

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$$L(y) = R_0 \frac{d^l y}{dx^l} + R_1 \frac{d^{l-1} y}{dx^{l-1}} + \ldots + R_{l-1} \frac{dy}{dx} + R_l y,$$

where the coefficients $R_0, R_1, \ldots, R_l$ are at our disposal, and

$$l = m - n.$$

Let these disposable coefficients be chosen, so as to make the order of the equation

$$M(y) - L\{N(y)\} = 0$$

as low as possible. By taking the $l + 1$ relations

$$P_0 = R_0 Q_0,$$

$$P_1 = R_1 Q_0 + R_0 (l Q_0' + Q_1),$$

$$P_2 = R_2 Q_0 + R_1 \{(l-1) Q_0' + Q_1\} + R_0 \{\tfrac{1}{2} l (l-1) Q_0'' + l Q_1' + Q_2\},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$P_l = R_l Q_0 + R_{l-1} (Q_0' + Q_1) + R_{l-2} (Q_0'' + 2Q_1' + Q_2) + \ldots,$$

which determine $R_0, \ldots, R_l$, we can secure that the terms involving derivatives of y of order higher than $n - 1$ disappear. Accordingly, writing

$$K(y) = S_0 \frac{d^k y}{dx^k} + S_1 \frac{d^{k-2} y}{dx^{k-2}} + \ldots + S_k y,$$

where $S_0, S_1, \ldots, S_k$ are determinate quantities and

$$k \leqslant n - 1,$$

we have

$$M - LN = K,$$

where K is of order less than N. Moreover, if $P_0, \ldots, P_m, Q_0, \ldots, Q_n$ are uniform functions of x, having $x = 0$ either an ordinary point or only a pole, the same holds of the coefficients R and the coefficients S; so that L and K are of the same generic character as M and N.

From this result several conclusions can be drawn.

I. Any integral, common to the equations $M = 0$, $N = 0$, is an integral of the equation $K = 0$. If, therefore, every integral of $N = 0$ is also an integral of $M = 0$, it follows that $K = 0$ must possess n linearly independent integrals; as its order is less than n, the equation is evanescent, and we then have

$$M(y) = L\{N(y)\}.$$

II. Any integral, common to the equations $N=0$, $K=0$, is an integral of the equation $M=0$; and therefore, in connection with the first part of the preceding result, the integrals common to $M=0$, $N=0$ constitute the integrals common to $N=0$, $K=0$.

The process of obtaining the integrals (if any), common to two given equations $M=0$ and $N=0$, can thus be made a kind of generalisation of the process of obtaining the greatest common measure of two given polynomials. Proceeding as above, we have

$$\left.\begin{aligned} M &= LN + K \\ N &= L_1K + K_1 \\ K &= L_2K_1 + K_2 \\ &\vdots \\ K_{s-2} &= L_sK_{s-1} + K_s \end{aligned}\right\},$$

where $K_1, K_2, \ldots, K_s$ are of successively decreasing orders. Then unless an evanescent quantity K of non-zero order is reached, sooner or later a quantity K is reached which is of order zero, that is, contains no derivative.

In the former case, let K_{r+1} be evanescent; then the integrals of the equation $K_r=0$ constitute the aggregate of integrals common to $M=0$, $N=0$.

In the latter case, let K_s be the quantity of order zero; then the integrals common to $M=0$, $N=0$ are integrals of

$$K_s = yf(z) = 0.$$

Now $f(z)$ is not zero, for otherwise K_s would be evanescent; and therefore we have

$$y=0,$$

the trivial solution common to all homogeneous linear equations. We then say that $M=0$, $N=0$ have no common integral.

III. An equation having regular integrals is reducible. For one such integral exists in the form

$$y = x^\theta f(x),$$

where $|\theta|$ is finite, and $f(x)$ is holomorphic in the vicinity of $x=0$, while $f(0)$ is not zero. We have

$$\begin{aligned} \frac{1}{y}\frac{dy}{dx} &= \frac{\theta}{x} + \frac{f'(x)}{f(x)} \\ &= \frac{1}{x} R(x), \end{aligned}$$

where $R(x)$ is a holomorphic function in the vicinity of $x=0$, such that $R(0)$ is not zero. Thus the given differential equation has an integral satisfying the equation

$$x\frac{dy}{dx} - yR(x) = 0,$$

that is, it has an integral common with an equation, which is of the first order and is of the same form as itself: in other words, the equation is reducible.

But it is not to be inferred that such equations are the only reducible equations.

IV. If an equation $M=0$ has p (and not more than p) linearly independent regular integrals, it can be expressed in the form

$$M(y) = L\{N(y)\} = 0,$$

where N is of order p, and L is of order $m-p$.

For the p regular integrals are known (§§ 25—28, 74) to satisfy an equation of the form

$$N = 0,$$

of order p. Every integral of $N=0$ is an integral of $M=0$; whence, by I., the result follows.

76. We proceed to utilise the last result in order to obtain some conclusions as regards the regular integrals (if any) of a given equation, say,

$$P(w) = \frac{d^m w}{dx^m} + p_1\frac{d^{m-1}w}{dx^{m-1}} + \ldots + p_{m-1}\frac{dw}{dx} + p_m w = 0.$$

The result of substituting x^ρ for w in $P(w)$, where ρ is a constant quantity, is

$$P(x^\rho) = x^\rho\left\{\frac{\rho(\rho-1)\ldots(\rho-m+1)}{x^m} + p_1\frac{\rho(\rho-1)\ldots(\rho-m+2)}{x^{m-1}} + \ldots \right.$$
$$\left. \ldots + p_{m-1}\frac{\rho}{x} + p_m\right\};$$

this is called* the *characteristic function* of the equation $P=0$ or of the operator P. We have

$$x^{-\rho}P(x^\rho) = \frac{\rho(\rho-1)\ldots(\rho-m+1)}{x^m} + p_1\frac{\rho(\rho-1)\ldots(\rho-m+2)}{x^{m-1}} + \ldots$$
$$\ldots + p_{m-1}\frac{\rho}{x} + p_m;$$

* Frobenius, *Crelle*, t. LXXX (1875), p. 318.

when the right-hand side is expanded in ascending powers of x, it contains (owing to the form of the coefficients p) only a limited number of powers with negative indices. The highest powers of x^{-1}, arising out of the $m+1$ terms in $x^{-\rho}P(x^{\rho})$, have exponents

$$m, \ \varpi_1 + m - 1, \ \varpi_2 + m - 2, \ \ldots, \ \varpi_{m-1} + 1, \ \varpi_m,$$

that is,

$$\Pi_0, \ \Pi_1, \ \ldots, \ \Pi_m.$$

Let n be the characteristic index of the equation, so that Π_n is the greatest integer in the set: if several of the quantities Π be equal to this greatest integer, then Π_n is the first that occurs as we proceed through the set from left to right. Denoting the value of Π_n by g, let

$$x^g \frac{p_r}{x^{m-r}} = q_r(x) = q_r, \qquad (r = 1, 2, \ldots, m),$$

so that $q_n(0)$ is not zero, and no one of the quantities $q_r(0)$ is infinite. Then

$$x^{-\rho}P(x^{\rho}) = x^{-g}G(\rho, x),$$

where G is a polynomial in ρ and is holomorphic in x in the vicinity of $x=0$. Moreover, expanding $G(\rho, x)$ in ascending powers of x, we have

$$G(\rho, x) = g_0(\rho) + xg_1(\rho) + \ldots,$$

where each of the coefficients g is a polynomial in ρ, of degree not higher than m; the degree of $g_0(\rho)$ is $m-n$, and the degree of $g_{g-m}(\rho)$ is m. Also, $g_0(\rho)$ is the quantity called (§ 39) the *indicial function*; the equation

$$g_0(\rho) = 0$$

is called the *indicial equation*.

Now take

$$\begin{aligned} N(w) &= x^g P(w) \\ &= q_0 x^m \frac{d^m w}{dx^m} + q_1 x^{m-1} \frac{d^{m-1} w}{dx^{m-1}} + \ldots + q_{m-1} x \frac{dw}{dx} + q_m w, \end{aligned}$$

where $q_0 = x^{g-m}$; the equation $P=0$ can manifestly be replaced by the equivalent

$$N(w) = 0,$$

which is taken to be the *normal form* for the present purpose. We have

$$x^{-\rho}N(x^{\rho}) = G(\rho, x) = g_0(\rho) + xg_1(\rho) + \ldots,$$

which thus contains only positive powers of x when the equation is in its normal form, and which has the indicial function for the term independent of x.

We have seen that, if $P(w)=0$ possess regular integrals, it is a reducible equation: and the operator P can then be represented as a product of operators. Consider, more generally in the first instance, two operators A and B, each in its normal form; and let C, also an operator, denote AB. Further, let the characteristic functions of A, B, C, respectively be

$$\left.\begin{aligned} A(x^\rho) &= x^\rho f(x, \rho) = x^\rho \sum_{\mu=0} f_\mu(\rho)\, x^\mu = \sum_{\mu=0} f_\mu(\rho)\, x^{\mu+\rho} \\ B(x^\rho) &= x^\rho g(x, \rho) = x^\rho \sum_{\mu=0} g_\mu(\rho)\, x^\mu = \sum_{\mu=0} g_\mu(\rho)\, x^{\mu+\rho} \\ C(x^\rho) &= x^\rho h(x, \rho) = x^\rho \sum h_\mu(\rho)\, x^\mu = \sum h_\mu(\rho)\, x^{\mu+\rho} \end{aligned}\right\},$$

where the summations in $f(x, \rho)$ and $g(x, \rho)$ include no negative powers of x, because A and B are in their normal forms. Now, as $C = AB$, we have

$$\begin{aligned} C(x^\rho) &= AB(x^\rho) \\ &= A\left\{\sum_{\mu=0} g_\mu(\rho)\, x^{\mu+\rho}\right\} \\ &= \sum_{\mu=0} g_\mu(\rho)\, A(x^{\mu+\rho}) \\ &= \sum_{\mu=0} \sum_{\lambda=0} g_\mu(\rho)\, f_\lambda(\mu+\rho)\, x^{\lambda+\mu+\rho}, \end{aligned}$$

and therefore

$$\Sigma h_\sigma(\rho)\, x^\sigma = \sum_{\mu=0} \sum_{\lambda=0} g_\mu(\rho)\, f_\lambda(\mu+\rho)\, x^{\lambda+\mu}.$$

As λ and μ are incapable of negative values, there are no negative values for σ; and therefore *C is in a normal form.* Also

$$h_0(\rho) = g_0(\rho)\, f_0(\rho),$$

so that *the indicial function of C is the product of the indicial functions of its component operators:* and

$$h_\sigma(\rho) = \sum_{\mu=0}^{\sigma} g_\mu(\rho)\, f_{\sigma-\mu}(\mu+\rho).$$

Further, if C be known to possess a component factor B which, when operated upon by A, produces C, then A can be obtained. For, take B and C in their normal forms: the equation

$$\sum_{\sigma=0} h_\sigma(\rho)\, x^\sigma = \sum_{\mu=0} \Sigma\, g_\mu(\rho)\, f_\lambda(\mu+\rho)\, x^{\lambda+\mu}$$

then holds. The values of λ are clearly 0, 1, ..., so that A is then in its normal form; and the successive quantities f_λ are given by the equation

$$h_\sigma(\rho) = \sum_{\lambda=0}^{\sigma} g_{\sigma-\lambda}(\rho) f_\lambda(\sigma-\lambda+\rho),$$

for $\sigma = 0, 1, \ldots, \rho$, the values obtained being polynomials in ρ, because C is known to be composite of A and B.

Of course, this merely gives the characteristic function of the operator; but the characteristic function uniquely determines the operator. For let $f(x, \rho)$ be a function, which is a polynomial in ρ, and the coefficients of which are functions of x: and let the degree of the polynomial be m. Then we have*

$$f(x, \rho) = \sum_{n=0}^{m-1} u_{m-n}\rho(\rho-1)\ldots(\rho-m+n+1) + u_0,$$

where, taking finite differences in the form

$$\Delta f(x, \rho) = f(x, \rho+1) - f(x, \rho),$$

we have

$$n!\, u_n = \{\Delta^n f(x, \rho)\}_{\rho=0}.$$

Thus

$$x^\rho f(x, \rho) = x^\rho \left\{u_m x^m \frac{\rho(\rho-1)\ldots(\rho-m+n+1)}{x^m} + \ldots + u_1 x \frac{\rho}{x} + u_0\right\},$$

which is the characteristic function of the operator

$$u_m x^m \frac{d^m}{dx^m} + u_{m-1} x^{m-1} \frac{d^{m-1}}{dx^{m-1}} + \ldots + u_1 x \frac{d}{dx} + u_0:$$

the operator is determined by the characteristic function.

Characteristic Index, and Number of Regular Integrals.

77. Now let the equation of order m, taken in its normal form, be

$$N(w) = q_0 x^m \frac{d^m w}{dx^m} + q_1 x^{m-1} \frac{d^{m-1} w}{dx^{m-1}} + \ldots + q_{m-1} x \frac{dw}{dx} + q_m w = 0;$$

and suppose that it possesses s (and not more than s) regular integrals, linearly independent of one another. These s integrals

* Boole's *Finite Differences*, 2nd ed., p. 35.

are a fundamental system of an equation, of order s and of Fuchsian type; when this equation is taken in its normal form, let it be

$$S(w) = x^s \frac{d^s w}{dx^s} + \sigma_1 x^{s-1} \frac{d^{s-1} w}{dx^{s-1}} + \dots + \sigma_{s-1} x \frac{dw}{dx} + \sigma_s = 0,$$

where $\sigma_1, \sigma_2, \dots, \sigma_s$ are holomorphic functions of x in the vicinity of $x=0$. As all the integrals of $S=0$ are possessed by $N=0$, there exists a differential operator T of order $m-s$, such that

$$N = TS;$$

because N and S are in their normal forms, T also is in its normal form, so that we can take

$$T = q_0 x^{m-s} \frac{d^{m-s}}{dx^{m-s}} + \tau_1 x^{m-s-1} \frac{d^{m-s-1}}{dx^{m-s-1}} + \dots + \tau_{m-s-1} x \frac{d}{dx} + \tau_{m-s},$$

where $\tau_1, \tau_2, \dots, \tau_{m-s}$ are holomorphic functions of x in the vicinity of $x=0$. If then

$$T(x^\rho) = x^\rho \theta(x, \rho),$$

the indicial function of T is the coefficient of x^0 in $\theta(x, \rho)$, which is a polynomial in ρ and contains no negative powers of x. This coefficient may be independent of ρ; in that case, the characteristic index of T is $m-s$. Or it may be a polynomial in ρ, say of degree k in ρ, where $k \geqslant 0$; the characteristic index of T then is $m-s-k$.

Because $N = TS$, the indicial function of N is the product of the indicial functions of T and S; so that *the indicial function of S, which gives all the regular integrals of N, is a factor of the indicial function of the original equation.* The degree of the indicial function of S is equal to s, because $S=0$ is an equation of order s of Fuchsian type; the degree of the indicial function of N is $m-n$, where n is the characteristic index of $N=0$. Hence

$$s + k = m - n,$$

that is,

$$s = m - n - k$$
$$\leqslant m - n;$$

so that (assuming for the moment that k may be either zero or greater than zero) *an upper limit for the number of regular integrals which an equation can possess is given by*

$$m - n,$$

where m is the order of the equation, and n is its characteristic index (supposed to be greater than zero). It is known that, when $n=0$, the number of regular integrals is equal to m.

COROLLARY I. *An equation, whose indicial function is a constant, so that its indicial equation has no roots, has no regular integrals;* for its characteristic index is equal to its order. But such equations are not the only equations devoid of regular integrals.

COROLLARY II. When k is equal to zero, then s is equal to $m-n$, so that the number of regular integrals of the equation is actually equal to the degree of the indicial function. The necessary and sufficient condition for this result is that the equation, which is reducible, must be capable of expression in the form

$$N = TS,$$

where the indicial function of T is a constant, and the degree of the indicial function of S is equal to the order of S.

This result, which is of the nature of a descriptive condition, appears to have been first given in this form by Floquet*. Other forms, of a similar kind, had been given earlier by Thomé† and by Frobenius‡ (see § 83, *post*).

Note. On the basis of the preceding analysis, it is easy to frame an independent verification that the characteristic index is not greater than $m-s$. For in the operator T, the quantity τ_{m-s-k} does not vanish when $x=0$; and all the quantities τ_λ, such that

$$\lambda < m-s-k,$$

do vanish when $x=0$. Hence, when we take N as expressed in the form

$$N = TS,$$

the coefficient of

$$x^{s+k}\frac{d^{s+k}w}{dx^{s+k}}$$

is the first (in the succession from left to right) in which τ_{m-s-k} occurs; it also contains $q_0, \tau_1, \ldots, \tau_{m-s-k-1}$, all of them occurring

* *Ann. de l'Éc. Norm. Sup.*, 2ᵉ Sér., t. VIII (1879), Suppl., pp. 63, 64.

† *Crelle*, t. LXXVI (1873), p. 285.

‡ *Crelle*, t. LXXX (1875), pp. 331, 332.

linearly. When $x=0$, all of these except τ_{m-s-k} vanish, and τ_{m-s-k} does not vanish; and therefore q_{m-s-k} does not vanish when $x=0$. In the coefficient of

$$x^\mu \frac{d^\mu w}{dx^\mu},$$

where $\mu > s+k$, the quantities $q_0, \tau_1, \ldots, \tau_{m-\mu}$ occur linearly: each of these vanishes when $x=0$, and therefore $q_{m-\mu}$ does vanish when $x=0$. As this holds for all values of μ, it follows that q_{m-s-k} is the first of the quantities q which does not vanish when $x=0$; hence the characteristic index of N is $m-s-k$, that is, it is $\leqslant m-s$, where s is the number of regular integrals possessed by the equation $N=0$.

Ex. 1. If $w=w_1$ be an integral, regular and free from logarithms, of an equation $P=0$, which is of order m and has s regular integrals, and if a new dependent variable u be given by

$$w=w_1\int u\,dz,$$

shew that u satisfies an equation $Q=0$, which is of order $m-1$ and has $s-1$ regular integrals; and obtain the relation between the characteristic index of $P=0$ and that of $Q=0$. (Thomé.)

Ex. 2. The equation

$$\frac{d^m w}{dz^m}+\sum_{r=1}^{m} p_r \frac{d^{m-r}w}{dz^{m-r}}=0$$

has $m-s$ integrals, regular in the vicinity of $z=0$ and linearly independent of one another, and $z=0$ is a pole for $p_1, \ldots, p_s$; shew that it is a pole (not an essential singularity) for each of the remaining coefficients p. (Thomé.)

Ex. 3. If, in the equation in the preceding example, $p_1, \ldots, p_s$ are arbitrarily assigned, subject to the condition that $z=0$ is a pole or an ordinary point, prove that the remaining coefficients p can be determined so as to permit the equation to possess $m-s$ arbitrarily assigned regular integrals, linearly independent of one another. (Thomé.)

Ex. 4. Prove that the condition, necessary and sufficient to secure that an equation $N=0$, of order m and having an indicial function of degree $m-\gamma$, shall have $m-\gamma-\delta$ linearly independent regular integrals, is that N shall be a product of the form QMD, where the indicial functions of Q, M, D are of degrees δ, 0, $m-\gamma-\delta$ respectively, and D is of order $m-\gamma-\delta$. Is there any limitation upon the order of M? (Cayley.)

Ex. 5. Shew that an equation $QD=0$ has at least as many regular integrals as $D=0$, and not more than $Q=0$ and $D=0$ together; and that, if all the integrals of $D=0$ are regular, then $QD=0$ has as many regular integrals as $Q=0$ and $D=0$ together.

Hence (or otherwise) shew that, if an equation $P=0$ has all its integrals regular, then P can be resolved into a product of operators, each of the first order and such that, equated to zero, it has a regular integral. Is this resolution unique? (Frobenius.)

78. In the two extreme cases, first, where the degree of the indicial function is equal to the order of the equation, and second, where its degree is zero, the number of regular integrals is equal to that degree. The preceding proposition shews that, in the intermediate cases, the degree merely gives an upper limit for the number of regular integrals. It is natural to enquire whether the number can fall below that upper limit.

As a matter of fact, it is possible* to construct equations, the number of whose regular integrals is less than the degree of the indicial function. Taking only the simplest case leading to equations of the second order, consider the two equations

$$U = \frac{dy}{dx} + ky + h = 0, \quad V = \frac{dy}{dx} + ky = 0,$$

of the first order; and form the equation

$$U\frac{dV}{dx} - V\frac{dU}{dx} = 0,$$

which manifestly is of the second order, say

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0,$$

where

$$p = k - \frac{1}{h}\frac{dh}{dx}, \quad q = \frac{dk}{dx} - \frac{k}{h}\frac{dh}{dx}.$$

If we can arrange so that $x=0$ is a pole of p of order n, where $n \geqslant 2$, then $x=0$ in general will be a pole of q of order $n+1$; and the indicial function will then be of the first degree.

Consider now the equation of the second order. Since

$$U = V + h,$$

it can be written

$$h\frac{dV}{dx} - V\frac{dh}{dx} = 0,$$

which is satisfied by

$$V = Ah,$$

where A is any arbitrary constant.

* Thomé, *Crelle*, t. LXXIV (1872), pp. 211—213.

Let Y be an integral of the equation of the second order. It may be an integral of $V=0$; if it is not, then, when we take

$$y_1 = -\frac{Y}{A},$$

we have

$$\frac{dy_1}{dx} + ky_1 + h = 0,$$

that is, y_1 is an integral of $U=0$. Thus any integral of the equation of the second order either is an integral of $V=0$ or is a constant multiple of an integral of $U=0$. If, then, $U=0$ and $V=0$ are such that they possess no regular integral, the differential equation of the second order can possess no regular integral; at the same time, its indicial function is of the first degree. The equation $V=0$ will not have a regular integral, if $x=0$ is a pole of k of order greater than unity; and the equation $U=0$ will then not have a regular integral, if h is a rational function of x.

Ex. 1. The aggregate of conditions can be satisfied simultaneously in many ways. For instance, take

$$p=\frac{1}{x^2}, \quad h=x;$$

then

$$k=\frac{1}{x^2}+\frac{1}{x}, \quad q=-\frac{3+2x}{x^3}.$$

The differential equation of the second order is

$$\frac{d^2y}{dx^2}+\frac{1}{x^2}\frac{dy}{dx}-\frac{3+2x}{x^3}y=0;$$

its indicial equation is of the first degree, and it has no regular integrals: or the number of its regular integrals is less than the degree of its indicial equation.

The conclusion can otherwise be verified; for it is easy to obtain two linearly independent integrals in the form

$$\frac{1}{x}e^{\frac{1}{x}}, \quad \frac{1}{x}e^{\frac{1}{x}}\int x^2 e^{-\frac{1}{x}}\,dx,$$

no linear combination of which gives rise to a regular integral.

Ex. 2. Shew that the equation

$$y''+\frac{1}{x^4}y'-\frac{5+2x^3}{x^5}y=0$$

has no regular integrals: and verify the result by obtaining the integrals of the equation. (Thomé, Floquet.)

Determination of such Regular Integrals as exist.

79. When the degree of the indicial function of an equation of order m is less than m, no precise information is given as to the number of regular integrals possessed by the equation. The further conditions, sufficient to determine whether a regular integral should or should not be associated with any root of the indicial equation, can be obtained in a form, which is mainly descriptive for the equation of general order and can be rendered completely explicit for any particular given equation.

Let the equation be

$$N(w) = q_0 x^m \frac{d^m w}{dx^m} + q_1 x^{m-1} \frac{d^{m-1} w}{dx^{m-1}} + \ldots + q_m w = 0,$$

of characteristic index n. Let $E(\theta)$ be the indicial function, and let σ be one of its zeros, so that

$$E(\sigma) = 0.$$

Then, if a regular integral is to be associated with σ, it must be of the form

$$u = x^\sigma (c_0 + c_1 x + c_2 x^2 + \ldots + c_p x^p + \ldots).$$

This expression, when substituted in the equation, must satisfy it identically, so that, after substitution, the coefficient of $x^{\sigma+p}$ must vanish for every value of p: and therefore

$$f_0(p)\, c_p + f_1(p)\, c_{p+1} + \ldots + f_\tau(p)\, c_{p+\tau} = 0,$$

where the number of terms in this difference-relation depends upon the actual forms of $q_0, q_1, \ldots, q_m$. Of the coefficients $f_0, f_1, \ldots, f_\tau$, the first is

$$f_0(p) = E(\sigma + p),$$

which is of degree $m - n$ in p; of the remainder, one at least, viz. f_{g-m}, is of degree m in p, where g has the same significance as in § 76.

The successive use of this difference-relation, together with the equations for the earlier coefficients, the first of which is

$$c_0 E(\sigma) = 0,$$

leads to the values of all the quantities $c_\mu \div c_0$, for the successive values of μ; and thus a formal expression for u is obtained that

satisfies the equation. If, however, the expression is an infinite series, it has no functional significance when it diverges: that this frequently, even generally, is the case, may be inferred as follows. For if $c_{\mu+1} \div c_\mu$, with indefinite increase of μ, tends to a limit that is not infinite, so also would $c_{\mu+2} \div c_{\mu+1}$, $c_{\mu+3} \div c_{\mu+2}$, and so on; and therefore

$$\frac{c_{\mu+a}}{c_\mu} = \frac{c_{\mu+1}}{c_\mu} \cdot \frac{c_{\mu+2}}{c_{\mu+1}} \cdot \ldots \cdot \frac{c_{\mu+a}}{c_{\mu+a-1}},$$

for finite values of a, also would tend to a limit that is not infinite. Now a number of the quantities

$$\frac{f_\theta(\mu)}{f_0(\mu)},$$

for various values of θ, undoubtedly tend to zero as μ increases indefinitely; some of them may have a finite limit: but one at least is infinite, viz.

$$\frac{f_{g-m}(\mu)}{f_0(\mu)},$$

because the numerator is of degree n higher than the denominator, both of them being polynomials in μ. Consequently, the expression

$$\frac{f_1(\mu)}{f_0(\mu)} \frac{c_{\mu+1}}{c_\mu} + \frac{f_2(\mu)}{f_0(\mu)} \frac{c_{\mu+2}}{c_\mu} + \ldots + \frac{f_\tau(\mu)}{f_0(\mu)} \frac{c_{\mu+\tau}}{c_\mu}$$

acquires an infinite value as μ increases without limit. The difference-relation requires the value of the expression to be always -1, so that the hypothesis leading to the wrong inference must be untenable. Therefore $c_{\mu+1} \div c_\mu$, with indefinite increase of μ, does not tend to a limit that is finite, and therefore the series diverges*. There is then no regular integral to be associated with the root σ.

* It is not inconceivable that, for special values of m and of n, and for special forms of the coefficients q, as well as for a special value of the limit $c_{\mu+1} \div c_\mu$, the infinite parts of the expression

$$\sum_{r=1}^{\tau} \frac{f_r(\mu)}{f_0(\mu)} \frac{c_{\mu+r}}{c_\mu}$$

might disappear, and the expression itself be equal to -1. In that case, the series would converge: and an exception to the general theorem would occur. But it is clear that such an exception is of a very special character: it will be left without further attempt to state the conditions explicitly.

As the series thus generally diverges when it contains an unlimited number of terms, the regular integral is thus generally illusory. The only alternative is that the series should contain a limited number of terms: and then the regular integral would certainly exist. Accordingly, let it be supposed that the series contains $k+1$ terms, so that

$$\frac{c_1}{c_0}, \quad \frac{c_2}{c_0}, \quad \ldots, \quad \frac{c_k}{c_0}$$

are quantities known from the difference-relation, and that

$$c_{k+1}, \quad c_{k+2}, \ \ldots \text{ ad inf.}$$

all vanish. If we secure that c_{k+1}, c_{k+2}, ..., $c_{k+\tau}$ all vanish, then every succeeding coefficient must vanish in virtue of the difference-relation; and these τ relations will then secure the existence of a regular integral to be associated with the exponent σ. Taking $p=k$, $k-1$, ..., $k-\tau+1$ in succession, we find the τ necessary conditions to be

$$f_0(k)\, c_k = 0, \text{ that is, } f_0(k) = 0,$$

and generally

$$\sum_{s=0}^{r} f_s(k-r)\, c_{k-r+s} = 0,$$

for values $r = 1, 2, \ldots, \tau - 1$. The first of these is

$$E(\sigma + k) = 0,$$

so that the indicial equation, which possesses a root σ, must possess also a root $\sigma + k$, where k is a positive integer. (In the special instance, when $k=0$, no condition is thus imposed: in the general instance, when k is a positive integer greater than zero, it is easy to verify that $E(\sigma+k)$ is the indicial function for $x=\infty$.)

When the aggregate of conditions, which will not be examined in further detail, is satisfied in connection with a root of the indicial equation, a regular integral exists, belonging to that root as its exponent; and there are as many regular integrals, thus determined, as there are sets of conditions satisfied for each root of the indicial equation.

Explicit expressions for the various coefficients c can be derived, when the explicit forms of the quantities q are known: but the general results involve merely laborious calculation, and would hardly be used in any particular case. The results are therefore,

as already remarked, mainly descriptive: and so, in any particular case, it remains chiefly a matter for experimental trial (to be completed) whether a regular integral is necessarily associated with a root of the indicial equation.

For this purpose, and also for the purpose of discussing the regular integrals associated with a multiple root of the indicial equation, a convenient plan is to adopt the process given by Frobenius (Chap. III) when all the integrals are regular. We substitute an expression

$$w = c_0 x^\rho + c_1 x^{\rho+1} + \ldots + c_\mu x^{\rho+\mu} + \ldots$$

in the equation

$$N(w) = q_0 x^m \frac{d^m w}{dx^m} + q_1 x^{m-1} \frac{d^{m-1} w}{dx^{m-1}} + \ldots + q_m w,$$

of characteristic index n. After the substitution, the first term is

$$c_0 E(\rho) x^\rho,$$

where $E(\rho)$ is the indicial function, of degree $m-n$; and we make all the succeeding terms vanish, by choosing the relations among the constants c appropriate for the purpose. We thus have

$$N(w) = c_0 E(\rho) x^\rho;$$

and the relations among the constants c are of the form

$$c_\mu E(\rho+\mu) = \alpha_{\mu,\mu-1} c_{\mu-1} + \alpha_{\mu,\mu-2} c_{\mu-2} + \ldots + \alpha_{\mu,0} c_0,$$

where the constants $\alpha_{\mu,\mu-1}, \ldots, \alpha_{\mu,0}$ are polynomials in μ and, when this relation is the general difference-relation between the coefficients c, one at least of these polynomials $\alpha_{\mu,r}$ is of degree m in μ. When the difference-relation is used for successive values of μ, we obtain expressions for the successive coefficients c, which give each of them as a multiple of c_0 by a quantity that is a rational function of μ. When these coefficients are used, we have the formal expression of a quantity w which satisfies the equation

$$N(w) = c_0 E(\rho) x^\rho.$$

Unfortunately for the establishment of the regular integrals, this formal expression does not necessarily (nor even generally) converge: for, in the difference-relation among the constants c, the right-hand side is a polynomial of degree m in μ, while the left-hand side is a polynomial of degree $m-n$ in μ, so that the series

$$\Sigma c_\mu x^{\rho+\mu}$$

would, as in the preceding investigation, generally diverge.

But while this is the fact in general, it may happen that the series would converge when ρ acquires a value occurring as a root of the equation

$$E(\rho) = 0.$$

In that case, the series satisfies the equation

$$N(w) = 0:$$

in other words, it is a regular integral of the differential equation. Further, if the particular value of ρ be a multiple root of the indicial equation, it can happen that the series

$$\frac{\partial w}{\partial \rho}$$

converges for this particular value of ρ; and then

$$N\left(\frac{\partial w}{\partial \rho}\right) = \frac{\partial}{\partial \rho}\{c_0 E(\rho)\, x^\rho\}$$
$$= 0,$$

because the value of ρ is a multiple root of $E = 0$: in other words, $\frac{\partial w}{\partial \rho}$ is then a regular integral of the differential equation. And so possibly for higher derivatives with regard to ρ, according to the multiplicity of the root of $E = 0$.

The whole test in this method is therefore as to whether the series

$$\Sigma c_\mu x^{\rho+\mu}$$

converges for the particular value (or values) of ρ given as the roots of the indicial equation. The method of dealing with a repeated root of the indicial equation has been briefly indicated. Corresponding considerations arise, when $E = 0$ has a group of roots differing among one another by integers. In fact, *all the processes adopted* (in Ch. III) *when all the integrals are regular, are applicable when only some of them are regular, provided the various series, whether original or derived, are converging series.* The deficiency, that arises through the occurrence of diverging series, represents the deficiency in the number of regular integrals below $m - n$. As already stated, the tests necessary and sufficient to discriminate between the convergence and divergence of the various series are not given in any explicit form, that admits of immediate application.

Ex. 1. Consider the equation

$$x^3y''+xy'-(3+2x)y=0,$$

constructed in § 78, Ex. 1. The indicial equation is

$$\rho-3=0,$$

so that there is not more than one regular integral; if it exists, it belongs to an exponent 3. To determine the existence, we substitute

$$y=c_0x^3+c_1x^4+c_2x^5+\dots$$

in the original equation; that it may be satisfied, we must have

$$0=\{(n+2)(n+1)-2\}c_{n-1}+nc_n,$$

for $n=1, 2, \dots$. We at once find

$$c_n=-(n+3)c_{n-1},$$

and therefore

$$c_n=\tfrac{1}{6}(-1)^n(n+3)!.$$

The series $\sum\limits_{n=0} c_nx^{3+n}$ diverges, and therefore the one possible regular integral does not exist; that is, the original equation possesses no regular integral, although the indicial equation is of the first degree.

If there were a regular integral, it would satisfy an equation

$$x\frac{dy}{dx}-uy=0,$$

where u is a holomorphic function of x; and the original equation could then be written

$$\left(x^2\frac{d}{dx}-v\right)\left(x\frac{dy}{dx}-uy\right)=0,$$

where v is some holomorphic function in the vicinity of $x=0$. It might be imagined that, as the indicial equation is of degree unity (a property that does not forbid the existence of a regular integral), it would be possible to obtain the regular integral through a determination of u, and that the divergence of the series in the preceding analysis is due to the operator

$$x^2\frac{d}{dx}-v,$$

which annihilates only expressions that are not regular. That this is not the case may easily be seen. We have

$$\left(x^2\frac{d}{dx}-v\right)\left(x\frac{dy}{dx}-uy\right)=x^3\frac{d^2y}{dx^2}+(x^2-vx-x^2u)\frac{dy}{dx}-\left(x^2\frac{du}{dx}-vu\right)y,$$

so that, if the resolution be possible, we have

$$x^2-vx-x^2u=x,$$

$$x^2\frac{du}{dx}-vu=3+2x.$$

Substituting in the second of these the value of v given by the first, we find

$$x^2\frac{du}{dx}+xu^2-xu+u=3+2x,$$

as an equation to determine u, supposed a holomorphic function of x. Let

$$u=a_0-a_1x+a_2x^2-\ldots$$

be substituted; in order that the equation for u may be satisfied, we have

$$a_0=3,$$
$$a_0^2-a_0-a_1=2,$$

and, for values of n higher than zero,

$$(n+2a_0-1)\,a_n+2\,(a_1a_{n-1}+a_2a_{n-2}+\ldots)-a_{n+1}=0.$$

Hence $a_0=3$, $a_1=4$, $a_2=24$, and so on. The relation giving a_{n+1}, when taken for successive values of n, shews that all the coefficients a are positive; hence

$$a_{n+1}>(n+2a_0-1)\,a_n$$
$$>(n+5)\,a_n,$$

that is,

$$30a_n>(n+4)\,!,$$

and so the series for u diverges: in other words, there is no function u, and the hypothetical resolution of the equation is not possible.

Note. This argument is general; it does not depend upon the particular coefficients for the special equation that has been discussed.

Ex. 2. Consider the equation

$$Dy=x^4y''+x^2(1-x-2x^2)\,y'-x\,(5+4x+4x^2)\,y'+(9+10x+4x^2)\,y=0,$$

which is in the normal form. The characteristic index is 1; the indicial equation is

$$\theta\,(\theta-1)-5\theta+9=0,$$

that is,

$$(\theta-3)^2=0,$$

so that the number of regular integrals cannot be greater than two, and such as exist belong to the exponent 3.

To determine these regular integrals (if any), we adopt the Frobenius method of Ch. III. Taking

$$y=c_0x^\rho+c_1x^{\rho+1}+\ldots+c_nx^{\rho+n}+\ldots,$$

we have

$$Dy=c_0\,(\rho-3)^2\,x^\rho,$$

provided

$$c_1=-c_0\frac{\rho^2-2\rho-5}{\rho-2},$$

and, for values of n greater than unity,

$$0=(\rho+n-3)\,c_n+\{\rho^2+\rho\,(2n-4)+n^2-4n-2\}\,c_{n-1}-2\,(\rho+n)\,c_{n-2},$$

a factor $\rho+n-3$ having been removed, because it does not vanish for these values of n. Let

$$(\rho+n-3)\,c_n-2c_{n-1}=k_n,$$

so that

$$k_1 = (\rho - 2) c_1 - 2c_0 = -(\rho - 3)(\rho + 1) c_0.$$

Also the difference-equation for the coefficients c becomes

$$k_n + (\rho + n) k_{n-1} = 0,$$

so that

$$\begin{aligned} k_n &= (-1)^{n-1} (\rho + n)(\rho + n - 1) \ldots (\rho + 2) k_1 \\ &= (-1)^n (\rho - 3)(\rho + 1)(\rho + 2) \ldots (\rho + n) c_0 \\ &= (-1)^n (\rho - 3) \frac{\Pi(\rho + n)}{\Pi(\rho)} c_0. \end{aligned}$$

Hence, writing

$$c_n = \frac{2^n}{\Pi(\rho + n - 3)} u_n$$

in the relation

$$(\rho + n - 3) c_n - 2c_{n-1} = k_n,$$

and substituting the value of k_n, we have

$$u_n - u_{n-1} = (-\tfrac{1}{2})^n \frac{\Pi(\rho + n)\, \Pi(\rho + n - 4)}{\Pi(\rho)} (\rho - 3) c_0.$$

Adding the sides of this equation, taken successively for $n, n-1, \ldots, 3, 2$, and noting that

$$\begin{aligned} u_1 &= \tfrac{1}{2} \Pi(\rho - 2) c_1 \\ &= \tfrac{1}{2} (5 + 2\rho - \rho^2) \Pi(\rho - 3) c_0, \end{aligned}$$

we have

$$u_n = c_0 \left[\tfrac{1}{2} (5 + 2\rho - \rho^2) \Pi(\rho - 3) + (\rho - 3) \sum_{m=2}^{n} (-\tfrac{1}{2})^m \frac{\Pi(\rho + m)\, \Pi(\rho + m - 4)}{\Pi(\rho)} \right].$$

We thus have a value of y in the form

$$Y = \sum_{n=0} c_n x^{\rho + n},$$

where

$$\frac{c_n}{c_0} = 2^{n-1} (5 + 2\rho - \rho^2) \frac{\Pi(\rho - 3)}{\Pi(\rho + n - 3)} + (\rho - 3) \frac{\sum_{m=2}^{n} (-\tfrac{1}{2})^m \Pi(\rho + m)\, \Pi(\rho + m - 4)}{\Pi(\rho)\, \Pi(\rho + n - 3)} 2^n;$$

and this satisfies the relation

$$DY = c_0 (\rho - 3)^2 x^\rho.$$

It is clear that formal solutions of the original differential equation are

$$[Y]_{\rho=3}, \quad \left[\frac{\partial Y}{\partial \rho}\right]_{\rho=3}.$$

Of these, the first is

$$y_0 = c_0 \sum_{n=0} x^{3+n} \frac{2^n}{\Pi(n)},$$

in effect, a constant multiple of $x^3 e^{2x}$; and the second is

$$y_0 \log x + \text{a diverging series},$$

because a series, in which

$$\frac{1}{6} \frac{\sum_{m=2}^{n} (-\tfrac{1}{2})^m \Pi(m+3)\Pi(m-1)}{\Pi(n)} 2^n$$

is the coefficient of x^{n+3}, manifestly diverges*.

It thus appears that, although the indicial equation for $x=0$ is of the second degree, the differential equation possesses only one integral which is regular in that vicinity; and this integral is a constant multiple of $x^3 e^{2x}$.

This regular integral satisfies the equation

$$x\frac{dy}{dx} - (3+2x)y = 0,$$

so that the original equation must be reducible. It is easy to verify that it can be expressed in the form

$$\left\{x^3 \frac{d^2}{dx^2} + x\frac{d}{dx} - (3+2x)\right\}\left\{x\frac{dy}{dx} - (3+2x)y\right\} = 0.$$

Ex. 3. As an example which allows the convergence of the series for the regular integral to occur in a different way, consider the equation

$$x^2 y'' - (1-2x+2x^2)y' + (1-2x+x^2)y = 0.$$

The indicial equation is

$$\rho = 0,$$

so that one regular integral may exist. To determine whether this is so or not, we substitute

$$y = a_0 + a_1 x + a_2 x^2 + \ldots,$$

which (if it exists) belongs to the exponent zero. Comparing coefficients, we find

$$a_1 = a_0, \quad 2a_2 = a_0,$$

and, for all values of n that are greater than unity,

$$(n+1)a_{n+1} = (n^2+n+1)a_n - 2na_{n-1} + a_{n-2}.$$

Let

$$c_m = ma_m - a_{m-1};$$

then

$$c_{n+1} = (n+1)c_n - c_{n-1}.$$

In general, the values of c (and the consequent values of a) as determined by the last equation, lead to diverging series; but in our particular case,

$$c_1 = a_1 - a_0 = 0,$$
$$c_2 = 2a_2 - a_1 = 0,$$

so that $c_3=0$, $c_4=0$, and generally $c_m=0$, that is,

$$ma_m = a_{m-1},$$

* The series in y_0 is saved from divergence because, in it, these coefficients are multiplied by the factor $\rho-3$, which vanishes for the special value of ρ and which therefore removes the quantities that cause the divergence in the second integral.

and therefore

$$a_m = \frac{a_0}{m!},$$

so that a regular integral exists. It is a constant multiple of e^x.

Ex. 4. Consider the equation

$$D(y) = (x^5 + x^6) y'''' + (x^3 + 4x^4 + 4x^5) y''' - (2x^2 + 3x^3 + 2x^4) y''$$
$$+ (3x + 6x^2 + 4x^3) y' - (3 + 6x + 4x^2) y = 0.$$

The characteristic index is unity; hence the number of regular integrals is not greater than three. To determine them, if they exist, we take an expression

$$y = c_0 x^\rho + c_1 x^{\rho+1} + \ldots + c_n x^{\rho+n} + \ldots,$$

and form $D(y)$, choosing relations among the coefficients c such that all terms after the first in the quantity $D(y)$ vanish. We thus find

$$D(y) = c_0 (\rho - 1)^2 (\rho - 3) x^\rho,$$

provided

$$c_1 \rho^2 (\rho - 2) + c_0 (\rho - 1)(\rho - 2)(\rho^2 + \rho - 3) = 0,$$

and, for values of n greater than unity,

$$c_n (\rho + n - 1)^2 (\rho + n - 3) + c_{n-1} (\rho + n - 2)(\rho + n - 3)\{(\rho + n)^2 - (\rho + n) - 3\}$$
$$+ c_{n-2} (\rho + n - 3)^2 (\rho + n - 4)(\rho + n) = 0.$$

The indicial equation is

$$(\rho - 1)^2 (\rho - 3) = 0,$$

of degree 3 as was to be expected $(= 4 - 1)$, because the characteristic index is 1. The roots form a single group; if a regular integral exists belonging to the root 3, it will be free from logarithms; if two regular integrals exist belonging to the root 1, one of them may or may not be free from logarithms, and the other will certainly involve logarithms.

Consider the root $\rho = 3$. As $\rho + n - 3$ then vanishes for no one of the values of n, we may remove it from the difference-equation, so that the latter becomes

$$c_n (\rho + n - 1)^2 + c_{n-1} (\rho + n - 2)\{(\rho + n)^2 - (\rho + n) - 3\}$$
$$+ c_{n-2} (\rho + n - 3)(\rho + n - 4)(\rho + n) = 0.$$

Taking

$$c_n (\rho + n - 1)^2 + c_{n-1} (\rho + n - 2)(\rho + n - 3) = k_n,$$

we at once find

$$k_n + (\rho + n) k_{n-1} = 0.$$

We require the value of k_2. We have, for $\rho = 3$,

$$c_1 = -2c_0,$$
$$16c_2 + 51c_1 + 10c_0 = 0,$$

so that

$$k_2 = 16c_2 + 6c_1 = 80c_0.$$

Now

$$k_n = (-1)^n (n + 3)(n + 2) \ldots 6k_2$$
$$= \tfrac{2}{3} (-1)^n (n + 3)!\, c_0;$$

so that, writing

$$c_n = (-1)^n a_n,$$

we have

$$(n+2)^2 a_n - n(n+1) a_{n-1} = (n+3)! \tfrac{2}{3} c_0.$$

As a_1 and a_2 are positive, it follows that all the coefficients a are positive; and clearly

$$a_n > (n+1)! \tfrac{2}{3} c_0,$$

so that the series

$$x^3 (c_0 - a_1 x + a_2 x^2 - \ldots)$$

diverges; and there is no regular integral belonging to the root 3. Moreover, the coefficient of c_n, being $(\rho+n-1)^2$, does not vanish when $\rho=3$ for any value of n; hence, if two regular integrals exist belonging to the root unity of the indicial equation, one of them will certainly be free from logarithms.

Consider now the repeated root $\rho=1$. As $\rho+n-3$ vanishes for this value of ρ when $n=2$, the difference-equation is then evanescent for $n=2$ and it does not determine c_2. For other values of n, the quantity $\rho+n-3$ does not then vanish, so that it may be removed. We then have, for values of $n \geqslant 3$, the same form of equation as before, viz.

$$c_n (\rho+n-1)^2 + c_{n-1} (\rho+n-2) \{(\rho+n)^2 - (\rho+n) - 3\} \\ + c_{n-2} (\rho+n-3)(\rho+n-4)(\rho+n) = 0.$$

Also

$$c_1 = -(\rho-1)(\rho^2+\rho-3) \frac{c_0}{\rho^2},$$

the value $\rho=1$ not yet being inserted because we have to differentiate with regard to ρ. The difference-equation for $n=3$ gives

$$9c_3 + 18c_2 = 0,$$

so that

$$c_3 = -2c_2.$$

For values of $n \geqslant 4$, let $\rho = \sigma - 2$, so that the value of σ is 3; take $n-2=m$, so that the values of m are $\geqslant 2$; and write

$$c_n = b_m;$$

then the difference-equation becomes

$$b_m (\sigma+m-1)^2 + b_{m-1} (\sigma+m-2) \{(\sigma+m)^2 - (\sigma+m) - 3\} \\ + b_{m-2} (\sigma+m-3)(\sigma+m-4)(\sigma+m) = 0.$$

Here $\sigma=3$, $m \geqslant 2$; $c_2 = b_0$, $c_3 = b_1 = -2b_0$: so that this equation is now exactly the same as in the former case for $\rho=3$. The series thence determined is

$$x^3 (b_0 - a_1 x + a_2 x^2 - \ldots)$$

with the earlier notation; it certainly diverges unless $b_0=0$. If $b=0$, every coefficient vanishes, and the series itself vanishes. As we require regular integrals, we shall therefore assume $b_0=0$, that is, $c_2=0$; and then all the remaining coefficients vanish, so that we have

$$Y = c_0 \left[x^\rho - x^{\rho+1} (\rho-1)(\rho^2+\rho-3) \frac{1}{\rho^2} \right],$$

an expression which is such that

$$DY = c_0 (\rho - 1)^2 (\rho - 3) x^\rho.$$

Accordingly,

$$[Y]_{\rho=1}, \quad \left[\frac{\partial Y}{\partial \rho}\right]_{\rho=1},$$

are integrals of the equation

$$Dw = 0.$$

The former is $c_0 x$: one regular integral thus is

$$w = x.$$

The latter is

$$c_0 (x \log x + x^2);$$

another regular integral is

$$w = x^2 + x \log x.$$

The original differential equation accordingly has two regular integrals.

Ex. 5. Shew that the equation

$$x^2 (1+x)^2 y'' - (1 + 2x + 2x^2 - x^4) y' - (1 + 2x + 3x^2 + 2x^3) y = 0$$

has one integral regular in the vicinity of $x = 0$; and express the equation in a reducible form.

Ex. 6. Shew that the equation

$$x^2 (1 + 2x + 2x^2 + x^4) y''' + (1 + 6x + 6x^2 - 3x^4 - 2x^6) y''$$
$$- (2 + 12x + 15x^2 + 6x^3 - x^6) y' + (1 + 6x + 8x^2 + 4x^3 + x^4) y = 0$$

has two regular integrals in the vicinity of $x = 0$, in the form

$$e^x, \quad xe^x;$$

and obtain the integral that is not regular.

Ex. 7. Shew that the equation

$$x^2 y'' + (3x - 1) y' + y = 0$$

has no integral, that is regular in the vicinity of $x = 0$; express the equation in a reducible form, and thence obtain the integral by quadratures. (Cayley.)

Ex. 8. An equation $P = 0$ can be expressed in the form

$$QD = 0,$$

where $D = 0$ has no regular integrals; can $P = 0$ have any regular integrals? Illustrate by a special case.

Ex. 9. In the equation

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + P_n y = 0,$$

the coefficients P are polynomials in x of degree p, and $p < n$: shew that it possesses $n - p$ integrals, which are integral functions of x. (Poincaré.)

Existence of Irreducible Equations.

80. We have seen that an equation is reducible when it is satisfied by one or more of the integrals of an equation of lower order, in particular, by the integral of an equation of the first order. The main use so far made of this property has been in association with the regular integrals of the equation: but it applies equally if the equation possesses non-regular integrals that satisfy an equation of lower order. It is superfluous to indicate examples.

It must not be assumed, however, that every equation is reducible by another, if only that other be chosen sufficiently general. On the contrary, it is possible to construct an irreducible equation of any order m, as follows*.

We construct an appropriate characteristic function which, as is known (§ 76), uniquely determines the equation. Take a polynomial in ρ of degree m, say

$$h(x, \rho);$$

let the coefficients of the powers of ρ be holomorphic functions of x, not all vanishing when $x=0$; and let the function, subject to these limitations, be so chosen that, when arranged in powers of x in the form

$$h(x, \rho) = h_0(\rho) + xh_1(\rho) + x^2h_2(\rho) + \dots,$$

$h_0(\rho)$ is independent of ρ and not zero, and $h_1(\rho)$ is of degree m in ρ. Then if $N=0$ is the equation determined by $h(x, \rho)$ as its characteristic function, $N=0$ is irreducible.

Were N reducible, an equation $S=0$ of lower order s would exist such that each of its integrals satisfies $N=0$; and then an operator Q, of order $m-s$, could be found such that

$$N = QD.$$

We take Q and D in their normal form; and so N is in its normal form. Now

$$Q(x^\rho) = x^\rho \{\eta_0(\rho) + x\eta_1(\rho) + x^2\eta_2(\rho) + \dots\},$$

$$D(x^\rho) = x^\rho \{\zeta_0(\rho) + x\zeta_1(\rho) + x^2\zeta_2(\rho) + \dots\},$$

* Frobenius, *Crelle*, t. LXXX (1875), p. 332.

the right-hand sides of which are polynomials in ρ of degrees $m-s$ and s respectively. Then, as in § 76, we have

$$h_0(\rho) = \zeta_0(\rho)\,\eta_0(\rho),$$
$$h_1(\rho) = \zeta_0(\rho)\,\eta_1(\rho) + \zeta_1(\rho)\,\eta_0(\rho+1).$$

Now $h_0(\rho)$ is a constant, being independent of ρ; hence, owing to the polynomial character of $Q(x^\rho)$ and $D(x^\rho)$ in terms of ρ, the two quantities $\zeta_0(\rho)$ and $\eta_0(\rho)$ are constants. Accordingly, $\eta_0(\rho+1)$ is a constant; and therefore the degree of

$$\zeta_0(\rho)\,\eta_1(\rho) + \zeta_1(\rho)\,\eta_0(\rho+1)$$

in ρ is the degree of $\eta_1(\rho)$ or $\zeta_1(\rho)$, whichever is the greater. But the degree of $\eta_1(\rho)$ is not greater than s, and that of $\zeta_1(\rho)$ is not greater than $m-s$; so that, as $s>0$, the degree is certainly less than m. But the expression is equal to $h_1(\rho)$, which is of degree m. Hence the hypothesis adopted is untenable; and the equation $N=0$, as constructed, is irreducible.

Equations having Regular Integrals are Reducible.

81. Suppose now that, by the preceding processes or by some equivalent process, the regular integrals of the equation $N=0$ have been obtained, s in number, and that the equation of which they constitute a fundamental system is $S=0$, of order s: a question arises as to the other $m-s$ integrals of a fundamental system of $N=0$. Let

$$N = TS,$$

where T and S (and therefore also N) are taken in their normal forms. The s regular integrals of N, say $y_1, y_2, \ldots, y_s$, all satisfy $S=0$; and no one of the $m-s$ non-regular integrals of N, say $w_1, w_2, \ldots, w_{m-s}$, satisfies $S=0$, for this equation has all its integrals regular. Let

$$S(w_r) = u_r, \qquad (r=1, \ldots, m-s);$$

then, as $N(w_r)=0$, we have

$$T(u_r) = 0.$$

Now w_r is not a regular expression; hence u_r is not regular, that is, it contains an unlimited number of positive and negative

exponents when it is expressed as a power-series. Accordingly, the $m-s$ quantities u are integrals of the equation

$$T(u)=0,$$

which is of order $m-s$ and has no regular integrals; and the $m-s$ non-regular integrals of $N=0$ are given by

$$S(w_r)=u_r,$$

it being sufficient for this purpose to take the particular integral and not the complete primitive of the latter equation.

The case which is next in simplicity to those already discussed arises when $s=m-1$, so that the original equation then possesses only one integral which is not regular. The equation $T=0$ is then of the first order.

With the limitations laid down, the normal form of T is

$$q_0 x \frac{d}{dx} + q_1,$$

where q_0 and q_1 do not become infinite when $x=0$. As the integral of $T(u)=0$ is not regular, it follows that q_1 does not vanish and that q_0 does vanish when $x=0$; so that, if

$$q_0 = x^a Q(x),$$

where α is a positive integer $\geqslant 1$ and $Q(x)$ is a holomorphic function in the vicinity of $x=0$, such that $Q(0)$ is not zero, the equation determining u is

$$\frac{1}{u}\frac{du}{dx} + \frac{1}{x^{a+1}}\frac{q_1}{Q(x)} = 0,$$

say

$$\frac{1}{u}\frac{du}{dx} + \frac{a_0\alpha}{x^{a+1}} + \frac{a_1(\alpha-1)}{x^a} + \ldots + \frac{\sigma}{x} + R(x) = 0,$$

where $R(x)$ is a holomorphic function of x in the vicinity of $x=0$. This gives

$$u = x^{-\sigma} e^{\frac{a_0}{x^a} + \frac{a_1}{x^{a-1}} + \ldots + \frac{a_{a-1}}{x}} P_1(x),$$

where P_1 is a holomorphic function of x in the vicinity of $x=0$; and then to determine w, the non-regular integral of $N=0$, we need only take the particular integral of

$$S(w)=u,$$

where

$$S = q_0 x^{m-1} \frac{d^{m-1}}{dx^{m-1}} + q_1 x^{m-2} \frac{d^{m-2}}{dx^{m-2}} + \dots + q_{m-2} x \frac{d}{dx} + q_{m-1},$$

in which $q_0, q_1, \dots, q_{m-1}$ denote holomorphic functions of x, and q_0 does not vanish. Writing

$$\Omega = \frac{a_0}{x^\alpha} + \frac{a_1}{x^{\alpha-1}} + \dots + \frac{a_{\alpha-1}}{x},$$

$$w = ve^\Omega,$$

the equation for v takes the form

$$q_0 \frac{d^{m-1}v}{dx^{m-1}} + \frac{p_1}{x^{\alpha+1}} \frac{d^{m-2}v}{dx^{m-2}} + \frac{p_2}{x^{2\alpha+2}} \frac{d^{m-3}v}{dx^{m-3}} + \dots$$

$$\dots + \frac{p_{m-1}}{x^{(m-1)(\alpha+1)}} v = x^{-\sigma-m+1} P_1(x),$$

where $q_0, p_1, p_2, \dots, p_{m-1}$ are holomorphic functions of x, such that q_0 and p_{m-1} do not vanish when $x = 0$.

In some cases it happens that a particular integral of this equation exists, in the form of a converging power-series represented by

$$x^{(m-1)\alpha-\sigma} P(x),$$

where $P(x)$ is a holomorphic function of x: in each such case, the non-regular integral of the original equation is

$$x^{(m-1)\alpha-\sigma} e^\Omega P(x).$$

But, in general, the particular integral of the v-equation is not of the same type as the regular integrals of the original equation: and then the non-regular integral of the preceding equation cannot be declared to be of that type.

Ex. An illustration is furnished by the equation in Ex. 6, § 79, viz.

$$x^2(1+2x+2x^2+x^4)y''' + (1+6x+6x^2-3x^4-2x^6)y''$$
$$-(2+12x+15x^2+6x^3-x^6)y' + (1+6x+8x^2+4x^3+x^4)y = 0.$$

It has two regular integrals, viz.

$$y_1 = e^x, \quad y_2 = xe^x;$$

and these constitute the fundamental system of

$$y'' - 2y' + y = 0,$$

or

$$x^2y'' - 2x^2y' + x^2y = 0,$$

in the normal form. To have the given equation in the normal form, we multiply throughout by x^2; and then it must be the same as

$$\left\{x^2(1+2x+2x^2+x^4)\frac{d}{dx}+p\right\}(x^2y''-2x^2y'+x^2y)=0,$$

when p is properly determined. We easily find that

$$p=1+4x+4x^2+x^4-2x^5\,;$$

and so the equation for determining u, where

$$u=x^2y''-2x^2y'+x^2y,$$

y being the non-regular integral, is

$$x^2(1+2x+2x^2+x^4)\frac{du}{dx}+(1+4x+4x^2+x^4-2x^5)\,u=0.$$

Hence

$$\begin{aligned}\frac{1}{u}\frac{du}{dx}&=-\frac{1+4x+4x^2+x^4-2x^5}{x^2(1+2x+2x^2+x^4)}\\&=-\frac{1}{x^2}-\frac{2}{x}+\frac{2+4x+4x^3}{1+2x+2x^2+x^4},\end{aligned}$$

so that

$$u=\frac{1+2x+2x^2+x^4}{x^2}e^{\frac{1}{x}}.$$

Hence the non-regular integral of the original equation arises as the particular integral of

$$y''-2y'+y=\frac{1+2x+2x^2+x^4}{x^4}e^{\frac{1}{x}}.$$

Let $y=ve^{\frac{1}{x}}$; the equation for v is easily found to be

$$v''-2v'\left(1+\frac{1}{x^2}\right)+v\,\frac{1+2x+2x^2+x^4}{x^4}=\frac{1+2x+2x^2+x^4}{x^4},$$

satisfied by $v=1$: and therefore the non-regular integral is

$$y=e^{\frac{1}{x}}.$$

The Adjoint Equation, and its Properties.

82. Of the properties characteristic of a linear equation, not a few are expressed by reference to the properties of an associated equation, frequently called Lagrange's adjoint equation. It is a consequence of the formal theory of our subject, as distinct from the functional theory to which the present exposition is mainly limited, that Lagrange's is only one of a number of covariantive equations associated with the original. As its properties have been studied, while those of the others remain largely undeveloped,

there may be an advantage in giving some indication of a few of its relations to the original linear equation.

The latter is taken in the customary form

$$P(w) = P_0 w^{(n)} + P_1 w^{(n-1)} + P_2 w^{(n-2)} + \ldots + P_n w = 0,$$

where $w^{(r)}$ is the rth derivative of w with respect to z; and from among the various definitions of the adjoint equation, we choose that which defines it to be *the relation satisfied by a quantity v in order that $vP(w)$ may be a perfect differential.* Now, on integrating by parts, we find

$$\begin{aligned}\int vP_r w^{(n-r)} dz = vP_r w^{(n-r-1)} &- \frac{d}{dz}(vP_r)\, w^{(n-r-2)} \\ &+ \frac{d^2}{dz^2}(vP_r)\, w^{(n-r-3)} - \ldots + (-1)^{n-r} \int w \frac{d^{n-r}}{dz^{n-r}}(vP_r)\, dz,\end{aligned}$$

for all the values of r; hence, writing

$$\begin{aligned} p_0 &= P_0 v \\ p_1 &= P_1 v - \frac{d}{dz}(vP_0), \\ p_2 &= P_2 v - \frac{d}{dz}(vP_1) + \frac{d^2}{dz^2}(vP_0), \\ &\vdots \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ p_{n-1} &= P_{n-1} v - \frac{d}{dz}(vP_{n-2}) + \ldots + (-1)^{n-1} \frac{d^{n-1}}{dz^{n-1}}(vP_0), \end{aligned}$$

$$R(w, v) = p_0 w^{(n-1)} + p_1 w^{(n-2)} + \ldots + p_{n-1} w,$$

$$p(v) = P_n v - \frac{d}{dz}(P_{n-1} v) + \frac{d^2}{dz^2}(P_{n-2} v) - \ldots + (-1)^n \frac{d^n}{dz^n}(vP_0),$$

we have

$$\int vP(w)\, dz = R(w, v) + \int wp(v)\, dz,$$

and therefore

$$vP(w) - wp(v) = \frac{d}{dz}\{R(w, v)\}.$$

It is clear that, in order to make $vP(w)$ a perfect differential, whatever be the value of w, it is necessary and sufficient that v should satisfy

$$p(v) = 0,$$

a linear equation of order n, commonly called *Lagrange's adjoint equation*; and further that, if v is regarded as known, then a first integral of the equation $P(w)=0$ is given by

$$R(w, v) = \alpha,$$

α being an arbitrary constant, and R being a function manifestly linear in w and its derivatives.

Further, since

$$\int wp(v)\,dz = -R(w, v) + \int vP(w)\,dz,$$

it is clear that $wp(v)$ is a perfect differential if

$$P(w) = 0,$$

shewing that the original equation is the adjoint of the Lagrangian derived equation: or the two equations are reciprocally adjoint to one another.

Ex. Shew that, if $w_1, \ldots, w_n$ be a fundamental system of integrals of the equation $P(w)=0$, then a fundamental system of integrals of the adjoint equation $p(v)=0$ is given by

$$v_1, \ldots, v_n = \frac{1}{P_0} e^{-\int \frac{P_1}{P_0} dz} \left\| \begin{matrix} w_1^{(n-2)}, & w_2^{(n-2)}, & \ldots, & w_n^{(n-2)} \\ w_1^{(n-3)}, & w_2^{(n-3)}, & \ldots, & w_n^{(n-3)} \\ w_1' \quad, & w_2' \quad, & \ldots, & w_n' \\ w_1 \quad, & w_2 \quad, & \ldots, & w_n \end{matrix} \right\|.$$

Shew also that the product of the respective determinants of the two sets of fundamental integrals depends only upon P_0.

One immediate corollary can be inferred from the general result, in the case when the equation $P(w)=0$ is reducible. Suppose that

$$P(w) = P_1 P_2(w) = P_1(W),$$

say, where $W = P_2(w)$; then we have

$$\int vP(w)\,dz = \int vP_1(W)\,dz$$

$$= R_1(W, v) + \int W\overline{P}_1(v)\,dz,$$

where $\overline{P}_1$ is the adjoint of P_1, and R_1 is of order in W and in v one unit less than P_1. Again, writing

$$V = \overline{P}_1(v),$$

we have

$$\int W\overline{P}_1(v)\,dz = \int VP_2(w)\,dz$$

$$= R_2(V, v) + \int w\overline{P}_2(V)\,dz,$$

where $\overline{P}_2$ is the adjoint of P_2, and R_2 is of order in V and in v one unit less than P_2. Combining these results, we have

$$\int vP(w)\,dz = R_1(W, v) + R_2(V, v) + \int w\overline{P}_2(V)\,dz$$

$$= R(w, v) + \int w\overline{P}_2\overline{P}_1(v)\,dz,$$

where R is of order one unit less than P in w and in v. It follows that

$$\overline{P}_2\overline{P}_1(v) = 0$$

is the adjoint of

$$P(w) = P_1P_2(w) = 0,$$

where P_1, $\overline{P}_1$ are adjoint to one another, and likewise P_2, $\overline{P}_2$.

By repeated application of this result, we see that the adjoint of

$$P(w) = P_1P_2\ldots P_r(w) = 0$$

is given by

$$\overline{P}_r\overline{P}_{r-1}\ldots\overline{P}_2\overline{P}_1(v) = 0.$$

Hence *the adjoint of a composite equation is compounded of the adjoints of the factors taken in the reverse order.* Manifestly an equation and its adjoint are reducible together, or irreducible together.

The expression $R(w, v)$ is linear in the derivatives of w, up to order $n-1$ inclusive, and also in those of v, up to the same order: it may be called the *bilinear concomitant** of the two mutually adjoint equations.

For further formal developments in respect to adjoint equations and the significance of the bilinear concomitant, reference may be made to Frobenius†, Halphen‡, Dini§, Cels‖, and Darboux¶.

* *Begleitender bilinearer Differentialausdruck*, with Frobenius.

† *Crelle*, t. LXXXV (1878), pp. 185—213; references are given to other writers.

‡ *Liouville's Journal*, 4ᵉ Sér., t. I (1885), pp. 11—85.

§ *Ann. di Mat.*, 3ª Ser., t. II (1899), pp. 297—324, *ib.*, t. III (1899), pp. 125—183.

‖ *Ann. de l'Éc. Norm.*, 3ᵉ Sér., t. VIII (1891), pp. 341—415.

¶ *Théorie générale des surfaces*, t. II, pp. 99—121.

Ex. 1. Prove that, if a linear equation of the second order is self-adjoint, it is expressible in the form

$$\frac{d}{dz}\left(P\frac{dw}{dz}\right)+Qw=0;$$

that if a linear equation of the third order, in the form

$$\frac{d^3w}{dz^3}+3P\frac{d^2w}{dz^2}+3Q\frac{dw}{dz}+Rw=0,$$

is effectively the same as its adjoint equation, then

$$P=0, \quad R-\tfrac{3}{2}\frac{dQ}{dz}=0;$$

and find the conditions that a linear equation of the fourth order should be self-adjoint.

Ex. 2. Prove that, if the equations

$$g_0w^{(n)}+ng_1w^{(n-1)}+\frac{n(n-1)}{2!}g_2w^{(n-2)}+\ldots=0,$$

$$\gamma_0v^{(n)}+n\gamma_1v^{(n-1)}+\frac{n(n-1)}{2!}\gamma_2v^{(n-2)}+\ldots=0,$$

are adjoint to one another, then

$$\begin{array}{ll}
\gamma_0=\quad g_0, & g_0=\quad \gamma_0,\\
\gamma_1=-g_1+g_0', & g_1=-\gamma_1+\gamma_0',\\
\gamma_2=\quad g_2-2g_1'+g_0'', & g_2=\quad \gamma_2-2\gamma_1'+\gamma_0'',\\
\gamma_3=-g_3+3g_2'-3g_1''+g_0''', & g_3=-\gamma_3+3\gamma_2'-3\gamma_1''+\gamma_0''',\\
\multicolumn{2}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}
\end{array}$$

and obtain the expression of the bilinear concomitant. (Halphen.)

Ex. 3. Let $z_1, z_2, \ldots, z_n$ denote any n arbitrary functions of x, such that the determinant

$$Q=\begin{vmatrix} z_1, & \dfrac{dz_1}{dx}, & \ldots, & \dfrac{d^{n-1}z_1}{dx^{n-1}} \\ z_2, & \dfrac{dz_2}{dx}, & \ldots, & \dfrac{d^{n-1}z_2}{dx^{n-1}} \\ \ldots & \ldots & \ldots & \ldots \\ z_n, & \dfrac{dz_n}{dx}, & \ldots, & \dfrac{d^{n-1}z_n}{dx^{n-1}} \end{vmatrix}$$

does not vanish identically; and suppose that these functions of x are regular in a given region of the variable, as well as the coefficients a of the equation

$$a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_ny=X.$$

Further, let a set of quantities p be constructed according to the law

$$p_0=za_0, \quad p_1=za_1-\frac{dp_0}{dx}, \quad p_2=za_2-\frac{dp_1}{dx}, \quad \ldots, \quad p_n=za_n-\frac{dp_{n-1}}{dx};$$

and let the last of them be denoted by $-Z$, so that there are n functions Z corresponding to the n functions z. Shew that, if $Q(c)$ is the value of Q when the last column of the latter is replaced by constants $c_1, \ldots, c_n$, if $Q(x, x_1)$ is its value when the last column is similarly replaced by $z_1(x_1), z_2(x_1), \ldots, z_n(x_1)$, and if $\overline{Q}(x, x_1)$ is its value when the last column is similarly replaced by $Z_1(x_1), Z_2(x_1), \ldots, Z_n(x_1)$, then

$$y = \frac{(-1)^{n-1}}{a_0 Q}\left\{Q(c) + \int_a^x X(x_1)\, Q(x, x_1)\, dx_1 + \int_a^x y(x_1)\, \overline{Q}(x, x_1)\, dx_1\right\},$$

where a is a value of x within the given region and the constants c are determined in association with a.

Indicate the form of this result when $z_1, \ldots, z_n$ are a fundamental system of the equation, which is the adjoint of the left-hand side of the above equation.

Also shew how, in even the most general case, it can be used as a formula of recurrence to obtain an infinite converging series of integrals as an expression for y. (Dini.)

83. Consider an expression $P(w)$ and its Lagrangian adjoint $p(v)$, and let $R(w, v)$ denote their bilinear concomitant; then

$$vP(w) - wp(v) = \frac{d}{dz}\{R(w, v)\},$$

which holds for all values of v and w. Accordingly, let

$$w = z^{-\rho-s-1}, \quad v = z^{\rho},$$

where s is any integer; then

$$z^{\rho}P(z^{-\rho-s-1}) - z^{-\rho-s-1}p(z^{\rho}) = \frac{d}{dz}\{R(z^{-\rho-s-1}, z^{\rho})\}.$$

Now the left-hand side is a series of powers of z, having integers for indices; as it is equal to the right-hand side, which is the first derivative of a similar series of powers, the left-hand side must be devoid of a term in z^{-1}.

Let

$$P(z^{\tau}), \ = \Sigma f_{\mu}(\tau)\, z^{\tau+\mu},$$

be the characteristic function of $P(w)$; then the coefficient of z^{-1} in $z^{\rho}P(z^{-\rho-s-1})$ is $f_s(-\rho-s-1)$. Further, let

$$p(z^{\rho}), \ = \Sigma \phi_{\mu}(\rho)\, z^{\rho+\mu},$$

be the characteristic function of $p(v)$; then the coefficient of z^{-1} in $z^{-\rho-s-1}p(z^{\rho})$ is $\phi_s(\rho)$. Hence

$$\phi_s(\rho) = f_s(-\rho-s-1),$$

and therefore

$$f_s(\rho) = \phi_s(-\rho - s - 1);$$

so that, if

$$\Sigma f_\mu(\rho) z^{\rho+\mu}$$

be the characteristic function of a given equation, then

$$\Sigma f_\mu(-\rho - \mu - 1) z^{\rho+\mu}$$

is the characteristic function of the adjoint equation.

When $P(w)$ is in its normal form, all the coefficients $f_\mu(\rho)$ vanish for negative values of μ, but $f_0(\rho)$ is not zero. Hence $f_\mu(-\rho-\mu-1)$ vanishes for negative values of μ, but not $f_0(-\rho-1)$; and therefore the adjoint expression $p(v)$ is in its normal form. Moreover, their indicial functions $f_0(\rho)$, $\phi_0(\rho)$ are such that

$$f_0(\rho) = \phi_0(-\rho - 1), \quad \phi_0(\rho) = f_0(-\rho - 1),$$

so that they are of the same degree*, or the characteristic indices are the same. Hence *if an equation has all its integrals regular in the vicinity of a singularity, the adjoint equation also has all its integrals regular in the vicinity of that singularity;* for the characteristic index is then zero for the original equation, and it therefore is zero for the adjoint equation. Similarly, *if an equation has all its integrals non-regular in the vicinity of a singularity, the adjoint equation also has all its integrals non-regular in the vicinity of that singularity;* for the characteristic index is then equal to the order of the original equation, and it therefore is equal to the (same) order of the adjoint equation.

On the basis of these two results, we can obtain a descriptive condition necessary and sufficient to secure that, if a differential equation of order m has an indicial function of degree $m-n$, the number of its regular integrals is actually equal to $m-n$.

Let $P=0$ be the differential equation, with an indicial function of degree $m-n$. Let $R=0$ be the differential equation of order $m-n$, which has the aggregate of regular integrals of $P=0$ for its fundamental system; its indicial function is of degree $m-n$. Then (§ 75, IV) the equation $P=0$ can be expressed in the form

$$P = QR = 0,$$

* Thomé, *Crelle*, t. LXXV (1873), p. 276; Frobenius, *Crelle*, t. LXXX (1875), p. 320.

where Q is a differential operator of order n. Because the degrees of the indicial functions of P and R are equal to one another, it follows (from § 76) that the degree of the indicial function of Q is zero, that is, the indicial function of Q is a constant, and therefore (§ 77, Cor. I) the equation $Q=0$ has no regular integral.

Now construct the equations which are adjoint to $P=0$, $Q=0$, $R=0$ respectively; and denote them by $p=0$, $q=0$, $r=0$. Because R and r are adjoint, and because all the integrals of $R=0$ are regular, it follows that all the integrals of $r=0$ are regular; and conversely. Similarly, because Q and q are adjoint, and because $Q=0$ has no regular integral, it follows that $q=0$ has no regular integral; and conversely. Further, by § 82, we have

$$p=rq,$$

so that the equation adjoint to $P=0$ is

$$p=rq=0,$$

and this equation possesses all the integrals of $q=0$, an equation whose indicial function is a constant. Hence it is necessary that the equation adjoint to $P=0$ should possess all the integrals of an equation of order n, having a constant for its indicial function, if $P=0$ is to have $m-n$ linearly independent regular integrals.

But this descriptive condition is also sufficient to secure this result. For, as the condition is satisfied, we have

$$p=rq,$$

where the indicial function of q is a constant; hence, with the preceding notation, we also have

$$P=QR,$$

and the indicial function of Q is a constant. Accordingly, as the indicial function of P is of degree $m-n$, it follows (§ 76) that the indicial function of R is of degree $m-n$; and therefore (Ch. III), as the order of $R=0$ is $m-n$, all its integrals are regular. But $P=0$ possesses all the integrals of $R=0$; and therefore it has $m-n$ regular integrals.

We therefore infer the theorem:—

In order that an equation of order m, having an indicial function of degree $m-n$, may possess $m-n$ regular integrals, it is necessary

and sufficient that the adjoint equation should possess all the integrals of an equation of order n, having an indicial function which is a constant.

This result was first established by Frobenius*; and it may be compared with the corresponding result obtained by Floquet (§ 77). The special case, when $n=1$, had been previously discussed by Thomé†, who obtained the result that *an equation of order m, having an indicial function of degree* $m-1$, *possesses* $m-1$ *regular integrals, if the adjoint equation has an integral of the form*

$$e^{G\left(\frac{1}{x}\right)} \sum_{n=0}^{\infty} c_n x^{n+a},$$

where $G\left(\dfrac{1}{x}\right)$ *is a polynomial in* $\dfrac{1}{x}$, *and* a *is a constant.*

We shall not pursue this part of the formal theory of linear differential equations further: we refer students to the authorities already (§ 82) quoted, as well as to Thomé‡, Floquet§, and Grünfeld||.

* *Crelle*, t. LXXX (1875), pp. 331, 332.

† *Crelle*, t. LXXV (1873), pp. 278, 279.

‡ A summary of many of the memoirs upon linear differential equations by Thomé, published in Crelle's Journal, will be found in *Crelle*, t. XCVI (1884), pp. 185—281.

§ *Ann. de l'Éc. Norm. Sup.*, 2e Sér., t. VIII (1879), Supplément, p. 132.

|| *Crelle*, t. CXV (1895), pp. 328—342, *ib.*, t. CXVII (1897), pp. 273—290, *ib.*, t. CXXII (1900), pp. 43—52, 88.

CHAPTER VII.

Normal Integrals; Subnormal Integrals.

84. It is now necessary to consider those integrals of the differential equation in the vicinity of a singularity, which are not of the regular type. Suppose that such an integral, or a set of such integrals, is associated with a root θ of the fundamental equation (§ 13) of the singularity which, as in the last chapter, will be transformed to the origin by the substitution

$$z - a = x, \quad z = \frac{1}{x},$$

according as it is in the finite part of the plane, or at infinity. Let ρ denote any one of the values of

$$\frac{1}{2\pi i} \log \theta;$$

then it is known that an integral exists in the form

$$x^\rho \phi,$$

where ϕ is a uniform function of x in the vicinity of the origin. As this integral is not of the regular type, the function ϕ will contain an unlimited number of negative powers, so that the origin is an essential singularity of ϕ: in the case of the integrals considered earlier, the origin was either a pole or an ordinary point. Accordingly, when ϕ is expressed as a power-series, it will contain an unlimited number of negative powers: it may contain an unlimited number of positive powers also, and in that case it has the form of a Laurent series.

Classification of such integrals might be effected in accordance with a classification of essential singularities; but the discrimina-

tion that thus far has been effected among essential singularities is of a descriptive type*, and has not led to functions whose general expressions are characteristic of various classes of singularities. Accordingly, it is possible to choose one function after another with differing forms of essential singularity, and to construct (where practicable) the corresponding linear equations possessing integrals with the respective types of singularity: but there is no guarantee that such a process will lead to a complete enumeration.

There is one such function, however, which is simpler than any other, and yet is general of its class. It suffices for the complete integration of the linear equation of the first order when the origin is a pole of the coefficient; and an indication has been given (§ 81) that it may serve for the expression of an integral of an equation higher than the first. The equation of the first order may be taken to be

$$\frac{dy}{dx} + Py = 0,$$

where $x^{1+s}P$ is a holomorphic function, s being some positive integer. Let

$$P = \frac{sa_1}{x^{s+1}} + \frac{(s-1)\,a_2}{x^s} + \ldots + \frac{2a_{s-1}}{x^3} + \frac{a_s}{x^2} - \frac{\rho}{x} - I'(x),$$

where $I'(x)$ is a holomorphic function; then we easily have

$$y = e^{\Omega} x^{\rho} e^{I(x)} = e^{\Omega} x^{\rho} \psi(x),$$

where $\psi(x)$ is a holomorphic function of x, and

$$\Omega = \frac{a_1}{x^s} + \frac{a_2}{x^{s-1}} + \ldots + \frac{a_s}{x}.$$

It is clear that $x=0$ is an essential singularity of the integral; and also that we thus have the complete primitive of the equation of the first order.

It appeared, in § 81 and the example there discussed, that such an expression, if not in general, still in particular cases, can be an integral of an equation of higher order.

As all expressions of the form

$$e^{\Omega} x^{\rho} \psi(x),$$

* *T. F.*, § 88.

where Ω is a polynomial in $\frac{1}{x}$, possess the same generic type of essential singularity, we proceed to the consideration of equations that may possess integrals of this form. Such an integral is called* a *normal elementary integral* or (where no confusion will occur) simply *normal*. The quantity e^{Ω}, through the occurrence of which the point $x=0$ is an essential singularity, is called the *determining factor* of the integral; the other part of the integral, being $x^{\rho}\psi(x)$ where ψ is holomorphic, is of the type of a regular integral, and so the quantity ρ is called the *exponent* of the integral.

Construction of Normal Integrals.

85. We proceed, in the first place, to indicate Thomé's method† of obtaining such normal integrals as the equation

$$\frac{d^m w}{dx^m} + p_1 \frac{d^{m-1} w}{dx^{m-1}} + \ldots + p_{m-1}\frac{dw}{dx} + p_m w = 0$$

may possess. (The method gives no criteria as to the actual existence of normal integrals: and therefore, if any criteria are to be obtained for equations of order higher than the first, they must be investigated otherwise.) If a normal integral exists, it is of the form

$$w = e^{\Omega} u,$$

where Ω is a polynomial in $\frac{1}{x}$; and Ω is determined so that, if possible, the equation satisfied by u may possess at least one regular integral. Let

$$\frac{d^n e^{\Omega}}{dx^n} = e^{\Omega} t_n,$$

so that

$$t_0 = 1, \quad t_1 = \Omega', \quad t_{p+1} = t_p' + \Omega' t_p, \qquad (p = 1, 2, \ldots);$$

then

$$\frac{d^n w}{dx^n} = e^{\Omega}\left(t_n u + n t_{n-1}\frac{du}{dx} + \ldots + n t_1 \frac{d^{n-1}u}{dx^{n-1}} + t_0 \frac{d^n u}{dx^n}\right).$$

* Thomé, *Crelle*, t. xcv (1883), p. 75. Cayley, *ib.*, t. c (1887), p. 286, suggested the name *subregular*; but the name *normal* is that which has generally been adopted.

† *Crelle*, t. LXXVI (1873), p. 292.

When these quantities, for the successive values of n, are substituted in the differential equation for w, the determining factor e^{Ω} can be removed; and the differential equation for u then is

$$\frac{d^m u}{dx^m} + q_1 \frac{d^{m-1}u}{dx^{m-1}} + \dots + q_{m-1}\frac{du}{dx} + q_m u = 0,$$

where

$$q_r = \frac{m!}{r!(m-r)!}t_r + \frac{(m-1)!}{(r-1)!(m-r)!}p_1 t_{r-1} + \frac{(m-2)!}{(r-2)!(m-r)!}p_2 t_{r-2} + \dots$$
$$\dots + (m-r+1)p_{r-1}t_1 + p_r t_0,$$

for $r = 1, 2, \dots, m$.

If the original equation possesses a normal integral, then, after the proper determination of Ω, the differential equation for u will possess at least one regular integral: its characteristic index cannot then be greater than $m-1$, which (after the results in the preceding chapter) is a necessary but not a sufficient condition.

As Ω is a polynomial in x^{-1}, its form and degree being unknown, let its degree be $s-1$, so that $s \geqslant 2$; we then have for Ω' an expression of the form

$$\Omega' = \frac{a_2}{x^2} + \frac{a_3}{x^3} + \dots + \frac{a_s}{x^s}.$$

Hence in t_1, the governing term (that is, the term with highest negative exponent of x) is $\frac{a_s}{x^s}$; in t_2, it is $\frac{a_s^2}{x^{2s}}$; and so on, so that, in t_n, it is $\frac{a_s^n}{x^{ns}}$. As in § 73, let ϖ_κ denote the multiplicity of $x=0$ as a pole of p_κ; then in q_r, the governing exponents of its respective parts are

$$rs,\ \varpi_1 + (r-1)s,\ \varpi_2 + (r-2)s,\ \dots,\ \varpi_{r-1} + s,\ \varpi_r.$$

Thus the governing exponents in q_r are, so far as they go, less than those in q_{r+1} by s, and $s \geqslant 2$. Hence, in forming the characteristic index for the equation in u, for the purpose of determining whether it may possess a regular integral, the governing exponent in q_m is certainly greater by s than that in any other coefficient; the characteristic index is m, the indicial function is a constant, and the equation has no regular integral. But, thus far, Ω is quite arbitrary; and it may be possible, by proper choice of its constant

coefficients, to secure that a number of the terms in q_m with the greatest exponents of x^{-1} shall disappear. If by thus utilising the governing exponent and the constants in Ω', we can secure that the characteristic index of the equation in u is less than m, the indicial function ceases to be a constant and the equation may have a regular integral.

In order that the indicial function may not be a constant, the governing exponent of q_{m-1} must be less than that of q_m by unity at the utmost, or that of q_{m-2} must be less than that of q_m by two at the utmost, or (for some value of r) the governing exponent of q_{m-r} must be less than that of q_m by r at the utmost; whereas at the present moment, these diminutions are s, $2s$, rs respectively, where $s \geqslant 2$. Hence an initial necessity is that the $s-1$ terms in q_m with the highest exponents of x^{-1} shall vanish. Now

$$q_m = t_m + p_1 t_{m-1} + \ldots + p_{m-1} t_1 + p_m.$$

The $s-1$ terms in t_μ with the highest exponents of x^{-1} are the same as in Ω'^{μ}, because of the form of Ω' and because

$$t_\mu = t'_{\mu-1} + \Omega' t_{\mu-1},$$

(but not more than those $s-1$ terms are the same); hence the $s-1$ terms with the highest exponents of x^{-1}, say the first $s-1$ terms, in

$$\Omega'^m + p_1 \Omega'^{m-1} + \ldots + p_{m-1}\Omega' + p_m$$

must vanish.

86. To render this result attainable, it is necessary that the greatest exponent must not occur in only a single term of the preceding expression, for then the term could vanish only by having $a_s = 0$; the greatest exponent must occur in at least two terms. Consequently no one of the numbers

$$ms,\ \varpi_1 + (m-1)s,\ \varpi_2 + (m-2)s,\ \ldots,\ \varpi_m,$$

may be greater than all the rest, that is, no one of the numbers

$$0,\ \varpi_1 - s,\ \varpi_2 - 2s,\ \ldots,\ \varpi_m - ms,$$

may be greater than all the rest. Of the quantities

$$\varpi_1,\ \tfrac{1}{2}\varpi_2,\ \tfrac{1}{3}\varpi_3,\ \ldots,\ \frac{1}{m}\varpi_m,$$

let g be the greatest. Evidently g is greater than unity; for the original differential equation has not all its integrals regular, and so $\varpi_n > n$ for at least one value of n. Now s cannot be greater than g; for any such value would make all the integers in the series

$$0, \quad \varpi_1 - s, \quad \varpi_2 - 2s, \quad \ldots, \quad \varpi_m - ms,$$

negative except the first, that is, the first would then be greater than all the rest. Hence $s \leqslant g$: and $s \geqslant 2$, from the nature of the case.

I. *When $g < 2$, no value of s is possible;* and then there is no normal integral of the type indicated. Such a case arises for the equation

$$y'' + \frac{1}{x} py' + \frac{1}{x^3} qy = 0,$$

when p and q are holomorphic in the domain of $x = 0$ and neither vanishes when $x = 0$. The quantity g is the greater of $1, \frac{3}{2}$, that is, it is less than 2; so that there is no normal integral. Moreover, as the indicial function of the particular equation is a constant, it has no regular integral.

II. When g is an integer (necessarily greater than unity), we manifestly might take $s = g$. For two at least of the numbers

$$0, \quad \varpi_1 - s, \quad \varpi_2 - 2s, \quad \ldots, \quad \varpi_m - ms,$$

would then be equal to the greatest among them, which is zero; and then two at least of the numbers

$$ms, \quad \varpi_1 + (m-1)s, \quad \varpi_2 + (m-2)s, \quad \ldots, \quad \varpi_{m-1} + s, \quad \varpi_m,$$

would be equal to the greatest among them, one of these being ms.

More generally, let n be the characteristic index of the original equation, so that

$$\varpi_n + m - n \geqslant \varpi_\mu + m - \mu,$$

for all values of μ that are greater than n; then, adding $(m-n)(s-1)$ to each side of the inequality, we have

$$\varpi_n + (m-n)s \geqslant \varpi_\mu + (m-\mu)s + (\mu - n)(s-1),$$

where $\mu > n$. In the case of all these numbers, $(\mu - n)(s-1)$ is certainly positive; so that the first $s-1$ terms in our expression

are not affected by the quantities corresponding to $\varpi_\mu + (m-\mu)s$, and they can occur only through the quantities corresponding to

$$\varpi_\lambda + (m-\lambda)s,$$

for $\lambda = 0, 1, \ldots, n$, where $\varpi_0 = 0$, and n is the characteristic index of the original equation. We thus consider the first $s-1$ terms in

$$\Omega'^{m-n}(\Omega'^n + p_1\Omega'^{n-1} + p_2\Omega'^{n-2} + \ldots + p_{n-1}\Omega' + p_n);$$

and this holds for any value of s equal to or greater than two.

As regards g, which is the greatest among the quantities

$$\varpi_1, \ \tfrac{1}{2}\varpi_2, \ \tfrac{1}{3}\varpi_3, \ \ldots, \ \frac{1}{m}\varpi_m,$$

it occurs only among the first n, in the present circumstances; for it certainly is greater than unity and if any one of the last $m-n$, (say $\frac{1}{\mu}\varpi_\mu$ is the greatest of these last $m-n$), is greater than unity, then because

$$\varpi_n + m - n \geqslant \varpi_\mu + m - \mu,$$

we have

$$\frac{\varpi_n}{n} - \frac{\varpi_\mu}{\mu} \geqslant \left(\frac{\varpi_\mu}{\mu} - 1\right)\left(\frac{\mu}{n} - 1\right),$$

that is,

$$\frac{\varpi_n}{n} > \frac{\varpi_\mu}{\mu},$$

for μ is greater than n. Thus g does not occur in the last $m-n$ of the quantities, if one or more than one of them is greater than unity; and it certainly does not occur among them, if no one of them is greater than unity. Hence g is the greatest among the quantities

$$\varpi_1, \ \tfrac{1}{2}\varpi_2, \ \tfrac{1}{3}\varpi_3, \ \ldots, \ \frac{1}{n}\varpi_n.$$

It may occur several times in this set; let $\frac{1}{\kappa}\varpi_\kappa$ be the first occurrence, in passing from left to right, and $\frac{1}{r}\varpi_r$ be the last. Take first $s = g$; then we have

$$\varpi_\kappa + (m-\kappa)s = mg, \quad \varpi_r + (m-r)s = mg,$$

$$\varpi_\lambda + (m-\lambda)s < mg, \ \text{ if } \lambda < \kappa, \ \text{ or if } \lambda > r;$$

so that the highest terms of all, being those with index mg, occur in

$$\Omega'^n, \quad p_\kappa \Omega'^{n-\kappa} + \ldots + p_r \Omega'^{n-r}.$$

If then

$$p_\sigma = x^{-\varpi_\sigma}(c_\sigma + d_\sigma x + \ldots), \qquad (\sigma = 1, 2, \ldots),$$

the equation which determines a_g, the coefficient of x^{-g} in Ω', is

$$a_g{}^r + c_\kappa a_g{}^{r-\kappa} + \ldots + c_r = 0.$$

The remaining $g-2$ coefficients in Ω' are given by equating to zero the coefficients of the next $g-2$ terms in

$$\Omega'^n + p_1 \Omega'^{n-1} + \ldots + p_{n-1}\Omega' + p_n.$$

Each set of values of the coefficients determines a possible form of Ω' and therefore a possible form of determining factor. The number of sets, different from one another, is $\leqslant r$.

The preceding cases arise through $s = g$; but if g, being an integer, is greater than 2, other values of s, less than g, may be admissible. They can be selected as follows*. Mark the points

$$0,\ n;\ \varpi_1,\ n-1;\ \varpi_2,\ n-2;\ \ldots;\ \varpi_n,\ 0;$$

in a plane referred to two rectangular axes; and taking a line through the first of them parallel to the axis of x, make it swing round that point in a clockwise direction, until it meets one or more of the other points; then make it swing in the same direction round the last of these, until it meets one or more of the remaining points; and so on, until the line passes through the last of the points. There thus will be obtained a broken line, outside which none of the marked points can lie.

If a line be drawn through any of the points, say $\varpi_\kappa,\ n-\kappa$, at an inclination $\tan^{-1}\mu$ to the negative direction of the axis of y, its distance from the origin is

$$(1+\mu^2)^{-\frac{1}{2}}\{\varpi_\kappa + (n-\kappa)\mu\},$$

so that, for a given direction μ, the distance is proportional to

$$\varpi_\kappa + (n-\kappa)\mu.$$

It therefore follows that an appropriate value of s is given by any portion of the broken line, which is inclined at an angle $\tan^{-1}\mu$ to

* The method is due to Puiseux; see *T. F.*, § 96.

the negative direction of the axis of y, where μ is a positive integer, $\geqslant 2$: the value of s being

$$s = \mu.$$

As many values of s are admissible as there are portions of the broken line with inclinations $\tan^{-1}\mu$, where μ is a positive integer, which is $\geqslant 2$.

For each admissible value of s, arising from a portion of the broken line, the terms in

$$\Omega'^n + p_1\Omega'^{n-1} + \ldots + p_{n-1}\Omega' + p_n,$$

which correspond to the points on that portion, give the terms of highest negative power in x. If, for instance, a portion of line, having as its extremities the points corresponding to

$$p_r\Omega'^{n-r} \text{ and } p_t\Omega'^{n-t}, \qquad (t > r),$$

gives a value g' (necessarily an integer, as being a value of s), then the coefficient $a_{g'}$ satisfies an equation

$$c_r a_{g'}{}^{t-r} + \ldots + c_t = 0,$$

and the remaining $g' - 2$ coefficients in Ω' are obtained in the same manner as before. Each set of values of the coefficients determines a form of Ω' and therefore also a possible determining factor; and the number of sets different from one another is $\leqslant t - r$.

And so on, with each piece of broken line that provides an admissible value of s.

III. When the greatest of the quantities

$$\varpi_1, \ \tfrac{1}{2}\varpi_2, \ \tfrac{1}{3}\varpi_3, \ \ldots$$

is greater than 2 but is not an integer, we construct a tableau of points as in the preceding case, and draw the corresponding line. Only such values of s (if any) are admissible as arise from portions of the line, which are inclined at an angle $\tan^{-1}\mu$ to the negative part of the axis of y, μ being an integer $\geqslant 2$.

87. In every case, where a possible form of Ω' and thence a possible form of Ω have been obtained, we take

$$w = e^{\Omega} u.$$

If a normal integral of the original equation exists, the equation for u must possess a regular integral; and each regular integral of the latter determines a normal integral of the former having the determining factor e^{Ω}. An upper limit to the number of integrals thus obtainable is furnished by the degree of the indicial function of u; but the investigations of the last chapter shew that, when the degree of the indicial function is less than the order of the differential equation, the number of regular integrals may be less than the degree and might indeed be zero. The simplest mode of settling the matter is to take a series of the appropriate form, determined by the indicial function of the u-equation, substitute it in the differential equation, and decide whether the coefficients thence determined make the series converge. The normal integral exists or is illusory, according as the series converges or diverges.

When the normal integral exists, we say that it is of *grade* equal to the degree of Ω as a polynomial in x^{-1}.

SUBNORMAL INTEGRALS.

88. In the preceding investigation of normal integrals, it was essential that the number s should be an integer $\geqslant 2$: and accordingly, such values of μ, as were given by the Puiseux diagram and did not satisfy the condition, were rejected. But though they are ineligible for the construction of normal integrals, they may be subsidiary to the construction of other integrals.

Let μ denote such a quantity, given by the Puiseux diagram in the form of a positive magnitude that is not an integer: its source in the diagram makes it a rational fraction which, being expressed in its lowest terms, may be denoted by $h \div k$. The terms which, for this quantity as representing a possible degree for Ω', have the highest index of x^{-1} in

$$\Omega'^n + p_1\Omega'^{n-1} + \dots + p_{n-1}\Omega' + p_n,$$

are those which correspond to points on the portion of the line that gives the value of μ. Hence, taking

$$\Omega' = \frac{A}{x^{\mu}} + \dots,$$

an equation is obtained by making the aggregate coefficient of this term of highest order disappear; the equation determines A.

Now take a new independent variable ξ such that

$$x = \xi^k,$$

and make it the independent variable for the differential equation; the expression for $\frac{d\Omega}{dx}$ is

$$\frac{d\Omega}{dx} = \frac{A}{\xi^h} + \ldots,$$

so that

$$\frac{d\Omega}{d\xi} = kA\xi^{-(h-k-1)} + \ldots,$$

and therefore

$$\Omega = -\frac{k}{h-k} A\xi^{-(h-k)} + \ldots.$$

Thus Ω is infinite when $x = 0$, provided $h > k$, that is, for values of μ that are greater than unity. Accordingly, when we proceed to consider the differential equation with ξ as the variable, values of Ω of the preceding form can be obtained by the earlier method: in fact, we may obtain a normal integral of the equation in its new form, the conditions being that the equation for v, which results from the substitution

$$w = e^{\Omega} v,$$

shall have a regular integral or regular integrals. When once the value of k is known and the transformation from x to ξ has been effected, the remainder of the investigation is the same as for the construction of normal integrals of the untransformed equation.

Examples will be given later, shewing that such integrals do exist. As they are of a normal type in a variable $x^{\frac{1}{k}}$, where k is a positive integer, they may be called *subnormal**. Their existence appears to have been indicated first by Fabry†.

89. We have seen that, if g denote the greatest of the quantities

$$\varpi_1,\ \tfrac{1}{2}\varpi_2,\ \tfrac{1}{3}\varpi_3,\ \ldots,$$

* Poincaré, *Acta Math.*, t. VIII, p. 304, calls them *anormales*.

† *Sur les intégrales des équations différentielles linéaires à coefficients rationels*, (Thèse, 1885, Gauthier-Villars, Paris), Section IV.

and if the equation possesses a normal or a subnormal integral of the form

$$e^{\Omega} z^{\sigma} \phi(z),$$

then Ω' is a polynomial in z^{-1} (or in some root of z^{-1}) of order equal to or less than g; and therefore Ω is a polynomial in z^{-1} of order equal to or less than $g-1$. Let

$$g - 1 = R;$$

then R is called the *rank* of the differential equation for $z=0$.

When R is an integer, the grade of a normal integral may be equal to R: if not, it is less than R. When R is not an integer, let p denote the integer immediately less than R; the grade of a normal integral may be equal to p or may be less than p. When R is a fraction, equal to $\frac{k}{l}$ when in its lowest terms, then a subnormal integral may exist having a determining factor e^{Ω}, where Ω is a polynomial of degree k in $z^{-\frac{1}{l}}$; it will still be said to be of grade $\frac{k}{l}$ in z, that is, of grade R. All subnormal integrals are of grade R or of grade less than R.

Ex. Obtain the rank of the equation

$$\sum_{r=0}^{n} p_r \frac{d^{n-r}w}{dz^{n-r}} = 0$$

for $z=\infty$, the coefficients p being polynomials in z.

90. The converse proposition, due* to Poincaré, is true as follows:—

If n normal or subnormal functions are of grade equal to or less than R, and have the origin for an essential singularity, they satisfy a linear differential equation of order n and rank not greater than R for $z=0$.

Any n functions satisfy a linear differential equation of order n: in the present case, let it be

$$\frac{d^n w}{dz^n} + P_1 \frac{d^{n-1}w}{dz^{n-1}} + \ldots + P_n w = 0.$$

* *Acta Math.*, t. VIII (1886), p. 305: the form has been somewhat altered, so as to admit the discussion of normal and subnormal integrals together.

Let the normal and the subnormal functions be arranged in a sequence of descending grade: when so arranged, let them be

$$w_1 = e^{\Omega_1} z^{\sigma_1} u_1,\quad w_2 = e^{\Omega_2} z^{\sigma_2} u_2,\quad \ldots,\quad w_n = e^{\Omega_n} z^{\sigma_n} u_n,$$

so that, if $R_1, R_2, \ldots, R_n$ be their respective grades,

$$R \geqslant R_1 \geqslant R_2 \geqslant \ldots \geqslant R_{n-2} \geqslant R_{n-1} \geqslant R_n.$$

Now

$$P_r \Delta + \Delta_{n,r} = 0,$$

where Δ is the fundamental determinant of the n functions, viz.

$$\Delta = \begin{vmatrix} \dfrac{d^{n-1}w_1}{dz^{n-1}}, & \dfrac{d^{n-1}w_2}{dz^{n-1}}, & \ldots, & \dfrac{d^{n-1}w_n}{dz^{n-1}} \\ \dfrac{d^{n-2}w_1}{dz^{n-2}}, & \dfrac{d^{n-2}w_2}{dz^{n-2}}, & \ldots, & \dfrac{d^{n-2}w_n}{dz^{n-2}} \\ \cdots & \cdots & \cdots & \cdots \\ w_1, & w_2, & \ldots, & w_n \end{vmatrix},$$

and $\Delta_{n,r}$ is obtained from Δ by substituting the derivatives of order n for the derivatives of order $n-r$ in the rth row. The value of P_r is

$$P_r = -\frac{\Delta_{n,r}}{\Delta}.$$

In order to obtain the degree of $z=0$ as an infinity of P_r, it will be sufficient to consider only the governing terms in Δ and $\Delta_{n,r}$; and the degree is determined through the differences between the two sets of most important terms in the rth rows. Now if

$$w_p = e^{\Omega_p} z^{\sigma_p} u_p,$$

we have

$$\frac{d^q w_p}{dz^q} = z^{-q(R_p+1)} e^{\Omega_p} z^{\sigma_p} \phi_{q,p},$$

where $\phi_{q,p}$ is finite (but not zero) when $z=0$. We take out of the pth column a factor

$$e^{\Omega_p} z^{\sigma_p},$$

for each of the n values of p; we take out of the mth row a factor

$$z^{-m(R_1+1)},$$

for each of the n values of m; and then every constituent in the surviving determinants Δ and $\Delta_{n,r}$ is finite. The initial terms in

these constituents are the same for all the rows except the $(r-1)$th: the difference there is that $\alpha_1^n, \alpha_2^n, \ldots, \alpha_n^n$ occur in $\Delta'_{n,r}$, while $\alpha_1^{n-r}, \alpha_2^{n-r}, \ldots, \alpha_n^{n-r}$ occur in Δ', where $\Delta'_{n,r}$ and Δ' are the modified determinants, and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the coefficients of the governing terms in $\Omega_1, \Omega_2, \ldots, \Omega_n$. Accordingly, if

$$\Delta' \quad = A z^\theta + \ldots,$$

then

$$\Delta'_{n,r} = A' z^\theta + \ldots,$$

where the other indices are higher than θ, and A, A' are constants; and therefore

$$\Delta \quad = z^{-\frac{1}{2}n(n-1)(R_1+1)} e^{\Sigma \Omega_p} z^{\Sigma \sigma_p} \Delta',$$

$$\Delta_{n,r} = z^{-\frac{1}{2}n(n-1)(R_1+1)+r(R_1+1)} e^{\Sigma \Omega_p} z^{\Sigma \sigma_p} \Delta'_{n,r},$$

the summation in the exponents being for values $1, 2, \ldots, n$ of p. Hence

$$P_r = -z^{-r(R_1+1)} \frac{A' + \ldots}{A + \ldots}.$$

Now Δ, being the fundamental determinant, does not vanish identically: and as $z=0$ is an essential singularity, and not merely an apparent singularity, Δ does not vanish when $z=0$; thus A is not zero. It might happen that $A'=0$; but in any case, if ϖ_r denote the order of $z=0$ as a pole of the coefficient P_r, we have $\varpi_r \leqslant r(R_1+1)$. Thus the largest of the numbers $\frac{1}{r}\varpi_r$ is $\leqslant R_1 + 1 \leqslant R+1$; and therefore, for $z=0$, the rank of the equation $\leqslant R$, which proves the proposition.

When all the integrals are normal, which is the circumstance contemplated by Poincaré, the quantities R are integers and the determinants $\Delta', \Delta'_{n,r}$ are uniform: so that the coefficients P then are uniform functions of z. The coefficients P are uniform also when the aggregate of subnormal integrals is retained: the proof of this statement is left as an exercise.

NOTE. *An equation, which has a number of normal integrals, is reducible; so also is an equation, which has a number of subnormal integrals.*

By the preceding proposition, the aggregate of the normal integrals (or of the subnormal integrals) satisfies a linear equation with uniform coefficients, say $N=0$, of which they are a

fundamental system. Denoting the original equation by $P=0$, we can prove, exactly as in § 75, that P can be expressed in the form

$$P=QN,$$

where Q is an appropriate differential operator. In other words, P is reducible.

The investigation of the detailed conditions, imposed upon the form of P by the possibility of such reducibility, will not be attempted here.

Further, it must not be assumed (and it is not the fact) that reducible equations are limited to equations, which have regular, or normal, or subnormal integrals.

Ex. 1. Consider the equation

$$y'''+\frac{1}{x^3}py''+\frac{1}{x^5}qy'+\frac{1}{x^7}ry=0,$$

where p, q, r are holomorphic functions of x that do not vanish when $x=0$.

To investigate the possible kinds of determining factor, we form the tableau of points

$$0,\ 3;\quad 3,\ 2;\quad 5,\ 1;\quad 7,\ 0;$$

and then construct the broken line. There are two pieces: one gives $\mu=3$, the other $\mu=2$; the former joins the first two points; the last three lie on the latter. The possible expressions for Ω' are therefore

$$\Omega'=\frac{a}{z^3}+\frac{\beta}{z^2},\qquad \Omega'=\frac{\gamma}{z^2},$$

where a and β are uniquely determinate, and γ is the root of a quadratic equation.

Of course, the actual existence of normal integrals depends upon the actual forms of p, q, r.

Ex. 2. Shew that the equation

$$y'''+\frac{1}{x^2}py''+\frac{1}{x^5}qy'+\frac{1}{x^6}ry=0$$

where p, q, r are holomorphic functions of x that do not vanish when $x=0$, possesses no normal integrals in the vicinity of $x=0$: but that it may possess subnormal integrals.

Ex. 3. Consider the equation

$$(1+6x^3)\,y'''+\frac{3}{x}y''+\frac{3}{x^2}y'+\frac{1+12x^3}{x^6}y=0,$$

which has no regular integral, because the indicial function is a constant. The numbers ϖ_1, ϖ_2, ϖ_3 are 1, 2, 6; so that $g=2$, and we therefore take $s=2$, so that

$$\Omega'=-\frac{a}{x^2}.$$

We have to make the single $(s-1)$ highest power of x^{-1} vanish, in the expansion of

$$\Omega'^3+\frac{3\Omega'^2}{x(1+6x^3)}+\frac{3\Omega'}{x^2(1+6x^3)}+\frac{1+12x^3}{x^6(1+6x^3)}$$

in ascending powers of x; hence

$$a^3=1,$$

so that a is a cube root of unity, and

$$\Omega=\frac{a}{x}.$$

Accordingly, we write

$$y=e^{\frac{a}{x}}u;$$

after reduction, the equation satisfied by u is found to be

$$u'''-\frac{3u''(a-x+6ax^3)}{x^2(1+6x^3)}$$
$$+\frac{3u'(a^2+x^2+6a^2x^3+12ax^4)}{x^4(1+6x^3)}-\frac{3u(a^2+ax-2x^2+12a^2x^3+12ax^4)}{x^5(1+6x^3)}=0.$$

The indicial equation for $x=0$ is

$$a^2(\theta-1)=0,$$

which has a single root $\theta=1$; so that the u-equation possibly may possess a single regular integral which, if it exists, will belong to the exponent 1, and so will be of the form

$$u=x(c_0+c_1x+c_2x^2+\dots).$$

As a matter of fact, the u-equation is satisfied by

$$u=c_0(x+a^2x^2),$$

as may easily be verified; and thus the original equation possesses a normal integral

$$y=e^{\frac{a}{x}}(x+a^2x^2),$$

where a is a cube-root of unity. But a may be any one of the three cube roots of unity; and therefore the original equation in y possesses the three normal integrals

$$e^{\frac{1}{x}}(x+x^2),\quad e^{\frac{a}{x}}(x+a^2x^2),\quad e^{\frac{a^2}{x}}(x+ax),$$

where a is now an imaginary cube-root of unity.

The singularities of the equation given by $1+6x^3=0$ are only apparent (§ 45).

Ex. 4. Prove that the equation

$$x^2y'-(a+bx^2)y=0$$

has, in the vicinity of $x=\infty$, two linearly independent normal integrals, provided a is of the form $p(p+1)$, where p is an integer $\geqslant 0$; and obtain them.

Ex. 5. Prove that each of the equations

$$x^3y'' + 2xy' - y = 0,$$
$$x^4y'' + 2x^3y' - (a^2 + 2x^2)\, y = 0,$$

has, in the vicinity of $x=0$, two linearly independent normal integrals; and obtain them.

Ex. 6. Prove that the equation

$$y''' - \frac{6}{x^2}y' + y = 0$$

has, in the vicinity of $x=\infty$, three linearly independent normal integrals; and obtain them.

Ex. 7. Prove that the equation

$$4x^4y'' - (4 + 12x + 3x^2)\, y = 0$$

possesses one normal integral in the vicinity of $x=0$; and that one normal integral is illusory in that vicinity.

Ex. 8. Shew that the equation

$$(x+2)\, x^6y''' + (x^2 + 3x - 2)\, x^4y'' - (x+2)\, x^2y' - (3x^2 - 5x - 2)\, y = 0$$

possesses three normal integrals in the vicinity of $x=0$.

[They are $xe^{\frac{1}{x}}$, $xe^{-\frac{1}{x}}$, $xe^{-\frac{1}{x}}\log x$.]

Ex. 9. Prove that a solution of the equation

$$y'' + \left(a + 2\frac{\sigma+1}{x}\right)y' + \left\{b + \frac{c}{x} + \frac{\sigma(\sigma+1)}{x^2}\right\}y = 0$$

is expressed by

$$e^{\frac{1}{2}(n-a)x}x^{-\sigma}\left[1 + \tfrac{1}{2}\lambda nx + \ldots + \frac{\lambda(\lambda-1)\ldots(\lambda-k+1)}{(k+1)(k!)^2}(nx)^k + \ldots\right],$$

where

$$n^2 = a^2 - 4b, \quad n(\lambda+1) = a(\sigma+1) - c.$$

(Math. Trip., Part I, 1896.)

Hamburger's Equations.

91. The conditions, sufficient to secure that an equation, of order m and not of the Fuchsian type, shall have a regular integral, have not been set out in completely explicit form (§§ 78, 79); and consequently, the conditions sufficient to secure that such an equation shall have a normal integral have not been set out in explicit form. The foregoing examples (§ 90) afford

illustrations of the detailed process of settling such questions in individual instances; and the following investigation* gives the appropriate tests for a particular class of equations, which afford an illustration of the general method of proceeding.

We consider the equation

$$w'' = \frac{\alpha + \beta z + \gamma z^2}{z^4} w,$$

in which α must be different from zero (§ 86) if the equation is to possess a normal integral. For any integral that occurs, $z = 0$ is an essential singularity. For large values of z, the integrals are regular; and a fundamental system for $z = \infty$ is composed of two regular integrals, which belong to exponents $-\rho_1$ and $-\rho_2$ arising as roots of the quadratic equation

$$\rho(\rho - 1) = \gamma.$$

These two regular integrals may be denoted by

$$z^{\rho_1} P_1\left(\frac{1}{z}\right), \quad z^{\rho_2} P_2\left(\frac{1}{z}\right),$$

where P_1, P_2 are converging power-series. As the origin is the only other singularity of the equation (and it is an essential singularity), it follows that P_1 and P_2 have $z = 0$ for an essential singularity; all other points in the plane are ordinary points for P_1 and P_2.

The expression of a uniform function having only a single essential singularity, say the origin, and no accidental singularity, is known by Weierstrass's theorem† to be of the form

$$P\left(\frac{1}{z}\right) e^{g\left(\frac{1}{z}\right)},$$

where $P\left(\frac{1}{z}\right)$ is a uniform function having all the zeros of the original function (the simplest form of P being admissible), and $g\left(\frac{1}{z}\right)$ is a holomorphic function of $\frac{1}{z}$ which is finite everywhere except at $z = 0$.

* It is due to Hamburger, *Crelle*, t. CIII (1888), pp. 238—273.

† *T. F.*, § 52.

The function g may be polynomial or it may be transcendental; the discrimination depends upon the character of the origin as an essential singularity for the original function. As the present application is directed towards the determination of normal integrals, the function $g\left(\frac{1}{z}\right)$ will be taken to be a polynomial in $\frac{1}{z}$.

If the original function has an unlimited number of assigned zeros in the plane outside any small circle round the origin, P is transcendental. When the number of zeros is limited, $P\left(\frac{1}{z}\right)$ is a polynomial in $\frac{1}{z}$, which can be taken in the form

$$P\left(\frac{1}{z}\right) = z^{-k} f(z),$$

where k is a finite positive integer, f is a polynomial in z of degree not greater than k, its degree being actually k when $z = \infty$ is not a zero.

The equations to be considered are those which have integrals

$$z^{\rho_1} P_1\left(\frac{1}{z}\right), \quad z^{\rho_2} P_2\left(\frac{1}{z}\right),$$

as above, one (or both) of the functions P_1 and P_2 having only a limited number of zeros outside any small circle round the origin, with the further condition that the essential singularity at the origin is of the preceding type. Thus an integral is to be of the form

$$\begin{aligned} w &= e^{g\left(\frac{1}{z}\right)} z^{\rho - k} f(z) \\ &= e^{\Omega} z^{\sigma} f(z) = e^{\Omega} u, \end{aligned}$$

say, where Ω is a polynomial in $\frac{1}{z}$, the exponent σ is a constant, and $f(z)$ is a polynomial in z; and the differential equation for u is to have a regular integral which, except as to a factor z^{σ}, is to be a polynomial in z. Let

$$\Omega = \frac{a}{z} + \frac{a_1}{z^2} + \ldots + \frac{a_m}{z^m};$$

then the equation for u is

$$u'' + 2u'\Omega' + u(\Omega'^2 + \Omega'') = u\,\frac{\alpha + \beta z + \gamma z^2}{z^4}.$$

After the earlier explanations, it is clear that we must take

$$m = 1, \quad a^2 = \alpha.$$

The equation for u then is

$$u'' - \frac{2a}{z^2} u' + \frac{2a - \beta - \gamma z}{z^3} u = 0,$$

which is to have a regular integral of the type

$$\begin{aligned} u &= z^\sigma f(z) \\ &= z^\sigma (c_0 + c_1 z + \ldots + c_n z^n + \ldots), \end{aligned}$$

there being only a limited number of terms on the right-hand side. The indicial equation for $z = 0$ is

$$-2a\sigma + 2a - \beta = 0,$$

so that

$$\sigma = 1 - \frac{\beta}{2a}.$$

Substituting the expression for u, and equating coefficients, we have, after a slight reduction,

$$\begin{aligned} \{(n+\sigma)(n+\sigma-1) - \gamma\} c_n &= \{2a(n+\sigma) + \beta\} c_{n+1} \\ &= 2a(n+1) c_{n+1}; \end{aligned}$$

and therefore

$$c_{n+1} = \frac{(n+\sigma)(n+\sigma-1) - \gamma}{2a(n+1)} c_n.$$

It is clear that, if the series with the coefficients c were to be an infinite series, it would diverge and the integral would be illusory. For this reason also, as well as by the initial condition, all the coefficients from and after some definite one, say after c_k, must vanish; and therefore we must have

$$(k+\sigma)(k+\sigma-1) = \gamma,$$

or substituting for σ its value, we see that *the quadratic equation*

$$\left(\theta + 1 - \frac{\beta}{2a}\right)\left(\theta - \frac{\beta}{2a}\right) = \gamma,$$

where $a^2 = \alpha$, must have a positive integer (*or zero*) *for a root.* This condition is sufficient to secure the significance of the series, and therefore sufficient to secure the existence of a normal integral of the equation

$$w'' = \frac{\alpha + \beta z + \gamma z^2}{z^4} w.$$

Clearly, there are two values of a. If for either value the condition is satisfied, there is a normal integral of the form

$$e^{\frac{a}{z}} u_1,$$

where a has the value for which the condition is satisfied.

The condition cannot be satisfied for both values, if the values of σ are different, and arise from different values of ρ; for if it could, we should have

$$\sigma_1 + \sigma_2 = 1 - \frac{\beta}{2a} + 1 + \frac{\beta}{2a} = 2.$$

Now $\rho_1 + \rho_2 = 1$; and therefore

$$k_1 + k_2 = \rho_1 - \sigma_1 + \rho_2 - \sigma_2 = -1,$$

which is impossible, as neither k_1 nor k_2 is negative.

The condition can be satisfied for both values of a, if the values of σ are the same, that is, if

$$\beta = 0:$$

for then the condition, that the equation $(\theta + 1)\theta = \gamma$ can have a positive integer as a root, shews that the equation

$$w'' = \frac{a^2 + \gamma z^2}{z^4} w$$

possesses two normal integrals of the form

$$e^{\frac{a}{z}} z (c_0 + c_1 z + \ldots + c_\theta z^\theta),$$

$$e^{-\frac{a}{z}} z (c_0 - c_1 z + \ldots \pm c_\theta z^\theta).$$

The condition can be satisfied for both values of a, if the values of σ arise through the same value of ρ, whether they are the same or not; and the equation then possesses two normal integrals. The limitations on the constants are given in the first of the succeeding examples.

Ex. 1. Prove that the equation

$$w'' = \frac{a + \beta z + \gamma z^2}{z^4} w$$

possesses two normal integrals, if

$$\frac{\beta}{a^{\frac{1}{2}}} = q, \quad 4\gamma + 1 = p^2,$$

where q is any integer, positive, negative, or zero, and p is an integer that may not vanish. (Hamburger.)

Ex. 2. Obtain the conditions sufficient to secure that the equation

$$w''+2\frac{a+bz}{z^2}w'+\frac{\alpha+\beta z+\gamma z^2+\delta z^3+\epsilon z^4}{z^4}w=0$$

may have a normal integral of the foregoing type. Can it have two normal integrals?

Ex. 3. Prove that the equation

$$w''+\frac{a}{z}w'+\frac{b}{z^4}w=0$$

possesses two normal integrals, if a is an integer (positive, negative, or zero).

Ex. 4. Prove that the equation

$$w''=\frac{\alpha+\beta z^2+\gamma z^4}{z^6}w$$

possesses a normal integral if the quadratic equation

$$\left(\theta+\tfrac{3}{2}-\frac{\beta}{2\sqrt{a}}\right)\left(\theta+\tfrac{1}{2}-\frac{\beta}{2\sqrt{a}}\right)=\gamma$$

has a positive integer (or zero) for one of its roots for either value of $\sqrt{a}$. What happens (i) when both its roots are integers for the same value of $\sqrt{a}$, (ii) when, for each value of $\sqrt{a}$, the equation has a positive integer for a root?

Ex. 5. Prove that the equation

$$w''''-2n(n+1)\frac{w''}{z^2}+4n(n+1)\frac{w'}{z^3}+\left\{\frac{1}{z^4}n(n+1)(n+3)(n-2)+a^4\right\}w=0,$$

where n is an integer and a is any constant, has four normal integrals of the form

$$e^{az}\phi\left(\frac{1}{z}\right),$$

where $\phi\left(\frac{1}{z}\right)$ is a polynomial in $\frac{1}{z}$. (Halphen.)

92. In an earlier paper, Cayley* had proceeded in a different manner. If

$$w=z^\rho\phi(z),$$

where $\phi(z)$ is a holomorphic function of z not vanishing with z, we have

$$\frac{w'}{w}=\frac{\rho}{z}+\frac{\phi'(z)}{\phi(z)}$$

$$=\frac{\rho}{z}+R(z),$$

* *Crelle*, t. c (1887), pp. 286—295; *Coll. Math. Papers*, vol. XII, pp. 444—452.

where $R(z)$ is a holomorphic function of z in the vicinity of the origin. Further, if

$$w = e^{\Omega} z^{\rho} \phi(z),$$

where $\phi(z)$ is a holomorphic function of z not vanishing with z, and Ω is a polynomial in $\frac{1}{z}$, we have

$$\begin{aligned}\frac{w'}{w} &= \Omega' + \frac{\rho}{z} + \frac{\phi'(z)}{\phi(z)} \\ &= \frac{a_m}{z^m} + \ldots + \frac{a_2}{z^2} + \frac{\rho}{z} + R(z),\end{aligned}$$

say, where $R(z)$ is holomorphic in the vicinity of the origin. Cayley transformed the equation by the substitution

$$\frac{w'}{w} = y,$$

and then proceeded to obtain, from the differential equation for y, an expansion in ascending powers of z. When once a significant expression for y has been obtained, the value of w can immediately be deduced.

Applying this method to the equation

$$w'' = \frac{\alpha + \beta z + \gamma z^2}{z^4} w,$$

the equation for y is at once found to be

$$y' + y^2 = \frac{\alpha}{z^4} + \frac{\beta}{z^3} + \frac{\gamma}{z^2}.$$

Hamburger's investigation shews that the integrals of the equation in w are

$$w_1 = z^{\rho_1} P_1\left(\frac{1}{z}\right), \quad w_2 = z^{\rho_2} P_2\left(\frac{1}{z}\right),$$

which are valid over the whole plane but have $z = 0$ for an essential singularity. If an integral, say w_1, has an unlimited number of zeros, the origin being its only essential singularity, then* any circle round the origin, however small, contains an unlimited

* *T. F.*, §§ 32, 33.

number of these zeros: so that if, in the vicinity of the origin, the expression of w_1 is

$$w_1 = z^\rho \phi(z),$$

$\phi(z)$ would have an unlimited number of zeros within the small circle so drawn. The expression for y is

$$y = \frac{\rho}{z} + \frac{\phi'(z)}{\phi(z)};$$

but the function $\dfrac{\phi'(z)}{\phi(z)}$ has an unlimited number of poles in the immediate vicinity of the origin, and so the right-hand side cannot be changed into an expression of the form

$$\frac{a_m}{z^m} + \ldots + \frac{a_2}{z^2} + \frac{\rho}{z} + R(z),$$

where m is a finite integer. Accordingly, the assumed expansion is not valid in this case: and the method does not lead to significant results.

But when the integral has only a limited number of zeros, so that $\phi(z)$ is expressible in the form

$$\phi(z) = z^{-k} f(z)\, e^{g\left(\frac{1}{z}\right)},$$

in the vicinity of $z = 0$, where $g\left(\dfrac{1}{z}\right)$ is a polynomial in $\dfrac{1}{z}$ and $f(z)$ is a polynomial in z that does not vanish with z, then $\dfrac{\phi'(z)}{\phi(z)}$ can be changed into an expansion

$$\frac{a_m}{z^m} + \ldots + \frac{a_2}{z^2} - \frac{k}{z} + R(z),$$

and so the assumed expansion for y is valid in this case. The method therefore does then lead to a significant result*.

Assuming the method applicable, and returning to the equation

$$y' + y^2 = \frac{\alpha}{z^4} + \frac{\beta}{z^3} + \frac{\gamma}{z^2},$$

* The discrimination between the cases, and the explanation, are due to Hamburger, *Crelle*, t. CIII (1888), p. 242.

we easily find

$$y = \frac{a_0}{z^2} + \frac{a_1}{z} + a_2 + a_3 z + \dots,$$

where

$$a_0^2 = \alpha,$$

$$2a_1 a_0 - 2a_0 = \beta,$$

$$2a_2 a_0 + a_1^2 - a_1 = \gamma,$$

and, for any value n which is greater than 2,

$$2(a_n a_0 + a_{n-1} a_1 + \dots) + (n-3) a_{n-1} = 0.$$

If the constants in the equation were unconditioned, the coefficients thus determined would give a diverging series for y. But we are assuming that the method is applicable, so that the conditions for convergence are to be satisfied; and then, as

$$\frac{w'}{w} = \frac{a_0}{z^2} + \frac{a_1}{z} + a_2 + \dots,$$

we have

$$w = e^{-\frac{a_0}{z}} z^{a_1} (c_0 + c_1 z + \dots),$$

where the last series converges. The method does not, however, give the tests for convergence of the series for y, at least without elaborate calculation: still less does it indicate that the convergence of the series for y is bound up with the polynomial character of the series in the expression for w. It can therefore be regarded only as a descriptive method, capable of partly indicating the form of integral when such an integral exists: manifestly, it is not so effective as Hamburger's.

But the method, if thus limited in utility, has the advantage of indicating an entirely different kind of integrals of the original differential equation, which are in fact subnormal integrals, though it does not establish the existence of such integrals: for the latter purpose, other processes are necessary. It will be sufficient to consider an equation, say of the fourth order, in the form

$$w'''' + p_1 w''' + p_2 w'' + p_3 w' + p_4 w = 0,$$

where the origin is a pole of p_μ of multiplicity ϖ_μ, for $\mu = 1, 2, 3, 4$. Taking

$$\frac{w'}{w} = y,$$

we have

$$\frac{w''}{w} = y' + y^2,$$

$$\frac{w'''}{w} = y'' + 3yy' + y^3,$$

$$\frac{w''''}{w} = y''' + 4yy'' + 3y'^2 + 6y^2y' + y^4,$$

so that the equation for y is

$$y''' + 4yy'' + 3y'^2 + 6y^2y' + y^4 + p_1(y'' + 3yy' + y^3) \\ + p_2(y' + y^2) + p_3 y + p_4 = 0.$$

If this equation is satisfied by an expression of the form

$$y = z^{-m}(a_0 + a_1 z + \ldots),$$

the coefficient of the lowest power of z must vanish. Now the governing exponents for the terms in succession are

$$-m-3,\ -2m-2,\ -2m-2,\ -3m-1,\ -4m,$$
$$-\varpi_1 - m - 2,\ -\varpi_1 - 2m - 1,\ -\varpi_1 - 3m,$$
$$-\varpi_2 - m - 1,\ -\varpi_2 - 2m,$$
$$-\varpi_3 - m,$$
$$-\varpi_4.$$

To determine which groupings of terms will give the lowest power of z, we use a Puiseux diagram*; and in connection with each quantity $\varpi_\mu + km + l$, for the various values of μ, k, l, mark a point $(\varpi_\mu + l, k)$ referred to two rectangular axes Ox, Oy. Through the point (0, 4) take a line parallel to the axis Ox, and make it swing in a clockwise sense until it meets one or more of the points: round the last of the points then lying in its direction, make it continue to swing until it meets some other point or points; and so on, until it passes through the point $(\varpi_4, 0)$. A broken line is thus obtained; the inclination of any portion to the negative direction of the axis Oy being $\tan^{-1}\mu$, the quantity μ is a possible value of m, and the terms giving rise to the lowest index of z in the differential equation for y are those which correspond to the points on that portion of the line. There are as many possible values of m thus suggested as there are portions of the line.

* See vol. II of this work, ch. V, *passim*.

It is not, however, a necessity of a Puiseux diagram that only integer values of m shall thus be provided: and it does, in fact, frequently happen that rational fractional values arise. Let such an one be $\frac{r}{s}$, where r and s are prime to each other; and take

$$z = \zeta^s,$$

so that

$$y = \zeta^r (a_0 + a_1 \zeta^s + \ldots).$$

When the independent variable is changed from z to ζ, an expression for y of this type can be constructed, and it will be a formal solution of the equation; if the series for y converges, then such an integral exists, expressed in the form of a series of fractional powers, and a corresponding integral w will be deducible. Such an integral, when it exists, is a subnormal integral.

It is easy to verify that the only points, which need be marked in the diagram for the purpose of obtaining the possible values of m, are those which correspond with the quantities

$$4m, \quad \varpi_1 + 3m, \quad \varpi_2 + 2m, \quad \varpi_3 + m, \quad \varpi_4,$$

as in § 86; but fractional values of m are now admissible in every case, instead of being so only under conditions as in the former use of the diagram.

Ex. 1. This indication of integrals in a series of fractional powers was applied by Cayley and Hamburger, in the memoirs already cited, to the equation*

$$w'' = \left(\frac{\beta'}{z^3} + \frac{\gamma'}{z^2}\right) w,$$

which possesses neither a regular integral nor a normal integral in the vicinity of $z=0$.

The only points to be marked for the Puiseux diagram are 0, 2; 3, 0; there is one portion of line, and it gives

$$m = \tfrac{3}{2}.$$

Accordingly, we take

$$z = \zeta^2;$$

and the equation for w then becomes

$$\frac{d^2 w}{d\zeta^2} - \frac{1}{\zeta}\frac{dw}{d\zeta} = w\left(\frac{4\beta'}{\zeta^4} + \frac{4\gamma'}{\zeta^2}\right),$$

or, writing

$$w = \zeta^{\frac{1}{2}} W,$$

* This equation is used only for purposes of illustration; its integrals are regular in the vicinity of $z=\infty$.

we have

$$\frac{d^2 W}{d\zeta^2} = W\left(\frac{4\beta'}{\zeta^4} + \frac{4\gamma' + \frac{3}{4}}{\zeta^2}\right),$$

which is a special form of the earlier equation in § 91. It possesses two integrals, normal in ζ, if the quadratic

$$\theta(\theta+1) = 4\gamma' + \tfrac{3}{4}$$

has one of its roots an integer, that is, if

$$\gamma' = \tfrac{1}{16}(2\theta - 1)(2\theta + 3),$$

where θ is any positive integer (or zero).

To find the integrals, we have merely to adapt the solution in § 91, by taking

$$a = 4\beta', \quad \beta = 0, \quad \gamma = 4\gamma' + \tfrac{3}{4} = \theta(\theta+1).$$

Thus $\alpha = a^{\frac{1}{2}} = 2\beta'^{\frac{1}{2}}$, $\sigma = 1$, and

$$\begin{aligned} 4\beta'^{\frac{1}{2}}(n+1)\, c_{n+1} &= \{(n+1)\, n - \theta\,(\theta+1)\}\, c_n \\ &= (n-\theta)(n+\theta+1)\, c_n\,; \end{aligned}$$

and so, taking $c_0 = 1$, we have

$$W = e^{\frac{2\beta'^{\frac{1}{2}}}{\zeta}}\, \zeta \sum_{n=0}^{\theta} \left(-\frac{1}{4\beta'^{\frac{1}{2}}}\right)^n \frac{(\theta+n)!}{n!\,(\theta-n)!}\, \zeta^n$$

as a normal integral of the equation in ζ. Accordingly, the equation

$$w'' = \frac{w}{z^3}\left\{a + \frac{z}{16}(2\theta-1)(2\theta+3)\right\},$$

where θ is a positive integer or zero, and a is a constant, has an integral

$$w = e^{2a^{\frac{1}{2}} z^{-\frac{1}{2}}}\, z^{\frac{3}{4}} \sum_{n=0}^{\theta} \left\{\left(-\frac{1}{4a^{\frac{1}{2}}}\right)^n \frac{(\theta+n)!}{n!\,(\theta-n)!}\, z^{\frac{1}{2}n}\right\}.$$

Manifestly, the other integral is given by

$$w = e^{-2a^{\frac{1}{2}} z^{-\frac{1}{2}}}\, z^{\frac{3}{4}} \sum_{n=0}^{\theta} \left\{\left(\frac{1}{4a^{\frac{1}{2}}}\right)^n \frac{(\theta+n)!}{n!\,(\theta-n)!}\, z^{\frac{1}{2}n}\right\},$$

the two constituting a fundamental system. Each of them is of the type of normal integral: but the series proceed in fractional powers of the variable.

It will be noted that the two values of σ are the same, and that only one value of ρ is used; the relation is

$$\rho = \sigma + \theta = 1 + \theta.$$

Ex. 2. Prove that the equation

$$v'' + \frac{\mu}{z} v' + \frac{\lambda}{z^3} v = 0,$$

where λ is a constant and 2μ is an odd integer, positive or negative, possesses two subnormal integrals.

Equations of Higher Order having Normal or Subnormal Integrals.

93. There is manifestly no reason why Hamburger's method should be restricted to equations of the second order; and he has applied it to obtain the corresponding class of equations of general order, the properties of the integrals defining the class being

(i) the integrals are of the regular type in the domain of $z = \infty$;

(ii) the origin is an essential singularity for each of the integrals, and at least one of the integrals must be of the normal type in the vicinity of $z = 0$;

(iii) all the points, except $z = 0$ and $z = \infty$, are ordinary points of the integrals and the equation;

(iv) the number of zeros of at least one integral, which lie outside any small circle round the origin, is limited;

the second and the fourth of which are not entirely independent.

Let the equation be of order n, and have its coefficients rational. The first of this set of properties requires the equation to be of the form

$$z^n \frac{d^n w}{dz^n} + z^{n-1} p_1 \frac{d^{n-1} w}{dz^{n-1}} + \ldots + z p_{n-1} \frac{dw}{dz} + p_n w = 0,$$

where $p_1, p_2, \ldots, p_n$ are holomorphic functions of z for large values of z, and thus are expressible in series of powers of z of the form

$$a_\mu + b_\mu \frac{1}{z} + c_\mu \frac{1}{z^2} + \ldots \qquad (\mu = 1, \ldots, n).$$

The third of the above set of properties requires that every value of z, except $z = 0$, shall be an ordinary point for each of the coefficients: and by the second of the properties, $z = 0$ is a singularity of the equation and therefore of some of the coefficients. Accordingly, the power-series for the coefficients p, which have been taken to be rational and are limited so that every point except $z = 0$ is ordinary for them, are polynomials in z^{-1}.

As the integrals are regular in the vicinity of $z=\infty$, one at least is of the form

$$w = z^{\rho} Q\left(\frac{1}{z}\right),$$

where Q is a series of powers of z^{-1}, which does not vanish when $z=\infty$ and converges for all values of z outside an infinitesimal circle round the origin, and where ρ is a root of the equation

$$\rho(\rho-1)\ldots(\rho-n+1)+a_1\rho(\rho-1)\ldots(\rho-n+2)+\ldots$$
$$\ldots+a_{n-1}\rho+a_n=0,$$

the indicial equation for $z=\infty$. The exponents to which the integrals belong, being regular in the vicinity of $z=\infty$, are the roots of this equation with their signs changed; and they exist in groups or are isolated, according to the character of the roots. Let the above integral be one which, under the second of the set of properties, is a normal integral in the vicinity of $z=0$, necessarily an essential singularity; in that vicinity, its expression is of the type

$$w = e^{\Omega} z^{\sigma} R(z),$$

where $R(z)$ is a function of z, which is holomorphic in the vicinity of $z=0$ and does not vanish when $z=0$, and where Ω is a polynomial in z^{-1}, say

$$\Omega = -\frac{1}{m}\frac{\alpha_m}{z^m} - \frac{1}{m-1}\frac{\alpha_{m-1}}{z^{m-1}} - \ldots - \frac{\alpha_1}{z},$$

and σ is a constant. Then, in the vicinity of $z=0$, we have

$$\begin{aligned}\frac{w'}{w} &= \frac{\alpha_m}{z^{m+1}} + \frac{\alpha_{m-1}}{z^m} + \ldots + \frac{\alpha_1}{z^2} + \frac{\sigma}{z} + \frac{R'(z)}{R(z)}\\ &= \Upsilon + R_1,\end{aligned}$$

where Υ is a polynomial in $\frac{1}{z}$, constituted by $\Omega' + \frac{\sigma}{z}$, and R_1 is the holomorphic function of z given by $R'(z) \div R(z)$. But as this arises through a form of the integral, postulated for the vicinity of $z=0$, while the integral is actually known to be

$$z^{\rho} Q\left(\frac{1}{z}\right),$$

the above form for w'/w must be deducible from this actual value. This is possible only if $Q\left(\frac{1}{z}\right)$, which has $z=0$ for an essential singularity, possesses at the utmost a limited number of zeros outside an infinitesimal circle round the origin; for if it had an unlimited number of zeros in the plane, other than $z=0$, any circle round the origin, however small, would include an infinite number, and then

$$\frac{1}{z^2}\frac{Q'\left(\frac{1}{z}\right)}{Q\left(\frac{1}{z}\right)}$$

would be incapable of such an expansion. The requirement, that thus arises, has been anticipated by the assignment of the fourth among the set of properties of the integrals; and so we may assume $Q\left(\frac{1}{z}\right)$ to have only a limited number of zeros. Accordingly, as in § 91, the form of $Q\left(\frac{1}{z}\right)$ must be

$$P\left(\frac{1}{z}\right)e^{g\left(\frac{1}{z}\right)},$$

where $P\left(\frac{1}{z}\right)$ is a polynomial in $\frac{1}{z}$ having as its roots all the zeros of $Q\left(\frac{1}{z}\right)$, and $g\left(\frac{1}{z}\right)$ is a holomorphic function of $\frac{1}{z}$, finite everywhere except at $z=0$.

Let k be the number of zeros of Q; then $P\left(\frac{1}{z}\right)$ is a polynomial of degree k, and so it can be represented in the form

$$z^{-k}G(z),$$

where $G(z)$ is a polynomial in z of degree k. Thus the integral is of the form

$$z^{\rho-k}G(z)e^{g\left(\frac{1}{z}\right)}.$$

The postulated form must agree with this form; hence $g\left(\frac{1}{z}\right)$ is the polynomial Ω of that form, and the holomorphic function $R(z)$ of that form is the polynomial $G(z)$: also

$$\sigma=\rho-k.$$

The expression for w'/w in ascending powers of z is thus valid, under the conditions assigned, provided $R(z)$ is a polynomial in z. Taking

$$\Upsilon + R_1 = z^{-m-1}P_1,$$

so that P_1 is a function of z, which is holomorphic in the vicinity of $z=0$ and is equal to α_m when $z=0$, we have

$$\frac{w'}{w} = z^{-m-1}P_1.$$

Then

$$\begin{aligned}\frac{w''}{w} &= \left(\frac{w'}{w}\right)^2 + \frac{d}{dz}\left(\frac{w'}{w}\right)\\ &= z^{-2m-2}(P_1{}^2 + z^m Q_1) = z^{-2m-2}P_2,\end{aligned}$$

say. Similarly,

$$\frac{w'''}{w} = z^{-3m-3}(P_1{}^3 + z^m Q_2) = z^{-3m-3}P_3,$$

say, and so on: where all the functions $P_2, P_3, \ldots, Q_1, Q_2, \ldots$ are holomorphic functions of z, and the first m terms in P_κ arise from $P_1{}^\kappa$. Substituting in the equation

$$z^n\frac{d^nw}{dz^n} + z^{n-1}p_1\frac{d^{n-1}w}{dz^{n-1}} + \ldots + zp_{n-1}\frac{dw}{dz} + p_nw = 0,$$

we have

$$P_n + z^m p_1 P_{n-1} + z^{2m}p_2P_{n-2} + \ldots + z^{nm}p_n = 0,$$

which must be identically satisfied. The coefficients p are polynomials in $\frac{1}{z}$; hence*

$$z^{\kappa m}p_\kappa$$

is expressible as a polynomial in z, and so the highest negative power in p_κ is $z^{-\kappa m}$ at the utmost. Accordingly, let

$$p_\kappa = a_{\kappa 0} + \frac{a_{\kappa,1}}{z} + \frac{a_{\kappa,2}}{z^2} + \ldots + \frac{a_{\kappa\ \kappa m}}{z^{\kappa m}},$$

for $\kappa = 1, \ldots, m$.

Now we have

$$P_1 = \alpha_m + \alpha_{m-1}z + \ldots + \alpha_1 z^{m-1} + z^mT = v + z^mT,$$

* If this were not the case, the assignment of a larger value of m could secure it: and so the assumption really is no limitation beyond that which is necessary for a normal integral, viz. m must be a finite integer.

say, where T is a holomorphic function of z; and

$$\begin{aligned} P_\mu &= P_1{}^\mu + z^m Q_{\mu-1} \\ &= v^\mu + z^m T_{\mu-1}, \end{aligned}$$

where $T_{\mu-1}$ is a holomorphic function of z; so that the first m terms in P_1, which give all the coefficients in the exponent of the determining factor e^Ω, are given as the first m terms of a root of the equation

$$v^n + z^m p_1 v^{n-1} + z^{2m} p_2 v^{n-2} + \ldots + z^{nm} p_n = 0,$$

when the root is expanded in ascending powers of z. When the first m terms in v are obtained, then the determining factor is known; for we have

$$\Omega = \int_\infty^z x^{-m-1} v\, dx.$$

Moreover, after this determination, the terms involving the powers $z^0, z^1, \ldots, z^{m-1}$ in

$$P_n + z^m p_1 P_{n-1} + z^{2m} p_2 P_{n-2} + \ldots + z^{nm} p_n$$

have disappeared, so that this quantity is divisible by z^m, leaving a holomorphic function of z as the quotient.

94. Having obtained the determining factor, let

$$w = e^\Omega u$$

be substituted in the differential equation, which can now be taken in the form

$$z^{mn+n} \frac{d^n w}{dz^n} + \sum_{r=1}^{n} \left\{ (z^{rm} p_r)\, z^{(n-r)(m+1)} \frac{d^{n-r} w}{dz^{n-r}} \right\} = 0.$$

For this purpose, derivatives of e^Ω are required. We have

$$e^{-\Omega} \frac{d}{dz} e^\Omega = \frac{v}{z^{m+1}};$$

let

$$e^{-\Omega} \frac{d^2}{dz^2} e^\Omega = \frac{v_1}{z^{2m+2}},$$

$$e^{-\Omega} \frac{d^3}{dz^3} e^\Omega = \frac{v_2}{z^{3m+3}},$$

and so on, where v is identical with the first m terms of P_1, v_1 is identical with the first m terms of P_2, and generally, v_λ is identical with the first m terms of $P_{\lambda+1}$. Now

$$\frac{d^\lambda w}{dz^\lambda} = e^\Omega \sum_{\kappa=0}^{\lambda} \left\{\frac{\lambda\,!}{\kappa\,!\,(\lambda-\kappa)\,!}\,\frac{v_{\kappa-1}}{z^{\kappa m+\kappa}}\,\frac{d^{\lambda-\kappa}u}{dz^{\lambda-\kappa}}\right\},$$

with the convention $v_0 = v$, $v_{-1} = 1$; and therefore the equation for u, after dropping the factor e^Ω, is

$$\sum_{r=0}^{n}\sum_{\kappa=0}^{n-r} \frac{(n-r)\,!}{\kappa\,!\,(n-r-\kappa)\,!}\, v_{\kappa-1}\, z^{rm} p_r\, z^{(n-r-\kappa)\,(m+1)}\, \frac{d^{n-r-\kappa}u}{dz^{n-r-\kappa}} = 0,$$

which can be written in the form

$$\sum_{s=0}^{n}\sum_{r=0}^{s} \left\{\frac{(n-r)\,!}{(n-s)\,!\,(s-r)\,!}\, v_{s-r-1}\, z^{rm} p_r\right\} z^{(n-s)\,(m+1)}\, \frac{d^{n-s}u}{dz^{n-s}} = 0,$$

where $p_0 = 1$. The coefficient of u is

$$= \sum_{r=0}^{n} v_{n-r-1} z^{rm} p_r$$

$$= v_{n-1} + z^m p_1 v_{n-2} + z^{2m} p_2 v_{n-3} + \ldots + v z^{(n-1)\,m} p_{n-1} + z^{nm} p_n.$$

Because the first m terms in $v_{\lambda-1}$ are the same as in P_λ, the first m terms in the preceding coefficient are the same as in

$$P_n + z^m p_1 P_{n-1} + \ldots + z^{nm} p_n,$$

and they are known to vanish, for the coefficients of $z^0, z^1, \ldots, z^{m-1}$ were made zero to determine v; hence the preceding coefficient is divisible by z^m, so that we can take

$$v_{n-1} + z^m p_1 v_{n-2} + \ldots + z^{nm} p_n = z^m (\theta_0 + \theta_1 z + \ldots),$$

where θ_0 is a determinate constant, because v is known.

The coefficient of $z^{m+1}\dfrac{du}{dz}$ is

$$= \sum_{r=0}^{n-1} (n-r)\, v_{n-r-2}\, z^{rm} p_r$$

$$= n v_{n-2} + (n-1)\, v_{n-3}\, z^m p_1 + \ldots + 2 v z^{(n-2)\,m} p_{n-2} + z^{(n-1)\,m} p_{n-1}.$$

The first m terms here are the same as the first m terms in

$$nP_{n-1} + (n-1)\, z^m p_1 P_{n-2} + \ldots + 2P_1 z^{(n-2)\,m} p_{n-2} + z^{(n-1)\,m} p_{n-1},$$

that is, the same as the first m terms in

$$n v^{n-1} + (n-1)\, v^{n-2} z^m p_1 + \ldots + 2 v z^{(n-2)\,m} p_{n-2} + z^{(n-1)\,m} p_{n-1}$$

The equation for v is

$$v^n + z^m p_1 v^{n-1} + z^{2m} p_2 v^{n-2} + \ldots + z^{nm} p_n = 0\,;$$

and, in particular, the equation determining α_m, the constant term in v, is

$$\alpha_m{}^n + \alpha_m{}^{n-1} a_{1,m} + \alpha_m{}^{n-2} a_{2,2m} + \ldots + a_{n,nm} = 0,$$

giving n values of α_m.

95. Let α_m denote a simple root of this equation, sometimes called the *characteristic* equation: then the quantity

$$n\alpha_m{}^{n-1} + (n-1)\,\alpha_m{}^{n-2} a_{1,m} + \ldots + a_{n-1,nm-m}$$

is not zero. The coefficient therefore of $z^{m+1}\dfrac{du}{dz}$, as given above, does not vanish when $z = 0$: let it be

$$\eta_0 + \eta_1 z + \ldots,$$

where η_0 is a determinate constant, because v is known.

It follows that the equation for u, in the form as obtained, is divisible throughout by z^m. Further, if it possesses (as, for the class of equations under consideration, it must possess) a regular integral, and if that regular integral belongs to the exponent σ, then σ is given by the indicial equation

$$\eta_0\sigma + \theta_0 = 0,$$

so that σ can now be regarded as a known constant.

Further, we had

$$\sigma = \rho - k,$$

where k is a positive integer (or zero), and ρ is a root of the equation

$$\rho(\rho-1)\ldots(\rho-n+1) + \rho(\rho-1)\ldots(\rho-n+2)\,a_{10} \\ + \rho(\rho-1)\ldots(\rho-n+3)\,a_{20} + \ldots + \rho a_{n-1,0} + a_{n0} = 0,$$

say, of

$$I(\rho) = 0.$$

Consequently, *the equation*

$$I\left(k - \frac{\theta_0}{\eta_0}\right) = 0,$$

regarded as an equation in k, must have at least one root equal to a positive integer or zero: if this root be denoted by κ, one condition that u should be of the form

$$u = z^{\sigma}(c_0 + c_1 z + \dots + c_\kappa z^\kappa)$$

(which is the form for u required by the earlier argument) *is satisfied.*

But while the condition is necessary, it is not sufficient for the purpose. When the value of u is substituted in the equation, the latter must be identically satisfied; and so we have relations among the coefficients c. The general relation is

$$I(\sigma + \alpha) c_\alpha + g_1(\alpha) c_{\alpha+1} + g_2(\alpha) c_{\alpha+2} + \dots + g_{mn-m}(\alpha) c_{\alpha+mn-m} = 0;$$

the relations for the first few coefficients are of a simpler form. When these relations are solved, so as to give successively the ratios of $c_1, c_2, \dots$ to c_0, a formal expression for u is obtained. In this formal expression, all the coefficients $c_{\kappa+1}, c_{\kappa+2}, \dots$ are to vanish; that this may be the case, we must (as in § 79) have

$$I(\sigma + \kappa) c_\kappa = 0,$$
$$I(\sigma + \kappa - 1) c_{\kappa-1} + g_1(\kappa - 1) c_\kappa = 0,$$
$$I(\sigma + \kappa - 2) c_{\kappa-2} + g_1(\kappa - 2) c_{\kappa-1} + g_2(\kappa - 2) c_\kappa = 0,$$

and so on, being $m(n-1)$ relations in all. Of these, the first is known to be satisfied as above; it is the first condition for the existence of u in the specified form. The aggregate of conditions is sufficient, as well as necessary: the last of them secures that $c_{\kappa+1}$ vanishes, the last but one secures that $c_{\kappa+2}$ vanishes, and so on: the first secures that $c_{\kappa+mn-m}$ vanishes; and then, in virtue of the general difference-relation among the constants c, every succeeding coefficient vanishes.

Thus when the $m(n-1)$ conditions are satisfied, in association with a simple root of the equation

$$\alpha_m{}^n + \alpha_m{}^{n-1} a_{1,1m} + \dots + a_{n,nm} = 0,$$

a normal integral of the original equation exists.

It may happen that the conditions are satisfied for more than one of the simple roots of the equation: then there will be a corresponding number of normal integrals of the equation.

The extreme case would be that in which every root of the equation

$$\alpha_m^n + \alpha_m^{n-1} a_{1,1m} + \ldots + a_{n,nm} = 0$$

is simple and the conditions are satisfied for each of the roots: there then would be n normal integrals. Let the n roots be denoted by $\theta_1, \theta_2, \ldots, \theta_n$, so that, if

$$\Omega_r = -\frac{1}{m}\frac{\theta_r}{z^m} + \ldots + \frac{\alpha_r}{z},$$

the normal integrals will be of the form

$$e^{\Omega_r} z^{\sigma_r} u_r,$$

where u_r is a polynomial, say of degree κ_r, in z. We have $\sigma_r = \rho - \kappa_r$; when these n indices σ_r are associated with n quantities ρ, it follows that

$$\rho_r = \sigma_r + \kappa_r,$$

for $r = 1, \ldots, n$. The distinct quantities ρ_r are the roots of $I(\rho) = 0$, so that, if they are all different from one another, we have n of them; also

$$\sum_{r=1}^{n} (\sigma_r + \kappa_r) = \tfrac{1}{2} n (n-1) - a_{10}.$$

The value of $\sum\limits_{r=1}^{n} \sigma_r$ can then be obtained as follows.

Construct the fundamental determinant

$$\Delta = \begin{vmatrix} w_1 & , & w_2 & , & \ldots, & w_n \\ \dfrac{dw_1}{dz} & , & \dfrac{dw_2}{dz} & , & \ldots, & \dfrac{dw_n}{dz} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{d^{n-1}w_1}{dz^{n-1}} & , & \dfrac{d^{n-1}w_2}{dz^{n-1}} & , & \ldots, & \dfrac{d^{n-1}w_n}{dz^{n-1}} \end{vmatrix},$$

which is equal to

$$Ae^{-\int \frac{p_1}{z} dz},$$

that is, to

$$Az^{-a_{10}} e^{\frac{a_{11}}{z} + \ldots + \frac{1}{m}\frac{a_{1,1m}}{z^m}},$$

where A is a constant. Now if $u_r = 1 + c_{r1} z + \ldots$, we have

$$\frac{dw_r}{dz} = \theta_r e^{\Omega_r} z^{\sigma_r - (m+1)} u_{r1},$$

where u_{r1} is a polynomial in z which is equal to 1 when $z=0$; also

$$\frac{d^2 w_r}{dz^2} = \theta_r{}^2 e^{\Omega_r} z^{\sigma_r - (2m+2)} u_{r2},$$

where u_{r2} is a polynomial in z which is equal to 1 when $z=0$; and so on. Thus

$$\Delta = e^{\Omega_1 + \ldots + \Omega_n} \begin{vmatrix} z^{\sigma_1} u_1 & , & z^{\sigma_2} u_2 & , & \ldots \\ \theta_1 z^{\sigma_1 - (m+1)} u_{11} & , & \theta_2 z^{\sigma_2 - (m+1)} u_{21} & , & \ldots \\ \theta_1{}^2 z^{\sigma_1 - (2m+2)} u_{12} & , & \theta_2{}^2 z^{\sigma_2 - (2m+2)} u_{22} & , & \ldots \\ \ldots & & \ldots & & \ldots \end{vmatrix}$$

$$= e^{\Omega_1 + \ldots + \Omega_n} z^{\sigma_1 + \ldots + \sigma_n - \frac{1}{2} n(n-1)(m+1)} \Phi(z),$$

where

$$\Phi(z) = \begin{vmatrix} 1 + \ldots, & 1 + \ldots, & \ldots \\ \theta_1 + \ldots, & \theta_2 + \ldots, & \ldots \\ \theta_1{}^2 + \ldots, & \theta_2{}^2 + \ldots, & \ldots \\ \ldots & \ldots & \ldots \end{vmatrix}.$$

As the roots θ are unequal to one another, $\Phi(z)$ does not vanish when $z=0$; and it is a polynomial. We thus have

$$e^{\Omega_1 + \ldots + \Omega_n} z^{\sigma_1 + \ldots + \sigma_n - \frac{1}{2} n(n-1)(m+1)} \Phi(z) = A z^{-a_{10}} e^{\frac{a_{11}}{z} + \ldots + \frac{1}{m} \frac{a_{1,1m}}{z^m}}.$$

Accordingly

$$\Omega_1 + \ldots + \Omega_n = \frac{a_{11}}{z} + \ldots + \frac{1}{m} \frac{a_{1,1m}}{z^m},$$

$$\sum_{r=1}^{n} \sigma_r - \tfrac{1}{2} n(n-1)(m+1) = -a_{10},$$

$$\Phi(z) = A,$$

that is, $\Phi(z)$ reduces to its constant non-vanishing term. Thus

$$\sum_{r=1}^{n} \sigma_r = \tfrac{1}{2} n(n-1)(m+1) - a_{10}.$$

We saw that

$$\sum_{r=1}^{n} (\sigma_r + \kappa_r) = \tfrac{1}{2} n(n-1) - a_{10};$$

and therefore

$$\sum_{r=1}^{n} \kappa_r = -\tfrac{1}{2} mn(n-1),$$

which is impossible because no one of the integers κ_r is negative. It therefore follows that *when the characteristic equation*

$$\alpha_m{}^n + \alpha_m{}^{n-1} a_{1,1m} + \ldots + a_{n,nm} = 0$$

has all its roots distinct from one another, and when the quantity denoted by σ has n distinct values, associated respectively with n distinct roots of $I(\rho)=0$, the differential equation

$$z^n \frac{d^n w}{dz^n} + \Sigma z^{n-\kappa} p_\kappa \frac{d^{n-\kappa} w}{dz^{n-\kappa}} = 0,$$

where

$$p_\kappa = \sum_{s=0}^{\kappa m} \frac{a_{\kappa, s}}{z^s},$$

cannot have more than $n-1$ normal integrals, linearly independent of one another.

If, however, the quantity denoted by σ has fewer than n distinct values, so that it could be the same for more than one of the n distinct quantities Ω, the relation

$$\sum_{r=1}^{n} \sigma_r = \tfrac{1}{2} n(n-1)(m+1) - a_{10}$$

would still hold, repetitions occurring on the left-hand side. But in that case not all the roots of the equation $I(\rho)=0$ are specified, for the same value of κ could be associated with the value of σ common to two integrals; and the relation

$$\Sigma(\sigma+\kappa) = \tfrac{1}{2} n(n-1) - a_{10}$$

no longer holds. The theorem then cannot be inferred as necessarily true: and it will appear from examples that an equation in such a case can have a number of normal integrals equal to its order.

Similarly, if σ has n distinct values, and if these values are not associated with n distinct roots of $I(\rho)=0$, the preceding theorem is not necessarily true; the differential equation can have a number of normal integrals equal to its order.

96. Next, let α_m denote a multiple root of the characteristic equation

$$\alpha_m{}^n + \alpha_m{}^{n-1} a_{1,1m} + \ldots + a_{n,nm} = 0;$$

then the quantity η_0 vanishes, where

$$\eta_0 = n\alpha_m{}^{n-1} + (n-1)\alpha_m{}^{n-2} a_{1,1m} + \ldots + a_{n-1,nm-m}.$$

The indicial equation is

$$\eta_0 \sigma + \theta_0 = 0,$$

and σ must be a finite quantity. If θ_0 is not zero, the latter

condition is not satisfied: and then the original equation has no normal integral to be associated with that multiple root. If θ_0 is zero, the preceding indicial equation is evanescent: and so further consideration is required. The differential equation for u, on division by z^{m+1}, becomes

$$(\theta_1 + \theta_2 z + \ldots)\, u + (\eta_1 + \eta_2 z + \ldots)\, z \frac{du}{dz}$$
$$+ z^{m+1} (\phi_0 + \phi_1 z + \ldots) \frac{d^2 u}{dz^2} + \ldots = 0,$$

where the coefficient of $\frac{d^r u}{dz^r}$ is of the form

$$z^{(m+1)(r-1)} (\psi_0 + \psi_1 z + \ldots),$$

for $r = 3, 4, \ldots$.

When $m = 1$, the indicial equation is

$$\theta_1 + \eta_1 \sigma + \phi_0 \sigma (\sigma - 1) = 0;$$

when $m > 1$, the indicial equation is

$$\theta_1 + \eta_1 \sigma = 0.$$

In either case, we can have a possible value for σ. A regular integral of the equation for u, and a consequent normal integral of the original equation, exist if the appropriate conditions, corresponding to those for a simple root, are satisfied: it is manifest that they become complicated in their expression*.

97. It might happen that, in determining v, one or more roots of the equation

$$\alpha_m{}^n + \alpha_m{}^{n-1} a_{1,1m} + \ldots + a_{n,nm} = 0$$

is zero, while some of the remaining coefficients in v do not vanish; the implication is that (other conditions being satisfied) a normal integral exists, having a determining factor of which the exponent is a polynomial with a number of terms less than m. It might even happen that, with a zero value of α_m, all the associable values of the rest of the coefficients are zero, so that $v = 0$, and the determining factor disappears. One possibility is the existence of a regular integral, and the possibility can be settled in the particular case by the method given in Ch. VI. If, however, the conditions for a regular integral are not satisfied, then there is the

* They are considered by Günther, *Crelle*, t. CV (1889), pp. 1—34, in particular, pp. 10 et seq.

possibility of a subnormal integral of the original equation: it arises as follows.

Let
$$w = e^{\Omega} u$$
be substituted in the equation
$$\frac{d^n w}{dz^n} + \frac{p_1}{z}\frac{d^{n-1} w}{dz^{n-1}} + \dots + \frac{p_n}{z^n} w = 0;$$
then the equation for u is (by § 85)
$$\frac{d^n u}{dz^n} + \left(\frac{p_1}{z} + n\Omega'\right)\frac{d^{n-1}}{dz^{n-1}} + \dots + q_n u = 0,$$
where
$$q_n = t_n + \frac{p_1}{z} t_{n-1} + \dots + \frac{p_n}{z^n},$$
and
$$e^{-\Omega}\frac{d^p}{dz^p}(e^{\Omega}) = t_p.$$

Now Ω is to be chosen so as to diminish the multiplicity of $z=0$ as a pole of q_n. After the preceding hypotheses, we shall not expect to have an expression of the form
$$\Omega' = \frac{a_1}{z^2} + \frac{a_2}{z^3} + \dots + \frac{a_m}{z^{m+1}},$$
where m is an integer; but after the indications in § 92, it is possible that Ω' may be a series of fractional powers. Accordingly, assume that the multiplicity of $z=0$ as an infinity of Ω' is μ, so that $z^{\mu}\Omega'$ is finite when $z=0$: then in q_n, we have a series of terms with infinities of orders
$$\begin{array}{ll}
n\mu \qquad\qquad\quad , & (n-1)\mu + 1 \qquad , \dots \\
(n-1)\mu + \ m + 1, & (n-2)\mu + m + 2, \dots \\
(n-2)\mu + 2m + 2, & (n-3)\mu + m + 3, \dots \\
\qquad\vdots & \\
n(m+1). &
\end{array}$$

Construct a Puiseux tableau by marking points, referred to two axes, and having coordinates
$$\begin{array}{llllll}
0, & n & ; & 1, & n-1; & \dots \\
m+1, & n-1; & & m+2, & n-2; & \dots \\
2m+2, & n-2; & & \dots & & \\
\vdots & & & & &
\end{array}$$

(it is easily seen to be necessary to mark only the first in each row), and construct the broken line for the tableau, as in § 92. If the inclination to the negative direction of the axis of y of any portion of the line is $\tan^{-1}\theta$, then θ is a possible value for μ. If θ be a positive integer $\geqslant 2$, we have a case which has already been dealt with. If $\theta = 1$, there may be a corresponding integral; but it is regular, not normal. If θ be a negative integer, Ω' is not infinite for $z = 0$, and the value is to be neglected. If θ be a positive quantity but not an integer, it must be greater than unity to be effective; for if it were less than unity, Ω would not be infinite for $z = 0$. Suppose, then, that θ has a value greater than unity; as it arises out of the Puiseux diagram, it must be commensurable: when in its lowest terms, let it be

$$\theta = \frac{q}{p},$$

where q and p are integers prime to one another, and $q > p$. Then take

$$z = x^p;$$

we have an equation in u and x, and a possible determining factor e^{Ω} can be found such that

$$\frac{d\Omega}{dz} = x^{-q}(a_0 + a_1 x + \ldots),$$

and so

$$\begin{aligned}\Omega &= x^{-(q-p)}(c_0 + c_1 x + \ldots)\\ &= \frac{c_0}{z^{\frac{q}{p}-1}} + \frac{c_1}{z^{\frac{q-1}{p}-1}} + \ldots,\end{aligned}$$

a series of fractional powers. The investigation of the integral of the new equation in u and x, that may exist in connection with this quantity Ω, is of the same character as the earlier investigations.

Equations of the Third Order with Normal or Subnormal Integrals.

98. The preceding general theory, and the methods of dealing with the cases when the equation for α_m has equal roots, or has zero roots, may be illustrated by the consideration of an equation

of the third order more clearly than by that of an equation of the second order, as in § 91. Taking the simplest value of m, which is unity, the equation is of the form

$$w''' + 3\frac{k_{10}z + k_{11}}{z^2} w'' + \frac{k_{20}z^2 + k_{21}z + k_{22}}{z^4} w' \\ + \frac{k_{30}z^3 + k_{31}z^2 + k_{32}z + k_{33}}{z^6} w = 0;$$

which, on using the substitution

$$y = we^{\int \frac{k_{10}z + k_{11}}{z^2} dz} = wz^{k_{10}}e^{-\frac{k_{11}}{z}},$$

becomes

$$y''' + \frac{a_{20}z^2 + a_{21}z + a_{22}}{z^4} y' + \frac{a_{30}z^3 + a_{31}z^2 + a_{32}z + a_{33}}{z^6} y = 0,$$

where the constants a are simple combinations of the constants k. The substitution adopted changes a normal integral of the one equation into a normal integral of the other, save for the very special case when it might be changed into a regular integral of the other: it therefore will be sufficient to discuss the form which is devoid of a term in y''.

In the present case, $m = 1$, we take

$$y = e^{\frac{\alpha}{z}} u,$$

and α is chosen so as to make the coefficient of the lowest power in the coefficient of u equal to zero. We thus have

$$\alpha^3 + \alpha a_{22} - a_{33} = 0;$$

and the equation for u then is

$$u''' - 3\frac{\alpha}{z^2} u'' + \frac{u'}{z^4}\{a_{20}z^2 + (a_{21} + 6\alpha) z + (a_{22} + 3\alpha^2)\} \\ + \frac{u}{z^5}\{a_{30}z^2 + (a_{31} - \alpha a_{20} - 6\alpha) z + (a_{32} - \alpha a_{21} - 6\alpha^2)\} = 0,$$

of which the indicial equation for $z = 0$ is

$$(a_{22} + 3\alpha^2)\sigma + a_{32} - \alpha a_{21} - 6\alpha^2 = 0.$$

It is clear that the equation in α will not have a triple root: if it could, we should have $a_{22} = 0$, $a_{33} = 0$, $\alpha = 0$, the last of which

values leads to the collapse of the process. (Account must, of course, be taken of the possibility that $a_{22} = 0 = a_{33}$, and this will be done later.) Meanwhile, we assume that α is either a simple root or a double root.

First, let α be a simple root; then $a_{22} + 3\alpha^2$ is not zero, and the foregoing indicial equation then gives a proper value for σ. If $-\rho$ is the exponent to which an integral in the vicinity of $z = \infty$ belongs, ρ is a root of the equation

$$f(\rho) = \rho(\rho - 1)(\rho - 2) + a_{20}\rho + a_{30} = 0.$$

The general investigation has shewn that this must have a root of the form $\rho = \sigma + \kappa$, where κ is a positive integer (or zero), and that, if this condition is satisfied, the form of u is

$$u = z^\sigma (c_0 + c_1 z + \ldots + c_\kappa z^\kappa).$$

We substitute this value, and compare coefficients. If

$$\begin{aligned} g_n &= (\sigma + n)(\sigma + n - 1)(\sigma + n - 2) + a_{20}(\sigma + n) + a_{30}, \\ h_n &= -3\alpha(\sigma + n)(\sigma + n + 1) + (a_{21} + 6\alpha)(\sigma + n + 1) \\ &\qquad\qquad + a_{31} - \alpha a_{20} - 6\alpha, \\ k_n &= (a_{22} + 3\alpha^2)(n + 2), \end{aligned}$$

then the difference-equation for the coefficients c is

$$g_n c_n + h_n c_{n+1} + k_n c_{n+2} = 0,$$

for values of $n \geqslant 0$, together with

$$h_{-1} c_0 + k_{-1} c_1 = 0.$$

As α is a simple root of its equation, $a_{22} + 3\alpha^2$ is not zero: thus all the quantities $k_{-1}, k_0, k_1, \ldots$ are different from zero, and the preceding equations thus determine $c_1, c_2, \ldots$ in succession, say in the form

$$c_n = c_0 l_n.$$

In order that the integral may not become illusory, the series is to be a terminating series: it would otherwise diverge, on account of the form of g_n. Let the series contain $\kappa + 1$ terms; then all the coefficients $c_{\kappa+1}, c_{\kappa+2}, \ldots$ must vanish. Now $c_{\kappa+1}$ vanishes if

$$g_{\kappa-1} c_{\kappa-1} + h_{\kappa-1} c_\kappa = 0;$$

then $c_{\kappa+2}$ vanishes if

$$g_\kappa c_\kappa = 0;$$

and then all the succeeding coefficients c vanish. The latter condition gives $g_\kappa = 0$ which, as $g_n = f(\sigma + n)$ for all values of n, is the same as

$$f(\sigma + \kappa) = 0,$$

a known condition; and the other gives

$$g_{\kappa-1} l_{\kappa-1} + h_{\kappa-1} l_\kappa = 0,$$

which is the new condition. When both conditions are satisfied, a normal integral exists for the equation in y. As that equation involves seven constants, which are thus subject to two conditions, there are effectively five constants left arbitrary, subject solely to a condition of inequality as regards the roots of the equation

$$\alpha^3 + \alpha a_{22} - a_{33} = 0;$$

moreover, κ may be any positive integer (or zero).

If the corresponding conditions hold for a second simple root of this cubic equation, the number of independent constants is reduced to three, while there are two integers such as κ; the differential equation for y then has two normal integrals.

If all the roots of the cubic equation are simple, and the corresponding conditions hold for each of them, there are three integers such as κ, and there is effectively one arbitrary constant: the differential equation for y would then have three normal integrals. This, however, is impossible, if there are three different values σ, σ', σ'' of σ, and three associated integers κ, κ', κ'', such that $\sigma + \kappa$, $\sigma' + \kappa'$, $\sigma'' + \kappa''$ are different roots of $f(\rho) = 0$. For then

$$\sigma + \kappa + \sigma' + \kappa' + \sigma'' + \kappa'' = 3.$$

Now we have

$$\sigma = \frac{6\alpha^2 + \alpha a_{21} - a_{32}}{3\alpha^2 + a_{22}}$$

$$= 2 + \frac{\alpha a_{21} - (a_{32} + 2a_{22})}{\dfrac{\partial h}{\partial \alpha}},$$

where

$$h = \alpha^3 + \alpha a_{22} - a_{33} = 0;$$

hence, summing for the three roots of h, we have

$$\sigma + \sigma' + \sigma'' = 6 + \Sigma \frac{\alpha a_{21} - (a_{32} + 2a_{22})}{\dfrac{\partial h}{\partial \alpha}}$$
$$= 6,$$

by a well known theorem in the theory of equations. We then should have the equation

$$\kappa + \kappa' + \kappa'' = -3,$$

which is impossible as no one of the integers κ, κ', κ'' can be negative. Hence, when the equation $\alpha^3 + \alpha a_{22} - a_{33} = 0$ has three distinct roots, and when there are three different values σ, σ', σ'' of σ, associated with three integers κ, κ', κ'', such that $\sigma + \kappa$, $\sigma' + \kappa'$, $\sigma'' + \kappa''$ are different roots of $f(\rho) = 0$, then the differential equation

$$y''' + \frac{a_{20}z^2 + a_{21}z + a_{22}}{z^4} y' + \frac{a_{30}z^3 + a_{31}z^2 + a_{32}z + a_{33}}{z^6} y = 0$$

cannot have more than two normal integrals. But, if the values of σ are fewer than three in number, or if the quantities $\sigma + \kappa$ are not different from one another, then the differential equation (the other conditions being satisfied) can have three normal integrals.

Next, let α be a double root of the equation

$$h = \alpha^3 + \alpha a_{22} - a_{33} = 0,$$

so that we have

$$3\alpha^2 + a_{22} = 0:$$

in order that this may be the case, the relation

$$27a_{33}^2 + 4a_{22}^3 = 0$$

must be satisfied. The quantity σ, given by

$$(3\alpha^2 + a_{22})\sigma + a_{32} - \alpha a_{21} - 6\alpha^2 = 0,$$

is infinite, unless $a_{32} - \alpha a_{21} - 6\alpha^2$ vanishes: if this condition is not satisfied, then the regular integral for the u-equation, and consequently the associated normal integral for the y-equation, cannot exist. Hence a further condition for the existence of the normal integral is, that the equation

$$a_{32} - \alpha a_{21} - 6\alpha^2 = 0$$

be satisfied, where α is the double root: that is,

$$4a_{22}^2 - 3a_{21}a_{33} + 2a_{22}a_{32} = 0.$$

Assuming this to be satisfied, the equation for u now is

$$u''' - \frac{3\alpha}{z^2} u'' + \frac{a_{20} z + a_{21} + 6\alpha}{z^3} u' + \frac{a_{30} z + a_{31} - \alpha a_{20} - 6\alpha}{z^4} u = 0.$$

Now

$$-3\alpha = -\frac{9a_{33}}{2a_{22}} = c_{10}, \text{ say};$$

$$a_{21} + 6\alpha = \frac{a_{32}}{\alpha} = \frac{2a_{22}a_{32}}{3a_{33}} = c_{21}, \text{ say};$$

$$a_{31} - \alpha (a_{20} + 6) = a_{31} - \frac{3a_{33}}{2a_{22}} (a_{20} + 6) = c_{31}, \text{ say};$$

so that the equation for u is

$$Du = u''' + \frac{c_{10}}{z^2} u'' + \frac{a_{20} z + c_{21}}{z^3} u' + \frac{a_{30} z + c_{31}}{z^4} u = 0.$$

The indicial equation for $z = 0$ is

$$c_{10}\sigma (\sigma - 1) + c_{21}\sigma + c_{31} = 0.$$

Substituting

$$u = z^\theta (c_0 + c_1 z + \ldots + c_n z^n + \ldots)$$

in the equation, we have

$$Du = c_0 z^\theta \{c_{10}\theta (\theta - 1) + c_{21}\theta + c_{31}\},$$

provided

$$g_n c_n + h_n c_{n+1} = 0,$$

for all values of $n \geqslant 0$, where

$$g_n = (\theta + n)(\theta + n - 1)(\theta + n - 2) + a_{20} (\theta + n) + a_{30},$$

$$h_n = c_{10} (\theta + n + 1)(\theta + n) + c_{21} (\theta + n + 1) + c_{31}.$$

First, let the roots of the indicial equation be unequal, say λ and μ, so that

$$Du = c_0 c_{10} z^\theta (\theta - \lambda)(\theta - \mu).$$

Then the value of u, when $\theta = \lambda$, gives an expression which formally satisfies the equation; but it has no functional significance unless the series converges. That this may happen, g_n must vanish for some value of n, say κ_1, when $\theta = \lambda$; that is, one root of

$$I(\rho) = \rho(\rho - 1)(\rho - 2) + a_{20}\rho + a_{30} = 0$$

must be

$$\rho = \lambda + \kappa_1,$$

where κ_1 is a positive integer or zero. If that condition is satisfied, then a regular integral of the u-equation and an associated normal integral of the y-equation exist.

Similarly, if $I(\rho)=0$ has another root

$$\rho = \mu + \kappa_2,$$

where κ_2 is a positive integer, then the value of u, when $\theta=\mu$, has significance. It is a regular integral of the u-equation; and a corresponding normal integral of the original equation then exists.

Let β denote the root of the cubic that is simple: then the earlier investigation shews that a corresponding normal integral may exist. If σ' be the exponent to which the regular u-integral belongs and if κ_3+1 be the number of terms it contains, then the equation $I(\rho)=0$ has a root

$$\rho = \sigma' + \kappa_3.$$

But the three normal integrals, each one of which is possible, cannot coexist, if $\lambda+\kappa_1$, $\mu+\kappa_2$, $\sigma'+\kappa_3$ are different roots of $I(\rho)=0$, supposed not to have equal roots. If they could, we should have

$$\lambda + \mu + \sigma' + \kappa_1 + \kappa_2 + \kappa_3 = \Sigma\rho = 3.$$

Now

$$\lambda + \mu = 1 - \frac{c_{21}}{c_{10}} = 1 - \frac{a_{32}}{a_{22}}.$$

Also

$$\sigma'(3\beta^2 + a_{22}) + a_{32} - \beta a_{21} - 6\beta^2 = 0,$$

and

$$\beta + 2\alpha = 0,$$

for α, α, β are the roots of the equation

$$\alpha^3 + \alpha a_{22} - a_{33} = 0;$$

so that

$$\sigma' = -\frac{a_{32} + 2\alpha a_{21} - 24\alpha^2}{a_{22} + 12\alpha^2}$$

$$= \frac{a_{32}}{a_{22}} + 4,$$

on reduction, after using the value of α and the relation

$$a_{32} - \alpha a_{21} - 6\alpha^2 = 0.$$

Hence

$$\lambda + \mu + \sigma' = 5,$$

and therefore

$$\kappa_1 + \kappa_2 + \kappa_3 = -2,$$

which is impossible, as no one of the integers κ can be negative. Hence, when the roots of the indicial equation

$$c_{10}\sigma(\sigma-1)+c_{21}\sigma+c_{31}=0$$

are unequal, and when $I(\rho)=0$ has not equal roots, the original equation cannot have more than two normal integrals, unless (in the preceding notation) there are equalities among the quantities $\lambda+\kappa_1$, $\mu+\kappa_2$, $\sigma'+\kappa_3$. If it possesses the two normal integrals associated with λ and μ, it is easy to see, from the expression for h_n, that, if $\lambda-\mu$ be a positive integer, it must be greater than κ_2+1: and that, if $\mu-\lambda$ be a positive integer, it must be greater than κ_1+1.

Next, let each of the roots of the indicial equation for σ be equal to τ: so that

$$Du=c_0c_{10}z^\theta(\theta-\tau)^2.$$

Thus the two quantities

$$[u]_{\theta=\tau},\quad \left[\frac{\partial u}{\partial\theta}\right]_{\theta=\tau},$$

are expressions that formally satisfy the equation: they have no significance unless the series converge. That this may happen, g_n must vanish for some value of n, say κ', when $\theta=\tau$; that is, one root of the equation

$$I(\rho)=\rho(\rho-1)(\rho-2)+a_{20}\rho+a_{30}=0$$

must be

$$\rho=\tau+\kappa',$$

where κ' is a positive integer or zero. (The quantity h_n never vanishes in this case and so imposes no condition.) On dropping the coefficient c_0, the expression for u in general is equal to

$$z^\theta\left\{1-\frac{g_0}{h_0}z+\frac{g_0g_1}{h_0h_1}z^2-\ldots+(-1)^{\kappa'}\frac{g_0g_1\ldots g_{\kappa'-1}}{h_0h_1\ldots h_{\kappa'-1}}z^{\kappa'}\right\},$$

so that the two integrals are of the form

$$v,\quad v\log z+v_1,$$

where $v=[u]_{\theta=\tau}$, and v_1 is an expression similar to v with different numerical coefficients, viz. the coefficient of $(-1)^r z^{\theta+r}$ in v_1 is

$$\left[\frac{g_0g_1\ldots g_{r-1}}{h_0h_1\ldots h_{r-1}}\left\{\sum_{s=0}^{r-1}\left(\frac{1}{g_s}\frac{\partial g_s}{\partial\theta}-\frac{1}{h_s}\frac{\partial h_s}{\partial\theta}\right)\right\}\right]_{\theta=\tau}.$$

The corresponding normal integrals are

$$e^{\frac{a}{z}}v,\quad e^{\frac{a}{z}}(v\log z+v_1).$$

A third normal integral can coexist with these two in the present case in the form

$$e^{-\frac{2a}{z}}u,$$

where u belongs to the exponent σ', $=\dfrac{a_{32}}{a_{22}}+4$, provided $I(\rho)=0$ has a root of the form $\sigma'+\kappa_3$, where κ_3 is a positive integer (or zero). The reason why three can coexist in this case is that only two quantities τ and σ' arise, and only two roots, not three roots, of $I(\rho)=0$ are assigned.

Ex. 1. Prove that, if the equation

$$y'''+\frac{1}{z^6}y'\sum_{r=0}^{4}a_{2r}z^r+\frac{1}{z^9}y\sum_{s=0}^{6}a_{3s}z^s=0$$

possesses a normal integral of the form

$$e^{-\frac{1}{2}\frac{\beta}{z^2}-\frac{a}{z}}z^\sigma(c_0+c_1z+\ldots+c_\kappa z^\kappa),$$

the constants β, a, σ are given by the relations

$$\beta^3+\beta a_{20}+a_{30}=0,$$
$$a(3\beta^2+a_{20})+\beta a_{21}+a_{31}=0,$$
$$\sigma(3\beta^2+a_{20})+3a^2\beta-9\beta^2+aa_{21}+\beta a_{22}+a_{32}=0\,;$$

and the equation

$$\rho(\rho-1)(\rho-2)+\rho a_{24}+a_{36}=0$$

must have one root equal to $\sigma+\kappa$, where κ is a positive integer (or zero). Obtain the relations sufficient to secure that the series $c_0+c_1z+\ldots$ shall contain only $\kappa+1$ terms.

Assuming that three values of σ, distinct from one another, correspond to three sets of values of a and β, prove that their sum is 9: and hence shew that, in this case, the differential equation cannot have more than two normal integrals.

In what circumstances can the differential equation possess three normal integrals?

Ex. 2. Obtain the constants, and the conditions of existence, of the normal integrals of the equation in the preceding example, when a_{30} vanishes and a_{20} does not vanish. How many normal integrals can the equation then have?

99. We now have to consider (i), the case in which one zero root for α occurs, so that $a_{33}=0$; and (ii), the case in which all the roots α are zero, so that $a_{33}=0$, $a_{22}=0$.

Taking $a_{33}=0$, the equation is

$$y'''+\frac{a_{22}+a_{21}z+a_{20}z^2}{z^4}y'+\frac{a_{32}+a_{31}z+a_{30}z^2}{z^5}y=0.$$

Two non-zero roots are given by

$$\alpha^2 + a_{22} = 0;$$

a normal integral may exist in connection with each of them. The indicial equation for $z = 0$ is

$$a_{22}\theta + a_{32} = 0;$$

in connection with this exponent, a regular integral may exist. The investigation of the respective conditions is similar to preceding investigations.

Now substitute in the equation

$$y = e^{\Omega} u;$$

the equation for u is

$$u''' + 3u''\Omega' + u'\left(3\Omega'^2 + \Omega'' + \frac{a_{22} + a_{21}z + a_{20}z^2}{z^4}\right)$$
$$+ u\left(\Omega'^3 + 3\Omega'\Omega'' + \Omega''' + \frac{a_{22} + a_{21}z + a_{20}z^2}{z^4}\Omega' + \frac{a_{32} + a_{31}z + a_{30}z^2}{z^5}\right) = 0:$$

and by proper choice of Ω, the multiplicity of $z = 0$ as a pole of u is to be diminished. Assume that $z^{-\mu}\Omega'$ is finite (but not zero) when $z = 0$, and form the tableau of points in a Puiseux diagram corresponding to

$$3\mu, \quad 2\mu + 1, \quad \mu + 2, \quad \mu + 4, \quad 5,$$

that is, insert the points

$$0, 3; \quad 1, 2; \quad 2, 1; \quad 4, 1; \quad 5, 0.$$

The broken line consists of two portions: one of them gives $\mu = 2$, the other gives $\mu = 1$. The former gives the possibility of two normal integrals: the latter gives the possibility of one regular integral as above.

But now let $a_{22} = 0$, as well as $a_{33} = 0$. The equation for α becomes

$$\alpha^3 = 0,$$

so that the method gives no normal integral. When we proceed to the equation for u, the coefficient of u is

$$\Omega'^3 + 3\Omega'\Omega'' + \Omega''' + \frac{a_{21} + a_{20}z}{z^3}\Omega' + \frac{a_{32} + a_{31}z + a_{30}z^2}{z^5}.$$

We form the tableau of points in a Puiseux diagram corresponding to

$$3\mu, \quad 2\mu+1, \quad \mu+2, \quad \mu+3, \quad 5,$$

that is, we insert the points

$$0, 3; \quad 1, 2; \quad 2, 1; \quad 3, 1; \quad 5, 0.$$

There is only a single portion of line; it gives

$$\mu = \tfrac{5}{3}.$$

Accordingly, we change the independent variable by the relation

$$z = x^3;$$

the form of Ω' is

$$\Omega' = \frac{\alpha'}{x^5} + \frac{\beta'}{x^4},$$

that is,

$$\frac{d\Omega}{dx} = \frac{3\alpha'}{x^3} + \frac{3\beta'}{x^2} = \frac{\beta}{x^3} + \frac{\alpha}{x^2},$$

say. The differential equation

$$y''' + \frac{a_{21} + a_{20}z}{z^3}y' + \frac{a_{32} + a_{31}z + a_{30}z^2}{z^5}y = 0,$$

with the substitution $y = vx^2$, becomes

$$\frac{d^3v}{dx^3} + \frac{9a_{21} + x^3(9a_{20} - 8)}{x^5}\frac{dv}{dx}$$

$$+ \frac{v}{x^9}\{27a_{32} + (27a_{31} + 18a_{21})x^3 + (27a_{30} + 18a_{20} + 8)x^6\} = 0.$$

If a determining factor exists, then (Ex. 1, § 98) it is of the form

$$e^{-\frac{1}{2}\frac{\beta}{x^2} - \frac{\alpha}{x}} = e^{\Omega},$$

where

$$\beta^3 + 27a_{32} = 0,$$

$$\alpha \,.\, 3\beta^2 + 9a_{21}\beta = 0,$$

that is,

$$\beta = -3a_{32}^{\frac{1}{3}}, \quad \alpha = -\frac{3a_{21}}{\beta} = a_{21}a_{32}^{-\frac{1}{3}}.$$

Substituting

$$v = ue^{\Omega},$$

and using these values of α and β, we find the equation for u in the form

$$\frac{d^3u}{dx^3} + 3\frac{\beta+\alpha x}{x^3}\frac{d^2u}{dx^2} + \frac{1}{x^6}\{3\beta^2 + (3\alpha^2 - 9\beta)x^2 - 6\alpha x^3 + (9a_{20} - 8)x^4\}\frac{du}{dx}$$

$$+ \frac{u}{x^7}[-9\beta^2 + (\alpha^3 + 63a_{21} + 27a_{31})x + \{(12a_{20} + 4)\beta - 6\alpha^2\}x^2$$

$$+ \alpha(9a_{20} - 2)x^3 + (27a_{30} + 18a_{20} + 8)x^4] = 0.$$

If the equation in v is to have a normal integral, this equation in u must have a regular integral belonging to an exponent σ, where it is easy to see that

$$\sigma = 3.$$

The regular integral for u is of the form

$$u = \sum_{n=0}^{\kappa} c_n x^{n+3}.$$

If

$$f_n = (n+3)(n+2)(n+1) + (9a_{20} - 8)(n+3) + 27a_{30} + 18a_{20} + 8,$$
$$g_n = 3\alpha(n+4)(n+3) - 6\alpha(n+4) + \alpha(9a_{20} - 2),$$
$$h_n = 3\beta(n+5)(n+4) + (3\alpha^2 - 9\beta)(n+5) + (12a_{20} + 4)\beta - 6\alpha^2,$$
$$k = \alpha^2 + 63a_{21} + 27a_{31},$$
$$l_n = 3\beta^2(n+4),$$

the difference-relation for the coefficients c is

$$f_n c_n + g_n c_{n+1} + h_n c_{n+2} + k c_{n+3} + l_n c_{n+4} = 0,$$

together with

$$kc_0 + l_{-3}c_1 = 0,$$
$$h_{-2}c_0 + kc_1 + l_{-2}c_2 = 0,$$
$$g_{-1}c_0 + h_{-1}c_1 + kc_2 + l_{-1}c_3 = 0.$$

The conditions, necessary and sufficient to ensure that the series for u terminates with (say) the $(\kappa+1)$th term, which is the generally effective manner of securing the convergence of the series, κ being some positive integer or zero, are

$$\left.\begin{aligned} f_\kappa &= 0 \\ f_{\kappa-1}c_{\kappa-1} + g_{\kappa-1}c_\kappa &= 0 \\ f_{\kappa-2}c_{\kappa-2} + g_{\kappa-2}c_{\kappa-1} + h_\kappa c_\kappa &= 0 \\ f_{\kappa-3}c_{\kappa-3} + g_{\kappa-3}c_{\kappa-2} + h_{\kappa-3}c_{\kappa-1} + kc_\kappa &= 0 \end{aligned}\right\},$$

four conditions in all.

Assuming these satisfied, we have

$$v = e^{-\frac{1}{2}\frac{\beta}{x^2} - \frac{a}{x}} x^3 \sum_{n=0}^{\kappa} c_n x^n,$$

and therefore

$$\begin{aligned} y &= vx^2 \\ &= e^{\frac{3}{2}\left(\frac{a_{32}}{z^2}\right)^{\frac{1}{3}} - \frac{a_{21}}{(a_{32}z)^{\frac{1}{3}}}} z^{\frac{5}{3}} \sum_{n=0}^{\kappa} c_n z^{\frac{1}{3}n}, \end{aligned}$$

a subnormal integral.

If the conditions are satisfied for more than one of the cube roots of a_{32}, then there is more than one integral of subnormal type. Moreover, the value of σ is the same for all three cube roots, and only one value of κ is required: so there may be even three subnormal integrals, each containing the same number of fractional powers.

In order that this analysis may lead to effective results, it is manifest that a_{32} should not vanish.

Ex. 1. Prove that the equation

$$y''' + \frac{3}{4z^2} y' + \frac{a - 95z^2}{108z^5} y = 0$$

possesses three subnormal integrals.

Ex. 2. Discuss the integrals of the equation

$$y''' + \frac{a_{21} + a_{20}z}{z^3} y' + \frac{a_{31} + a_{32}z}{z^4} y = 0.$$

Normal Integrals of Equations with Rational Coefficients.

100. In the discussion at the beginning of this chapter, the only requirement exacted from the coefficients was as regards their character in the vicinity of the singularity considered: and a special limitation was imposed upon them, so as to constitute Hamburger's class of equations in §§ 91—99. More generally, we may take those equations in which the coefficients are rational functions of z, not so restricted that the equations shall be of Fuchsian type; we then have

$$p_0 \frac{d^n w}{dz^n} + p_1 \frac{d^{n-1} w}{dz^{n-1}} + \dots + p_n w = 0,$$

where $p_0, p_1, \ldots, p_n$ are polynomials in z, of degrees $\varpi_0, \varpi_1, \ldots, \varpi_n$ respectively. The singularities of the equation are, of course, the roots of $p_0 = 0$ and possibly $z = \infty$; owing to the form of all the other coefficients, it is natural to consider* the integrals for large values of $|z|$.

It will be assumed that the integrals are not regular in the vicinity of $z = \infty$. When a normal integral exists in that vicinity, it is of the form

$$e^{\Omega} z^{\sigma} \phi,$$

where ϕ is a uniform function of z^{-1} that does not vanish when $z = \infty$, and Ω is a polynomial in z of degree (say) m, so that the integral can be regarded as of grade m. As in §§ 85—87, the value of Ω' is obtained, by making the m highest powers in the expression

$$p_0 \Omega'^n + p_1 \Omega'^{n-1} + \ldots + p_n$$

acquire vanishing coefficients: and a Puiseux diagram at once indicates whether a quantity Ω' of such an order can be constructed. The value of $m - 1$ is the greatest among the magnitudes

$$\varpi_1 - \varpi_0, \ \tfrac{1}{2}(\varpi_2 - \varpi_0), \ \tfrac{1}{3}(\varpi_3 - \varpi_0), \ \ldots,$$

provided two at least of them have that greatest value, which may be denoted by h. Then for such normal integrals as exist, we have

$$m - 1 \leqslant h,$$

when h is an integer, and

$$m - 1 \leqslant [h],$$

where $[h]$ is the integral part of h, when h is not an integer. The integrals are of grade $\leqslant h + 1$, or $\leqslant [h] + 1$, in the respective cases; and the equation is of rank $h + 1$.

Take the simplest general case, when the equation is of rank unity, and when, in the vicinity of $z = \infty$, it may possess n normal integrals which, accordingly, must be of grade unity. No one of the polynomials $p_1, \ldots, p_n$ is of degree higher than p_0; assume the degree of p_0 to be κ, and let

$$p_r = a_r z^{\kappa} + b_r z^{\kappa - 1} + \ldots + k_r,$$

* See Poincaré, *Amer. Journ. Math.*, t. VII (1885), pp. 203—258; *Acta Math.*, t. VIII (1886), pp. 295—344.

where some (but not all) of the coefficients a may be zero and, in particular, where it will be assumed that a_0 and a_n differ from zero. The determining factor for any normal integral is of the form $e^{\theta z}$: θ satisfies the equation

$$U_0(\theta) = a_0\theta^n + a_1\theta^{n-1} + \ldots + a_{n-1}\theta + a_n = 0.$$

The preceding theory then shews that, if the roots of this equation are unequal and are denoted by $\theta_1, \theta_2, \ldots, \theta_n$, the normal integrals are of the form

$$e^{\theta_1 z} z^{\sigma_1}\phi_1, \quad e^{\theta_2 z} z^{\sigma_2}\phi_2, \quad \ldots, \quad e^{\theta_n z} z^{\sigma_n}\phi_n;$$

the quantities σ_r are given by the equations

$$\sigma_r \frac{\partial U_0}{\partial \theta_r} + U_1(\theta_r) = 0, \qquad (r = 1, \ldots, n),$$

where

$$U_1(\theta) = b_0\theta^n + b_1\theta^{n-1} + \ldots + b_{n-1}\theta + b_n;$$

and $\phi_1, \phi_2, \ldots, \phi_n$ are uniform functions of z^{-1}, which do not vanish or become infinite when $z = \infty$. Special relations among coefficients are necessary in order to secure the convergence of the infinite series ϕ; unless these conditions are satisfied, the foregoing expressions only formally satisfy the differential equation and, as integrals, they are illusory.

Ex. 1. Prove that the equation

$$x^3 w''' + (1-a)\, x w' - (1 - a^2 + bx^3)\, w = 0$$

possesses three normal integrals in the vicinity of $x = \infty$, when a is a positive integer not divisible by 3; and obtain them.

Ex. 2. Prove that the equation

$$x^p w''' = w$$

possesses three subnormal integrals in the vicinity of $x = \infty$, when

$$p = 3\left(1 - \frac{1}{n}\right),$$

n being an integer not divisible by 3; and obtain them. (Halphen.)

Ex. 3. Shew that the equation

$$y'' - \frac{2x}{x^2-1} y' = \left\{\frac{2a}{x^2-1} + \frac{n(n+1)}{x^2} + (a-n)(a+n+1)\right\} y$$

has two normal integrals in the vicinity of $x = \infty$; and, by obtaining them, verify that the points $x = 1$, $x = -1$ are only apparent singularities.

(Halphen.)

Ex. 4. Shew that the equation

$$y''' - 2\frac{n+1}{x}y'' + \left(\frac{6n}{x^2} - a\right)y' + \frac{2a}{x}y = 0$$

possesses one integral, which is a polynomial in x, and two other integrals, normal in the vicinity of $x = \infty$. (Halphen.)

Ex. 5. Prove that, if normal integrals exist for the equation

$$y'''' - \frac{2a}{x^2}y'' + \frac{4a}{x^3}y' + \left\{\frac{a(a-b)}{x^4} + c\right\}y = 0,$$

the constant a must be the product of two consecutive integers. (Halphen.)

Ex. 6. Prove that, if all the singularities for finite values of z which are possessed by the integrals of the equation

$$p_0\frac{d^n w}{dz^n} + p_1\frac{d^{n-1}w}{dz^{n-1}} + \ldots + p_n w = 0$$

are poles, and if $p_0, p_1, \ldots, p_n$ be polynomials in z such that the degree of p_0 is not less than the greatest among the degrees of $p_1, \ldots, p_n$, then the primitive of the equation can be obtained in the form

$$w = \sum_{r=1}^{n} A_r e^{a_r z}\phi_r(z),$$

where the constants $a_1, \ldots, a_n$ are determinate, and all the functions $\phi_1, \ldots, \phi_n$ are rational functions of z. (Halphen.)

Ex. 7. Apply the preceding theorem in Ex. 6 to obtain the primitive of the equation

$$\text{(i)}\quad w'' - \left\{\frac{n(n+1)}{z^2} + a^2\right\}w = 0,$$

where n is an integer; also the primitive of the equation

$$\text{(ii)}\quad w''' + \frac{1-n^2}{z^2}w' - \left(\frac{1-n^2}{z^3} + c\right)w = 0,$$

where n is an integer prime to 3. (Halphen.)

Ex. 8. Similarly obtain the primitive of the equation

$$x^2(x^2-1)y'' - 2x^3y' - 6(x^4+x^2-1)y = 0,$$

in the form

$$x^2y = Ae^{x\sqrt{6}}\{(x^3+3x)\sqrt{6} - 7x^2 - 3\} + Be^{-x\sqrt{6}}\{(x^2+3x)\sqrt{6} + 7x^2 + 3\}.$$

(Math. Trip., Part II, 1895.)

POINCARÉ'S DEVELOPMENT OF LAPLACE'S DEFINITE-INTEGRAL SOLUTION.

101. Several instances, both general and particular, have occurred in the preceding investigations in which formal solutions, expressed as power-series, have been obtained for linear differential equations and have been rejected because the power-series diverged. These instances have occurred, either directly, in association with an original equation, or indirectly, in association with a subsidiary equation, when an attempt was made to obtain regular integrals of an equation, some at least of whose integrals were not regular; and they have arisen when an attempt has been made to obtain normal integrals of an equation, which is of the requisite form but the coefficients of which do not satisfy the latent appropriate conditions.

In such instances, the expressions obtained for formal solutions do not possess functional significance. But Poincaré has shewn that it is possible to assign a different kind of significance to such solutions in a number of cases. In particular, there is a theorem*, due to Laplace, according to which a solution of the given differential equation with rational coefficients can be obtained in the form of a definite integral; this solution has been associated† by Poincaré with the preceding results in § 100 relating to normal integrals. For this purpose, let

$$w = \int e^{tz} T\, dt,$$

where the contour of the integral (taken to be independent of z) will subsequently be settled, and T is a function of t the form of which is to be obtained. If this is to be a solution of our equation, we must have

$$\int (p_0 t^n + p_1 t^{n-1} + \dots + p_n)\, e^{tz} T\, dt = 0:$$

or, if

$$\left.\begin{array}{l} U_0 = a_0 t^n + a_1 t^{n-1} + \dots + a_n \\ U_1 = b_0 t^n + b_1 t^{n-1} + \dots + b_n \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ U_k = k_0 t^n + k_1 t^{n-1} + \dots + k_n \end{array}\right\},$$

* See my *Treatise on Differential Equations*, § 140.

† In the memoirs quoted in the footnote on p. 314. The following exposition is based partly upon these memoirs, partly upon Picard's *Cours d'Analyse*, t. III, ch. XIV.

the necessary condition is

$$\int (U_0 z^k + U_1 z^{k-1} + \dots + U_k)\, e^{tz} T\, dt = 0.$$

Let

$$V_r = z^{r-1} T U_{k-r} - z^{r-2} \frac{d}{dt}(T U_{k-r}) + \dots + (-1)^{r-1} \frac{d^{r-1}}{dt^{r-1}} (T U_{k-r}),$$

for $r = 1, 2, \dots, k$. Then

$$\int z^r T U_{k-r} e^{tz} dt = [e^{tz} V_r] + \int (-1)^r \frac{d^r}{dt^r} (T U_{k-r})\, e^{tz} dt,$$

for each of the k values of r; and the value of $[e^{tz} V_r]$ depends upon the contour of the definite integral. Using this result, the above condition becomes

$$\left[\sum_{r=1}^{k} e^{tz} V_r\right] + \int e^{tz} \left\{ T U_k - \frac{d}{dt}(T U_{k-1}) + \dots + (-1)^k \frac{d^k}{dt^k}(T U_0) \right\} dt = 0,$$

which will be satisfied, if T be a solution of the equation

$$T U_k - \frac{d}{dt}(T U_{k-1}) + \dots + (-1)^k \frac{d^k}{dt^k}(T U_0) = 0,$$

and if the contour of the integral be such that

$$\left[\sum_{r=1}^{k} e^{tz} V_r\right] = 0.$$

The equation for T is

$$U_0 \frac{d^k T}{dt^k} + \left(k \frac{dU_0}{dt} - U_1\right) \frac{d^{k-1} T}{dt^{k-1}}$$
$$+ \left\{\tfrac{1}{2} k (k-1) \frac{d^2 U_0}{dt^2} - (k-1) \frac{dU_1}{dt} + U_2 \right\} \frac{d^{k-2} T}{dt^{k-2}} + \dots = 0,$$

so that its singularities are the roots of $U_0 = 0$, that is, are the points $\theta_1, \theta_2, \dots, \theta_n$, and possibly infinity. Writing the equation in the form

$$\frac{d^k T}{dt^k} + P_1 \frac{d^{k-1} T}{dt^{k-1}} + P_2 \frac{d^{k-2} T}{dt^{k-2}} + \dots = 0,$$

the value of P_1 when t is infinite is $-\frac{b_0}{a_0}$, that of P_2 is $\frac{c_0}{a_0}$, and so on. Further, the quantity $\sum_{r=1}^{k} V_r$ involves derivatives of T up to order $k-1$ inclusive.

This equation for T has its integrals regular in the vicinity of each of its singularities $\theta_1, \theta_2, \dots, \theta_n$: their actual form will be

considered later. Let Ψ_s denote the most general integral of the equation for T in the vicinity of θ_s; then, assuming that the conditions connected with the limits of the definite integral can be satisfied, we have an integral of the original differential equation in the form

$$w_s = \int e^{tz} \Psi_s \, dt,$$

and this result is true for $s = 1, 2, \ldots, n$. Now Ψ_s is certainly significant, because it is a linear combination of k regular integrals of the equation for T; hence we have a system of n integrals of the original differential equation.

102. This system of n significant integrals can be transformed into the system of n normal integrals, when the latter exist. They can be associated with the formal expression of the n normal integrals, when the latter are illusory.

A preliminary proposition, relating to the given differential equation, must first be established*. In the first place, let it be assumed that all the constants in the equation for T are real, and that T and t are restricted to real values. That equation can be replaced by the system

$$\frac{dT}{dt} = T_1,$$
$$\frac{dT_1}{dt} = T_2,$$
$$\cdots\cdots\cdots\cdots$$
$$\frac{dT_{k-1}}{dt} = -P_1 T_{k-1} - P_2 T_{k-2} - \ldots - P_k T.$$

When we substitute

$$T_r = \Theta_r e^{-\lambda t}, \qquad (r = 0, 1, \ldots, k-1),$$

with the conventions that $T_0 = T$ and $\Theta_0 = \Theta$, the modified system is

$$\frac{d\Theta}{dt} = \lambda\Theta + \Theta_1,$$
$$\vdots$$
$$\frac{d\Theta_r}{dt} = \lambda\Theta_r + \Theta_{r+1},$$
$$\vdots$$
$$\frac{d\Theta_{k-1}}{dt} = -P_k\Theta - P_{k-1}\Theta_1 - \ldots - P_2\Theta_{k-2} - (P_1 - \lambda)\Theta_{k-1}.$$

* It is due to Liapounoff (1892); see Picard, *Cours d'Analyse*, t. III, p. 363, note.

Hence

$$\tfrac{1}{2}\frac{d}{dt}(\Theta^2+\Theta_1^2+\ldots+\Theta^2_{k-1})=\lambda(\Theta^2+\Theta_1^2+\ldots+\Theta^2_{k-2})+(\lambda-P_1)\Theta^2_{k-1}$$
$$+\Theta\Theta_1+\Theta_1\Theta_2+\ldots+\Theta_{k-2}\Theta_{k-1}$$
$$-P_k\Theta\Theta_{k-1}-\ldots-P_2\Theta_{k-2}\Theta_{k-1}.$$

Take a real quantity t_0, smaller than the least real root of $U_0=0$; as t ranges along the axis of real quantities between $-\infty$ and t_0, all the quantities $P_1, P_2, \ldots, P_k$ remain finite. Hence, by taking a sufficiently large value of λ, the quadratic form on the right-hand side can be made positive for that range of values of t; and therefore, as t increases from $-\infty$ to t_0, the quantity

$$\Theta^2+\Theta_1^2+\ldots+\Theta^2_{k-1}$$

steadily increases in value. Consequently, when t decreases from t_0 to $-\infty$, the quantity

$$\Theta^2+\Theta_1^2+\ldots+\Theta^2_{k-1}$$

steadily decreases in value. As t_0 is not a singularity of the equation, the values of $\Theta, \Theta_1, \ldots, \Theta_{k-1}$ for any integral that exists at t_0 are finite there; their initial values are finite, and therefore each of the quantities $|\Theta|, |\Theta_1|, \ldots, |\Theta_{k-1}|$ remains finite and decreases steadily, as t decreases from t_0 to $-\infty$. Hence the quantities

$$Te^{\lambda t},\ T_1e^{\lambda t},\ \ldots,\ T_{k-1}e^{\lambda t}$$

all remain finite within that range, that is, no one of them can become infinite, for a value of λ sufficiently large* to make the quadratic form positive.

Next, suppose that the constants are complex, so that $T, T_1, \ldots$ can have complex values; but let t still be real. Then we write

$$T_r=\phi_r+i\psi_r,$$

…sts, see Williamson's *Differential Calculus*, 3rd ed., p. 408. In the

the value of P_1 …onditions are:

$$\lambda>0,$$

on. Further, the qu…

$$\lambda^2-\tfrac{1}{4}>0,$$

$$\lambda(\lambda^2-\tfrac{1}{2})>0,$$

order $k-1$ inclusive.

$$\ldots^2+P_3^2+P_4^2\}-\tfrac{1}{4}\lambda P_3(1-P_2-P_4)+\tfrac{1}{16}(1-P_2+P_4)^2>0;$$

This equation for T ha… λ greater than the greatest positive value which

each of its singularities θ_1, θ_2 … inequality vanish.

for all values of r, where ϕ_r and ψ_r are real; the system of equations takes the form

$$\left.\begin{aligned}
&\frac{d\phi_r}{dt} = \phi_{r+1}, \quad \frac{d\psi_r}{dt} = \psi_{r+1}, \qquad (r = 0, 1, \ldots, k-2) \\
&\frac{d\phi_{k-1}}{dt} = -p_1\phi_{k-1} + q_1\psi_{k-1} - p_2\phi_{k-2} + q_2\psi_{k-2} - \ldots - p_k\phi + q_k\psi \\
&\frac{d\psi_{k-1}}{dt} = -p_1\psi_{k-1} - q_1\phi_{k-1} - p_2\psi_{k-2} - q_2\phi_{k-2} - \ldots - p_k\psi - q_k\phi
\end{aligned}\right\},$$

where $\phi_0 = \phi$, $\psi_0 = \psi$, and $P_s = p_s + iq_s$. We now have $2k$ equations; on substituting

$$\phi_r = \Phi_r e^{-\lambda t}, \quad \psi_r = \Psi_r e^{-\lambda t},$$

they give the modified set

$$\left.\begin{aligned}
&\frac{d\Phi_r}{dt} = \lambda\Phi_r + \Phi_{r+1}, \quad \frac{d\Psi_r}{dt} = \lambda\Psi_r + \Psi_{r+1} \\
&\frac{d\Phi_{k-1}}{dt} = -(p_1 - \lambda)\,\Phi_{k-1} + q_1\Psi_{k-1} + \ldots - p_k\Phi + q_k\Psi \\
&\frac{d\Psi_{k-1}}{dt} = -q_1\Phi_{k-1} - (p_1 - \lambda)\,\Psi_{k-1} + \ldots - q_k\Phi - p_k\Psi
\end{aligned}\right\}.$$

Hence

$$\tfrac{1}{2}\frac{d}{dt}\left(\sum_{r=1}^{k-1}\Phi_r^2 + \sum_{r=1}^{k-1}\Psi_r^2\right)$$

$$= \lambda\sum_{r=1}^{k-2}(\Phi_r^2 + \Psi_r^2) + (\lambda - p_1)(\Phi^2_{k-1} + \Psi^2_{k-1}) + \text{bilinear terms.}$$

As before, by choosing a sufficiently large value of λ, the right-hand side can be made always positive. Then, by taking a value t_0 smaller than the least real root of $U_0 = 0$, and by making t decrease from t_0 to $-\infty$, so that all the quantities p and q are finite, it follows that, for such a variation of t,

$$\sum_{r=1}^{k-1}\Phi_r^2 + \sum_{r=1}^{k-1}\Psi_r^2$$

steadily decreases, and therefore that each of the magnitudes

$$|\Phi_r + i\Psi_r|$$

remains finite within the range from t_0 to $-\infty$. Hence each of the quantities

$$Te^{\lambda t}, \; T_1e^{\lambda t}, \; \ldots, \; T_{k-1}e^{\lambda t}$$

remains finite within the range of t from t_0 to $-\infty$, for a value of λ sufficiently large to make the quadratic form positive.

Lastly, let the constants be complex, so that T, $T_1, \ldots$ can have complex values; and now let t be complex in such a way that, in the variation from t_0 towards $-\infty$, where

$$t = t_0 + \tau e^{\alpha i},$$

α remains unaltered. The independent variable now is τ, a real quantity, varying from 0 to $-\infty$; and the preceding argument applies. A finite number λ can be found such that each of the quantities

$$Te^{\lambda\tau},\quad T_1 e^{\lambda\tau},\quad \ldots,\quad T_{k-1}e^{\lambda\tau}$$

remains finite within the range of t. But

$$e^{\lambda\tau} = e^{\lambda(t-t_0)(\cos\alpha - i\sin\alpha)};$$

hence a finite quantity λ' can be chosen, so that each of the quantities

$$Te^{\lambda' t},\quad T_1 e^{\lambda' t},\quad \ldots,\quad T_{k-1}e^{\lambda' t}$$

remains finite within the range of t from t_0 towards $-\infty$.

In the first and the second cases, let

$$\mu = \lambda + \sigma,$$

where σ is any real positive quantity that is not infinitesimal; and in the third case, let

$$\mu = \lambda' + \sigma e^{-\alpha i},$$

where σ is any real positive quantity that is not infinitesimal. Then, because

$$e^{(\mu-\lambda)t} \quad \text{and} \quad e^{(\mu-\lambda')t}$$

in the respective cases tend to zero, as t becomes infinite in its assigned range, it follows that a quantity μ of finite modulus can be obtained, such that

$$Te^{\mu t},\quad T_1 e^{\mu t},\quad \ldots,\quad T_{k-1}e^{\mu t}$$

all become zero when t becomes infinite in its assigned range.

This is true, *a fortiori*, when μ is replaced by another quantity of the same argument and greater modulus.

It also is true when any one (or any number) of the quantities T should happen to be multiplied by a polynomial in t. For all that is necessary is to take a value $\mu+\rho$, where ρ has the same argument as μ; then

$$e^{\rho t} P,$$

where P is a polynomial in t, is zero in the limit, when t is infinite in its assigned range. Thus a quantity μ can be chosen so that

$$TPe^{\mu t},\quad T_1 P_1 e^{\mu t},\ \ldots,\ T_{k-1} P_{k-1} e^{\mu t},$$

where P, P_1, ..., P_{k-1} are polynomials, all become zero when t becomes infinite in its assigned range from t_0, which is not a singularity of the equation, to $-\infty$.

103. This result is now to be applied to the equation which determines T. Let $t=\theta_r$ be any one of the roots of $U_0=0$, and consider a fundamental system of integrals in that vicinity. If

$$\rho = \left[\frac{U_1}{\dfrac{\partial U_0}{\partial t}}\right]_{t=\theta_r} - k + (k-1) = \left[\frac{U_1}{\dfrac{\partial U_0}{\partial t}}\right]_{t=\theta_r} - 1,$$

the indicial equation for θ_r is

$$\phi(\phi-1)\ldots(\phi-k+2)(\phi-\rho)=0.$$

Suppose that ρ is not an integer. The integrals which belong to the exponents 0, 1, ..., $k-2$ are holomorphic functions of $t-\theta_r$ in the vicinity of θ_r (Ex. 12, § 40); and the integer which belongs to ρ is of the form

$$(t-\theta_r)^\rho P(t-\theta_r),$$

where P is a holomorphic function of its argument.

The contour of integration has yet to be settled. In connection with the value θ_r, we draw a straight line from that point towards $-\infty$, either parallel to the axis of real quantities by preference, or not deviating far from that parallel, choosing the direction so that the line does not pass through, or infinitesimally near, any of the other roots of $U_0=0$; and we draw a circle with θ_r as centre, of such a radius that no one of those other roots lies within or upon the circumference. The path of t is made to be (i) in the line from $-\infty$ towards θ_r, as far as the circumference of

the circle, (ii) then the complete circumference of the circle, described positively, (iii) then in the line from the circumference back towards $-\infty$. So far as concerns the conditions imposed upon T by the relation

$$\left[\sum_{r=1}^{k} e^{tz} V_r\right] = 0$$

at the limits, we have only to take the values at the two extremities $t = -\infty$. Now V_r is a linear function of $T, T_1, \ldots, T_{k-1}$, the coefficients of these quantities in that linear function being polynomials in z and t; hence, taking z as equal to the quantity μ of the preceding investigation, or as equal to any other quantity of the same argument as μ and with a greater modulus, we have

$$\sum_{r=1}^{k} e^{tz} V_r = 0$$

at each of the two infinities for t; and so the conditions at the limits are satisfied.

In these circumstances, the complete primitive of the equation for T is

$$T = A\,(t-\theta_r)^\rho\, P\,(t-\theta_r) + Q\,(t-\theta_r),$$

where Q is a holomorphic function of $t-\theta_r$, involving $n-1$ arbitrary constants linearly. The corresponding integral of the original equation then arises in the form

$$\int e^{tz}\, T\, dt,$$

taken round the chosen contour.

104. We proceed to discuss this integral for large values of $|z|$. Let a be the radius of the circle in the contour, so that the series P and Q converge for values of t such that $|t-\theta_r| \leqslant a$. For simplicity of statement, we shall assume* that the duplicated rectilinear part of the contour passes parallel to the axis of real quantities from $t = \theta_r - a$ to $t = -\infty$. From the nature of the integral T, we know that a finite positive quantity λ exists, such that the value of

$$e^{\lambda t}\, T$$

* The alternative would be merely to take

$$t = \theta_r + t'' e^{\alpha i},$$

with a suitable constant value of α, and then make t'' vary from $-a$ to $-\infty$.

remains finite, as t decreases from $\theta_r - a$ to $-\infty$. Let δ denote the maximum value within this range; then

$$e^{\lambda t} T \leqslant \delta,$$

for all the values of t, and then

$$\int_{-\infty}^{\theta_r - a} e^{tz} T \, dt < \delta \int_{-\infty}^{\theta_r - a} e^{(z-\lambda)t} \, dt$$

$$< \delta \left[\frac{e^{(z-\lambda)t}}{z-\lambda} \right]_{-\infty}^{\theta_r - a}.$$

Let z have the same argument* as λ, and have a modulus greater than $|\lambda|$, that is, with the present hypothesis, let z be positive; then the part corresponding to the lower limit is zero, and we have

$$\int_{-\infty}^{\theta_r - a} e^{tz} T \, dt < \frac{\delta}{z-\lambda} e^{(z-\lambda)(\theta_r - a)},$$

for values of z that have the same argument as λ and have a modulus greater than $|\lambda|$; and δ is a finite quantity.

Similarly, if, after t has described the circle, δ' denote the maximum value of $e^{\lambda t} T$ for $\theta_r - a > t > -\infty$, then the second description of the linear part of the contour gives an integral, such that

$$\int_{-\infty}^{\theta_r - a} e^{tz} T \, dt < \frac{\delta'}{z-\lambda} e^{(z-\lambda)(\theta_r - a)},$$

for similar values of z; and δ' is a finite quantity.

If, then, these two parts of the integral be denoted by I' and I''' respectively, we have

$$z^q e^{-z\theta_r} I' < \delta \frac{z^q}{z-\lambda} e^{-az-\lambda(\theta_r - a)},$$

where a is a positive quantity; hence for any constant quantity q, however large, we have

$$\text{Limit } (z^q e^{-z\theta_r} I') = 0,$$

when z tends to an infinitely large positive value. Similarly, in the same circumstances, we have

$$\text{Limit } (z^q e^{-z\theta_r} I''') = 0.$$

* This form of statement is suited also for the variation of t indicated in the preceding note.

Now consider the integral round the circular part of the contour. As $Q(t-\theta_r)$ is a holomorphic function over the whole of the circle, we have

$$\int e^{tz} Q(t-\theta_r)\,dt = 0,$$

taken round the circle; and therefore the portion of the integral $\int e^{tz} T\,dt$ contributed by this part is I'', where

$$I'' = \int (t-\theta_r)^\rho e^{tz} P(t-\theta_r)\,dt,$$

on taking $A = 1$. The function P is holomorphic everywhere within and on the circumference, so that we may take

$$P(t-\theta_r) = c_0 + c_1(t-\theta_r) + \ldots + c_m(t-\theta_r)^m + R_m,$$

where $|R_m|$ can be made as small as we please by sufficiently increasing m; for if g be the radius of convergence of $P(t-\theta_r)$, so that $g > a$, and if M denote the greatest value of $|P(t-\theta_r)|$ within or on the circumference of a circle of radius c, where

$$g > c > a,$$

then*

$$|c_p| < \frac{M}{c^p},$$

and

$$|R_m| < M\frac{|t-\theta_r|^{m+1}}{c^{m+1}}\left\{1 + \frac{|t-\theta_r|}{c} + \frac{|t-\theta_r|^2}{c^2} + \ldots\right\}$$

$$< M\left\{\frac{|t-\theta_r|}{c}\right\}^{m+1} \frac{1}{1-\dfrac{|t-\theta_r|}{c}},$$

for values of t such that

$$|t-\theta_r| \leqslant a < c.$$

The value of the integral taken round the circumference can be obtained as follows. Draw an infinitesimal circle with θ_r as centre, and make a section in the plane from the circumference of this circle to that of the outer circle of radius a along the linear direction in which t decreases towards $-\infty$. The subject of integration is holomorphic over the area of this slit ring: and therefore the integral taken round the complete boundary is zero. Let

* *T. F.*, § 22.

J' denote the value along the upper side of the slit, J'' the value along the lower side, K the value round the small circle which is described negatively; so that

$$J' = \int_{\theta_r - a}^{\theta_r} (t - \theta_r)^\rho e^{tz} P(t - \theta_r)\, dt,$$

$$J'' = \int_{\theta_r}^{\theta_r - a} e^{-2\pi i \rho} (t - \theta_r)^\rho e^{tz} P(t - \theta_r)\, dt$$

$$= - e^{-2\pi i \rho} J',$$

and, if the real part of ρ be greater than -1, then*

$$K = 0.$$

Hence, beginning at the point on the outer circumference which is on the lower edge of the slit, we have

$$I'' + J' + K + J'' = 0,$$

that is,

$$I'' = (e^{-2\pi i \rho} - 1) J'.$$

Let u denote the integral

$$u = \int_{\theta_r - a}^{\theta_r} (t - \theta_r)^\kappa e^{tz} dt,$$

and consider the value of u for large values of z. Let

$$t - \theta_r = -\tau = \tau e^{\pi i};$$

then

$$u = e^{\pi i \kappa + z\theta_r} \int_0^a \tau^\kappa e^{-z\tau} d\tau.$$

Taking real positive values of z, write

$$\tau z = y,$$

so that, as z is to have very large values, the upper limit for u with the new variable is effectively $+\infty$; thus

$$u = e^{\pi i \kappa + z\theta_r} z^{-(\kappa+1)} \int_0^\infty y^\kappa e^{-y}\, dy$$

$$= (-1)^\kappa e^{z\theta_r} z^{-(\kappa+1)} \Gamma(\kappa + 1).$$

* *T. F.*, § 24.

Also, if v denote the integral

$$\int_{\theta_r - a}^{\theta_r} (t-\theta_r)^\rho \, e^{tz} R_m \, dt,$$

then

$$v = (-1)^\rho \, e^{z\theta_r} \int_0^a \tau^\rho \, e^{-z\tau} R_m \, d\tau$$

$$= (-1)^\rho \, e^{z\theta_r} \, z^{-(\rho+1)} \int_0^\infty y^\rho \, e^{-y} R_m \, dy.$$

Further,

$$\left| \int_0^\infty y^\rho e^{-y} R_m \, dy \right| < M \left(\frac{a}{c}\right)^{m+1} \frac{1}{1-\dfrac{a}{c}} \int_0^\infty y^\rho e^{-y} \, dy$$

$$< M \left(\frac{a}{c}\right)^m \frac{1}{1-\dfrac{a}{c}} \Gamma(\rho+1),$$

which, when the real part of $\rho+1$ is positive, can be made less than any assigned finite quantity as m increases without limit, because $a<c$.

Using these results, we have

$$J' = \int_{\theta_r - a}^{\theta_r} \left\{ \sum_{\alpha=0}^{m} c_\alpha (t-\theta_r)^\alpha + R_m \right\} (t-\theta_r)^\rho \, e^{tz} \, dt$$

$$= (-1)^\rho e^{z\theta_r} z^{-(\rho+1)} \sum_{\alpha=0}^{m} (-1)^\alpha z^{-\alpha} c_\alpha \, \Gamma(\rho+\alpha+1),$$

when m is made as large as we please, and the real part of ρ is greater than -1. Hence I'' is a constant multiple of this quantity.

105. If now w_r denote the integral of the original equation, we have

$$w_r = \int e^{tz} T \, dt$$

$$= I' + I'' + I''',$$

so that

$$w_r e^{-z\theta_r} z^{\rho+1} = e^{-z\theta_r} z^{\rho+1} I' + e^{-z\theta_r} z^{\rho+1} I''' + e^{-z\theta_r} z^{\rho+1} I''.$$

For very large values of z, the first term on the right-hand side tends to the value zero; so also does the second term. The third is a constant multiple of

$$\sum_{\alpha=0}^{\infty} (-1)^\alpha z^{-\alpha} c_\alpha \, \Gamma(\rho+\alpha+1).$$

Hence, dropping the constant factor, we have

$$w_r = e^{z\theta_r} z^{-\rho-1} \sum_{\alpha=0}^{\infty} (-1)^\alpha z^{-\alpha} c_\alpha \Gamma(\rho + \alpha + 1),$$

for very large values of z. If the coefficients, of which $c_\alpha \Gamma(\rho+\alpha+1)$ is the type, constitute a converging series, then this expression has a functional significance. If they constitute a diverging series, the result is illusory from the functional point of view.

Now we have

$$\rho + 1 = \left[\frac{U_1}{\dfrac{\partial U_0}{\partial t}}\right]_{t=\theta_r} = -\sigma_r;$$

and therefore the preceding integral, when it exists, is of the form

$$w_r = e^{z\theta_r} z^{\sigma_r} \sum_{\alpha=0}^{\infty} (-1)^\alpha z^{-\alpha} c_\alpha \Gamma(\alpha - \sigma_r).$$

When the series converges, this expression agrees with the form in § 100, which is

$$w_r = e^{z\theta_r} z^{\sigma_r} \phi_r,$$

where ϕ_r is a holomorphic function of z^{-1} for large values of z.

It thus appears that, when Laplace's solution of the equation, originally obtained as a definite integral, can be expressed explicitly as a function of z, which is valid for large values of z, it becomes a normal integral of the equation.

This normal integral has arisen through the consideration of the root θ_r of the equation $U_0 = 0$. When the corresponding conditions are satisfied for any other root of that equation, there is a normal integral associated with that root. Hence, when n normal integrals exist, they can be associated with the roots of the equation $U_0 = 0$, which comprise all the finite singularities of the equation in T.

Note. It has been assumed that ρ is not an integer. When ρ is an integer, logarithms may enter into the expression of the primitive of the equation for T, and they must enter if ρ has any one of the values $0, 1, \ldots, k-2$. There is a corresponding investigation, which leads from the definite integral to the explicit expression as a normal integral. When the normal integral exists,

it can always be obtained by the process in § 100. If logarithms enter into the expression of $e^{\Omega}u$, they enter into the expression of u in the usual mode of constructing the regular integrals of the equation satisfied by u.

Ex. 1. The preceding method of obtaining the normal integral gives a test as to the convergence of the series in its expression. If the infinite series

$$\sum_{a=0} (-1)^a z^a c_a \Gamma(\rho+a+1)$$

converges, which must be the case if the expression for the developed definite integral is not to prove illusory, its radius of convergence r is given by the relation*

$$\operatorname*{Lim}_{a=\infty} |c_a \Gamma(\rho+a+1)|^{\frac{1}{a}} = \frac{1}{r}.$$

But, from Stirling's theorem for the approximation to the value of $\Gamma(n)$, when n is infinitely large, we have

$$\operatorname*{Lim}_{a=\infty} |\Gamma(\rho+a+1)|^{\frac{1}{a}} = \infty;$$

hence

$$\operatorname*{Lim}_{a=\infty} |c_a|^{\frac{1}{a}} = 0.$$

Thus the series

$$c_0 + c_1(t-\theta_r) + c_2(t-\theta_r)^2 + \dots$$

must converge over the whole of the t-plane; and therefore the integral T_r is of the form

$$(t-\theta_r)^{\rho}\phi(t),$$

where $\phi(t)$ is holomorphic over the whole plane: a result due to Poincaré.

Ex. 2. Prove that, if the condition in Ex. 1 is satisfied, a normal integral certainly exists. (Poincaré.)

Ex. 3. Consider Bessel's equation

$$x^2 w'' + xw' + (x^2 - n^2) w = 0,$$

for large values of $|x|$. The integrals in the vicinity of $x=\infty$ may be normal—they are not regular—and, if normal, must be of grade unity. Accordingly, let

$$w = e^{\theta x} u;$$

then the equation for u is

$$x^2 u'' + (x + 2\theta x^2) u' + \{x^2(1+\theta^2) + \theta x - n^2\} u = 0.$$

We take

$$\theta^2 + 1 = 0,$$

* *T. F.*, § 26.

and then seek for a regular integral (if any) of the equation

$$x^2u'' + (x + 2\theta x^2)\, u' + (\theta x - n^2)\, u = 0.$$

If an integral, regular in the vicinity of $x = \infty$, can exist, it is of the form

$$u = \sum_{m=0} c_m x^{\rho - m}.$$

Substituting, and making the coefficients in the resulting equation vanish, we have

$$c_0\,(2\theta\rho + \theta) = 0,$$

and, for all values of m,

$$c_m \{(\rho - m)^2 - n^2\} + \theta c_{m+1}\,(2\rho - 2m - 1) = 0.$$

The former gives

$$\rho = -\tfrac{1}{2};$$

and the latter then gives

$$c_m = \frac{\{(m-\frac{1}{2})^2 - n^2\}\,\{(m-\frac{3}{2})^2 - n^2\}\ldots\{(\frac{1}{2})^2 - n^2\}}{m!\,(2\theta)^m}\, c_0.$$

Hence, taking $c_0 = 1$, and $\theta = i$, a formal solution of the original equation is

$$w_1 = x^{-\frac{1}{2}} e^{ix} \sum_{m=0} \frac{\{(m-\frac{1}{2})^2 - n^2\}\ldots\{(\frac{3}{2})^2 - n^2\}\,\{(\frac{1}{2})^2 - n^2\}}{m!}\, \frac{1}{(2ix)^m};$$

and taking $\theta = -i$, $c_0 = 1$, another formal solution is

$$w_2 = x^{-\frac{1}{2}} e^{-ix} \sum_{m=0} \frac{\{(m-\frac{1}{2})^2 - n^2\}\ldots\{(\frac{3}{2})^2 - n^2\}\,\{(\frac{1}{2})^2 - n^2\}}{m!}\, \frac{1}{(-2ix)^m}.$$

If $2n$ is an odd integer, positive or negative, both series terminate; and the formal solutions constitute two normal integrals of the equation. It is not difficult to obtain an expression given by Lommel* for J_n, in a form that is the equivalent of

$$\pi^{\frac{1}{2}} J_n = i^{-n-1} w_1 + i^{n+1} w_2.$$

If $2n$ is not an odd integer, both series diverge; and the formal solutions are then illusory as functional solutions.

When Laplace's method of solution is adopted, so as to give an integral of the form

$$w = \int e^{tx}\, T\, dt,$$

the equation for T is

$$-n^2 T - \frac{d}{dt}(tT) + \frac{d^2}{dt^2}\{(t^2+1)\, T\} = 0,$$

that is,

$$(t^2+1)\, T'' + 3tT' + (1 - n^2)\, T = 0.$$

On writing

$$t = i - 2iv,$$

where v is a new independent variable, the equation for T is

$$v\,(1-v)\frac{d^2T}{dv^2} + (\tfrac{3}{2} - 3v)\frac{dT}{dv} - (1 - n^2)\, T = 0.$$

* *Math. Ann.*, t. IV (1871), p. 115.

This is the differential equation of a hypergeometric function, whose (Gaussian) elements are given by

$$\gamma=\tfrac{3}{2},\quad a+\beta+1=3,\quad a\beta=1-n^2.$$

The contour of the integral consists of (i) a circle round i as centre with radius less than 2 (so as to exclude $-i$, the other finite singularity of the equation in T), and then (ii) a duplicated line from a point in the circumference passing in the direction of a diameter continued towards $-\infty$. The argument of t and the argument of x must be such that the real part of xt is negative. In order to construct the integral, we need the complete primitive of the T-equation in the vicinity of $v=0$: it is

$$T=AF(a,\beta,\gamma,v)+Bv^{-\frac{1}{2}}F(a-\gamma+1,\beta-\gamma+1,2-\gamma,v),$$

where A and B are arbitrary constants. The part multiplying A, being a holomorphic function, merely contributes a zero term to w; and we need therefore substitute only the other part. Manifestly, we may write $B=1$. Now

$$\begin{aligned}F(a-\gamma+1,\beta-\gamma+1,2-\gamma,v)&=F(a-\tfrac{1}{2},\beta-\tfrac{1}{2},\tfrac{1}{2},v)\\&=\sum_{m=0} c_m v^m,\end{aligned}$$

where

$$c_m=\frac{(a-\frac{1}{2})(a+\frac{1}{2})\ldots(a+m-\frac{3}{2})(\beta-\frac{1}{2})(\beta+\frac{1}{2})\ldots(\beta+m-\frac{3}{2})}{m!\,\frac{1}{2}\cdot\frac{3}{2}\cdot\ldots\cdot\frac{2m+1}{2}}.$$

But

$$(a+p)(\beta+p)=p^2+2p+1-n^2=(p+1)^2-n^2;$$

also

$$\tfrac{1}{2}\cdot\tfrac{3}{2}\cdot\ldots\cdot\frac{2m+1}{2}=\Pi(m+\tfrac{1}{2})\div\Pi(-\tfrac{1}{2});$$

so that

$$c_m=\frac{\{(\frac{1}{2})^2-n^2\}\{(\frac{3}{2})^2-n^2\}\ldots\{(m-\frac{1}{2})^2-n^2\}}{m!\,\Pi(m+\frac{1}{2})}\Pi(-\tfrac{1}{2}).$$

Taking this value of c_m, we substitute

$$T=v^{-\frac{1}{2}}\sum_{m=0} c_m v^m$$

in the definite integral. In the preceding notation, we have

$$\theta_r=i,\quad \rho=-\tfrac{1}{2},\quad \sigma_r=-(\rho+1)=-\tfrac{1}{2},$$
$$\Pi(m-\sigma_r)=\Pi(m+\tfrac{1}{2});$$

so that, when the solution

$$w=\int e^{tx}\Big(v^{-\frac{1}{2}}\sum_{m=0} c_m v^m\Big)\,dt,$$

where $t=i-2iv$, is expanded into explicit form, it becomes a constant multiple of

$$e^{xi}x^{-\frac{1}{2}}\sum_{m=0}\left\{(-1)^m x^{-m}\frac{c_m}{(-2i)^m}\Pi(m+\tfrac{1}{2})\right\}.$$

But

$$c_m \Pi(m+\tfrac{1}{2}) = \frac{\{(\frac{1}{2})^2 - n^2\}\{(\frac{3}{2})^2 - n^2\}\ldots\{(m-\frac{1}{2})^2 - n^2\}}{m!} \Pi(-\tfrac{1}{2});$$

so that, after substituting for c_m and rejecting the constant factor $\Pi(-\frac{1}{2})$, the integral becomes a constant multiple of

$$e^{xi} x^{-\frac{1}{2}} \sum_{m=0} \frac{\{(\frac{1}{2})^2 - n^2\}\{(\frac{3}{2})^2 - n^2\}\ldots\{(m-\frac{1}{2})^2 - n^2\}}{m!} \frac{1}{(2ix)^m},$$

which agrees formally with the expression earlier obtained.

The corresponding integral, associated with the primitive of the T-equation in the vicinity of $t=-i$ as a singularity, can be similarly deduced*.

Ex. 4. Shew that the equation

$$xy'' + (a_1 x + b_1) y' + (a_2 x + b_2) y = 0,$$

where $a_1^2 - 4a_2$ is not zero, can be transformed to

$$xw'' + (\lambda_1 + \lambda_2 + 2) w' + \{x + i(\lambda_1 - \lambda_2)\} w = 0.$$

Assuming $\lambda_1, \lambda_2, \lambda_1 + \lambda_2$ not to be integers, prove that the latter equation is satisfied by

$$w = \int (t-i)^{\lambda_1} (t+i)^{\lambda_2} e^{xt} dt,$$

for an appropriate contour independent of x; and deduce the normal series which formally satisfy the equation. (Horn.)

Double-loop Integrals.

106. Before proceeding further with the investigation in §§ 101—105, which is concerned partly with the precise determination of a definite integral satisfying the linear differential equation, we shall interrupt the argument, in order to mention another application of definite integrals to the solution of certain classes of linear equations. It is due to Jordan† and to Pochhammer‡, who appear to have devised it independently of one

* In connection with the solution of Bessel's equation by means of definite integrals, papers by Hankel, *Math. Ann.*, t. I (1869), pp. 467—501; Weber, *ib.*, t. XXXVII (1890), pp. 404—416; Macdonald, *Proc. Lond. Math. Soc.*, t. XXIX (1898), pp. 110—115, *ib.*, t. XXX (1899), pp. 165—179; and the treatise by Graf u. Gubler, *Einleitung in die Theorie der Besselschen Funktionen*, (Bern), t. I (1898), t. II (1900); may be consulted.

† *Cours d'Analyse*, 2e éd., t. III (1896), pp. 240—276; it had appeared in the earlier edition of this work.

‡ *Math. Ann.*, t. XXXV (1890), pp. 470—494, 495—526; *ib.*, t. XXXVII (1890), pp. 500—511.

another. A brief sketch is all that will be given here: for details and for applications, reference may be made to the sources just quoted, and to a memoir by Hobson*, who gives an extensive application of the method to harmonic analysis.

As indicated by Jordan, the method is most directly useful in connection with an equation of the form

$$\Delta(w) = Q(z)\frac{d^n w}{dz^n} - \alpha Q'(z)\frac{d^{n-1}w}{dz^{n-1}} + \frac{\alpha(\alpha+1)}{2}Q''(z)\frac{d^{n-2}w}{dz^{n-2}} - \dots$$
$$- R(z)\frac{d^{n-1}w}{dz^{n-1}} + (\alpha+1)R'(z)\frac{d^{n-2}w}{dz^{n-2}} - \dots = 0,$$

where $Q(z)$ and $zR(z)$ are polynomials, one of degree n, the other of degree $\leqslant n$ in z, $R(z)$ also being a polynomial. For simplicity, we shall assume $Q(z)$ to be of degree n.

Consider an integral

$$W = \int T(t-z)^{\alpha+n-1}\,dt,$$

where T is a function of t alone: this function of t has to be determined, as well as the path of integration. We have

$$\Delta W \div (-1)^n(\alpha+n-1)(\alpha+n-2)\dots(\alpha+1)$$
$$= \int\Big[\alpha(t-z)^{\alpha-1}\left\{Q(z) + (t-z)Q'(z) + \frac{(t-z)^2}{2\,!}Q''(z) + \dots\right\}$$
$$+ (t-z)^\alpha\left\{R(z) + (t-z)R'(z) + \frac{(t-z)^2}{2\,!}R''(z) + \dots\right\}\Big]\,Tdt$$
$$= \int[\alpha(t-z)^{\alpha-1}Q(t) + (t-z)^\alpha R(t)]\,Tdt,$$

the summation being possible because Q and R are polynomials of the specified degrees. The integral will be capable of simplification, if the integrand is a perfect differential; accordingly, we choose T so that

$$TR(t) = \frac{d}{dt}\{TQ(t)\},$$

which gives

$$T = \frac{1}{Q(t)}e^{\int\frac{R(t)}{Q(t)}dt}.$$

* *Phil. Trans.*, 1896 (A), pp. 443—531.

The preceding integral then becomes

$$\int dV,$$

where

$$V = (t-z)^a TQ(t)$$
$$= (t-z)^a e^{\int \frac{R(t)}{Q(t)} dt}.$$

Hence the original differential equation will be satisfied if

$$\int dV = 0;$$

and this will be the case, if the path of integration is either

(i) a closed contour such that the initial and the final values of V are the same: or

(ii) a line, not a closed contour, such that V vanishes at each extremity*.

Each such distinct path of integration gives an integral. It is proved by Jordan that there is a path of the first kind, for each root of Q; and that, when there is a multiple root of Q, paths of the second kind are to be used.

Again, restricting $Q(z)$ for the sake of simplicity, we assume that each of its n zeros is simple; let them be $a_1, a_2, \ldots, a_n$. As the polynomial $R(z)$ is of degree less than n, we have

$$\frac{R(t)}{Q(t)} = \sum_{r=1}^{n} \frac{\gamma_r}{t-a_r},$$

where $\gamma_1, \ldots, \gamma_n$ are constants; and then

$$V = (t-z)^a \prod_{r=1}^{n} (t-a_r)^{\gamma_r}.$$

To obtain the paths desired, take any initial point in the plane; from it, draw loops† round the points $a_1, \ldots, a_n, z$, and denote these by $A_1, A_2, \ldots, A_n, Z$. Take any determination of

$$T(t-z)^{a+n-1},$$

* A third possibility would arise, if the path were a line such that V has the same value at its extremities: but this case is of very restricted occurrence.

† *T. F.*, § 90.

that is, of

$$I = (t - z)^{a+n-1} \prod_{r=1}^{n} (t - a_r)^{\gamma_r - 1},$$

which is the subject of integration in W, as an initial value; and let the values of W, for the various loops $A_1, \ldots, A_n, Z$ with this as the initial value, be denoted by $W(a_1), \ldots, W(a_n), W(z)$ respectively.

An integral of the original differential equation will be obtained, if the path of integration gives to V a final value the same as its initial value. Such a path can be made up of $A_r A_s A_r^{-1} A_s^{-1}$, that is, first the loop A_r, then the loop A_s, then the loop A_r reversed, then the loop A_s reversed. Let $W(a_r, a_s)$ denote the value of the integral for this path; then $W(a_r, a_s)$ is a solution of the differential equation. Taking the above initial value (say I_0) for I, we have

$$\begin{aligned} W(a_r, a_s) &= W(a_r) + e^{2\pi i \gamma_r} W(a_s) - e^{2\pi i \gamma_s} W(a_r) - W(a_s) \\ &= \{1 - e^{2\pi i \gamma_s}\} W(a_r) - \{1 - e^{2\pi i \gamma_r}\} W(a_s); \end{aligned}$$

for after the description of A_r, the initial value of I is $e^{2\pi i \gamma_r} I_0$ for the description of A_s; it is $e^{2\pi i (\gamma_r + \gamma_s)} I_0$ for the description of A_r^{-1}, and it is $e^{2\pi i \gamma_s} I_0$ for the description of A_s^{-1}.

It is clear that

$$W(a_r, a_s) = - W(a_s, a_r),$$

$$\{1 - e^{2\pi i \gamma_t}\} W(a_s, a_r) = \{1 - e^{2\pi i \gamma_r}\} W(a_s, a_t) + \{1 - e^{2\pi i \gamma_s}\} W(a_t, a_r);$$

and therefore all these values of the integrals, for the various appropriate paths, can be expressed linearly in terms of any n of the quantities $W(a_r, a_s)$, in particular, in terms of

$$W(z, a_1), \; W(z, a_2), \; \ldots, \; W(z, a_n).$$

Each such quantity is an integral of the original equation; and we therefore have n integrals of that equation.

Note. For the special cases when a or any of the constants γ is an integer; for the cases when $Q(t)$ has multiple roots; and for the cases when $R(t)$ is of degree $n-1$, while $Q(t)$ is of degree less than $n-1$; reference may be made to the authorities previously cited. As already stated, all that is given here is merely a brief indication of the method of double-loop integrals.

Ex. 1. Consider the equation of the quarter-period in elliptic functions, viz.

$$z(z-1)\frac{d^2w}{dz^2}+(2z-1)\frac{dw}{dz}+\tfrac{1}{4}w=0.$$

Here we have

$$Q(z)=z(z-1),$$

$$a=-\tfrac{3}{2},\quad n=2,$$

$$R(z)=z-\tfrac{1}{2};$$

thus

$$\frac{R(z)}{Q(z)}=\frac{\frac{1}{2}}{z}+\frac{\frac{1}{2}}{z-1},$$

so that

$$\gamma_0=\tfrac{1}{2}=\gamma_1,$$

and

$$T=\frac{1}{t(t-1)}t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}=t^{-\frac{1}{2}}(t-1)^{-\frac{1}{2}}.$$

Accordingly, we have

$$W=\int t^{-\frac{1}{2}}(t-1)^{-\frac{1}{2}}(t-z)^{-\frac{1}{2}}\,dt,$$

and the path of integration has to be settled.

We have

$$W(0)=2\int_a^0 dW,$$

$$W(1)=2\int_a^1 dW,$$

$$W(z)=2\int_a^z dW,$$

where a marks the initial point of the loops. Hence

$$W(0,1)=2W(0)-2W(1)=4\int_0^1 dW,$$

$$W(0,z)=2W(0)-2W(z)=4\int_0^z dW;$$

and thus two integrals of the equation are given by

$$\int_0^1 dW,\quad \int_0^z dW.$$

The comparison with the known results is immediate.

Ex. 2. Integrate in the same way the equation

$$(1-z^2)\frac{d^2w}{dz^2}-2az\frac{dw}{dz}+bw=0,$$

where a and b are constants. (This is another form of the equation

$$(1-z^2)\frac{d^2w}{dz^2}-2(m+1)z\frac{dw}{dz}+(n-m)(n+m+1)w=0,$$

discussed by Hobson (*l.c.*) for unrestricted values of the constants m and n.)

Ex. 3. Prove that when the equation

$$\frac{d^2w}{du^2} = w\{n(n+1)\wp(u) + h\},$$

where h is a constant, is subjected to the transformation

$$z = \wp(u),$$

the transformed equation (which is of Fuchsian type, § 54) can, under a certain condition, be treated by the foregoing method: and assuming the condition to be satisfied, obtain the integral.

Ex. 4. Apply the method to the equation

$$z\frac{d^2w}{dz^2} + (2n+1)\frac{dw}{dz} + zw = 0;$$

apply it also to the equation of the hypergeometric series.

(Jordan; Pochhammer.)

Ex. 5. Apply the method to solve the equation

$$(1-z^2)w'' - 2zw' - \tfrac{1}{4}w = 0,$$

for real values of z such that $-1 < z < 1$. Shew that the equation is transformed into itself by the relations

$$(z-1)(Z-1) = 4, \quad w(z+1)^{\frac{1}{2}} = W(Z+1)^{\frac{1}{2}};$$

and deduce the solution for real values of z such that $1 < z < \infty$.

(Math. Trip., Part II, 1900.)

Poincaré's Asymptotic Representations of an Integral.

107. After this digression, we resume the consideration of the investigations in §§ 101—105.

In those cases when the infinite series in a normal integral diverges, the normal integral has been rejected as illusory from the functional point of view. There are, however, cases belonging to a general class which, while certainly illusory as functions of the variable, are still of considerable use in another aspect: they are *asymptotic* representations of the integral, to use Poincaré's phrase*.

A diverging series of the form

$$c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \ldots + \frac{c_n}{x^n} + \ldots,$$

* *Acta Math.*, t. VIII, p. 296.

is said to represent a function $J(x)$ asymptotically when, if S_n denote the sum of the first $n+1$ terms, the quantity

$$x^n \{J(x) - S_n\}$$

tends towards zero when x increases indefinitely: so that, when x is sufficiently large, we have $x^n \{J(x) - S_n\} < \epsilon$, where $|\epsilon|$ is a small quantity. The error, committed in taking S_n as the value of J, is less than

$$\frac{\epsilon}{x^n},$$

which is much smaller than

$$\frac{c_n}{x^n},$$

that is, the error in taking S_n as the value is much smaller than in taking S_{n-1}. (The definition, though stated only for large values of x, applies also to the vicinity of any point in the finite part of the plane, *mutatis mutandis*.)

The asymptotic representation is, however, not effective for all values of the argument of the independent variable. If $x^n \{J(x) - S_n\}$ tended uniformly to zero for all infinitely large values of x, the function $J(x)$ would be holomorphic, and the series would converge: the permissible values of the argument of the independent variable are therefore restricted. It is manifest from the nature of the case that, when such a series is an asymptotic representation of a function, the series can be used for the numerical calculation of the approximate value of the function for large values of x with a permissible argument: the error at any stage is much less than the magnitude of the term last included. Without entering upon any discussion of the question why a diverging series, which is functionally invalid, can yet, when it is an asymptotic representation of a function, be of utility for the numerical calculation of the function, it is proper to mention one conspicuous example of the use of such series, as found in their application to dynamical astronomy*.

The normal series, derived from the solution of the equation as represented accurately by the definite integrals, are proved by Poincaré to give this type of asymptotic representation of the

* In particular, see Poincaré, *Mécanique Céleste*, t. II.

solution. For, denoting the solution by w, and the sum of the first $m+1$ terms of the series

$$\sum_{\alpha=0} (-1)^\alpha z^{-\alpha} c_\alpha \Gamma(\rho+\alpha+1)$$

by S_m, we have

$$z^m \left\{we^{-z\theta_r} z^{\rho+1} - S_m\right\} = z^{\rho+1+m}\int_{\theta_r-a}^{\theta_r} R_m (t-\theta_r)^\rho e^{z(t-\theta_r)}\, dt.$$

Now

$$R_m = Mu \frac{(t-\theta_r)^{m+1}}{c^{m+1}} \frac{1}{1-\dfrac{|t-\theta_r|}{c}},$$

where

$$|t-\theta_r| \leqslant a < c,$$

and

$$|u| < 1.$$

Then, as before, we have

$$z^{\rho+1+m}\int R_m (t-\theta_r)^\rho e^{z(t-\theta_r)}\, dt$$

$$= \int \frac{M}{c^{m+1}} \frac{u}{1-\dfrac{|t-\theta_r|}{c}} z^{\rho+1+m} (t-\theta_r)^{m+1+\rho} e^{z(t-\theta_r)}\, dt,$$

which is a multiple of

$$\int_0^a \frac{u}{1-\dfrac{|t-\theta_r|}{c}} z^{\rho+1+m} \tau^{m+1+\rho} e^{-z\tau}\, d\tau$$

by a quantity independent of z. When we take

$$\tau z = y,$$

so that, as z is to have large values, the limits of y effectively are 0 to $+\infty$, the last definite integral is a multiple of

$$\frac{1}{z}\int_0^\infty \frac{u}{1-\dfrac{|t-\theta_r|}{c}} y^{\rho+1+m} e^{-y}\, dy.$$

This definite integral is finite. Denoting its value by I, we have

$$z^m(we^{-z\theta_r} z^{\rho+1} - S_m) = \frac{\alpha I}{z},$$

where α is a quantity independent of z, and I is finite. Hence, when z is sufficiently large, we have

$$z^m(we^{-z\theta_r} z^{\rho+1} - S_m) < \epsilon,$$

where $|\epsilon|$ is a small quantity; and so we can say that S_m asymptotically represents $we^{-z\theta_r}z^{\rho+1}$, or we can say that the normal series is an asymptotic representation of the actual integral, the representation being valid (on the hypotheses adopted earlier) for large positive real values of z.

Note. For further discussion of these asymptotic expansions in connection with linear differential equations, reference may be made to Poincaré's memoir*, which initiated the idea. Among other memoirs, in which the subject is developed and new applications are made, special mention should be made of those† by Kneser, and those|| by Horn. Picard's chapter‡ on the subject may also be consulted with advantage: and a corresponding discussion on integration by definite integrals is given by Jordan§.

Ex. 1. Shew that the complete primitive of the differential equation

$$\frac{d^2w}{dx^2}+w\left(a^2+\frac{a_1}{x}+\frac{a_2}{x^2}+\dots\right)=0,$$

in the vicinity of $x=\infty$, can be asymptotically represented by

$$\left(\alpha_0+\frac{\alpha_1}{x}+\frac{\alpha_2}{x^2}+\dots\right)\cos X+\left(\beta_0+\frac{\beta_1}{x}+\frac{\beta_2}{x^2}+\dots\right)\sin X,$$

where

$$X=ax+\tfrac{1}{2}\frac{a_1}{a}\log x,$$

and α_0, β_0 are arbitrary constants. (Kneser.)

Ex. 2. In the differential equation

$$\frac{d}{dx}\left(A\frac{dy}{dx}\right)+(k^2B+C)\,y=0,$$

k^2 is an arbitrary parameter, A, B, C are real functions of x and (with their derivatives) are holomorphic when $a \leqslant x \leqslant b$; moreover, A and B are positive. Prove that an integral of the equation, determined by initial values that are independent of k, is a holomorphic transcendental function of k; and shew that, for large values of k, its asymptotic expansion is of the form

$$y=\left(\phi_0+\frac{\phi_2}{k^2}+\dots\right)\cos kw+\left(\frac{\phi_1}{k}+\frac{\phi_3}{k^2}+\dots\right)\sin kw,$$

where ϕ_0, ϕ_1, ϕ_2, ..., w are functions of x. (Horn.)

* *Acta Math.*, t. VIII (1886), pp. 295—344.

† *Crelle*, t. CXVI (1896), pp. 178—212; *ib.*, t. CXVII (1897), pp. 72—103; *ib.*, t. CXX (1899), pp. 267—275; *Math. Ann.*, t. XLIX (1897), pp. 383—399.

|| *Math. Ann.*, t. XLIX (1897), pp. 452—472, 473—496; *ib.*, t. L (1898), pp. 525—556; *ib.*, t. LI (1899), pp. 346—368; *ib.*, t. LII (1899), pp. 271—292, 340—362.

‡ *Cours d'Analyse*, t. III, ch. XIV.

§ *Cours d'Analyse*, t. III, ch. II, § IV.

Ex. 3. Shew that the equation

$$\frac{d^2y}{dx^2}+(k^2-k_1\cos 2x)\,y=0$$

has a solution of the form

$$\sum_{m=0}^{\infty} k_1{}^m\,(A_m\cos kx+B_m\sin kx),$$

where A_m, B_m are rational functions of k, and that it has an asymptotic solution of the form

$$\left(\phi_0+\frac{\phi_2}{k^2}+\dots\right)\cos kx+\left(\frac{\phi_1}{k}+\frac{\phi_3}{k^3}+\dots\right)\sin kx\,;$$

and indicate the relation of the solutions to one another. (Poincaré: Horn.)

Equations of Rank greater than Unity replaced by Equations of Rank Unity.

108. When the differential equation

$$p_0\frac{d^nw}{dz^n}+p_1\frac{d^{n-1}w}{dz^{n-1}}+\dots+p_nw=0$$

possesses, in the vicinity of $z=\infty$, normal integrals which are of grade m, then, denoting the degree of the polynomial p_0 by ϖ_0, it follows (as in § 85) that the degree ϖ_r of the polynomial p_r is such that

$$\varpi_r \leqslant \varpi_0+r\,(m-1),$$

the sign of equality holding for some at least of the degrees. Also, if e^{Ω} be the determining factor of any such integral, then Ω' is the aggregate of the first m terms in the expansion, in descending powers of z, of a root of the equation

$$p_0Z^n+p_1Z^{n-1}+\dots+p_n=0.$$

The existence of the normal integral then depends upon the possession of regular integrals by the linear equation in u, where

$$w=e^{\Omega}u.$$

In the case where $m=1$, the method of Laplace certainly gives the integrals of the differential equation, even when the normal series diverge; but it is not applicable, when m is greater than unity. Poincaré, however, devised a method by which the given equation is associated with an equation of grade unity: Laplace's method is applicable to the new equation, so that its primitive is

known: and from this primitive, an integral of the original equation can be obtained by means of one quadrature. The new equation is of order n^m; and the investigation leads to an expression for

$$\frac{1}{w}\frac{dw}{dz},$$

which, when it exists, can be obtained more directly by Cayley's process (§ 92).

Poincaré's method is as follows. Let the given equation be supposed to possess n normal integrals of grade m, say, in the form

$$e^{\Omega_1(z)}\phi_1(z),\quad e^{\Omega_2(z)}\phi_2(z),\ \ldots,\ e^{\Omega_n(z)}\phi_n(z);$$

let these be denoted by $f_1(z), f_2(z), \ldots, f_n(z)$.

Let α denote a primitive mth root of unity, say $e^{\frac{2\pi i}{m}}$; and consider, in connection with any integral $f(z)$ of the original equation, a product

$$y = \prod_{r=0}^{m-1} f(\alpha^r z).$$

Then y satisfies an equation of order n^m, which possesses n^m normal integrals

$$f_a(z) f_b(\alpha z) f_c(\alpha^2 z) \ldots f_k(\alpha^{m-1} z),$$

where $a, b, c, \ldots, k$ are the numbers $1, 2, \ldots, n$ or some of them, any number of repetitions being permitted; and these normal integrals are of grade m. Let

$$n^m = N,$$

and let the equation for y be

$$Q_N \frac{d^N y}{dz^N} + Q_{N-1}\frac{d^{N-1}y}{dz^{N-1}} + \ldots + Q_0 y = 0,$$

where, if Q_N be of degree θ in z, then the degree of Q_{N-r} in general is equal to $\theta + r(m-1)$, because of the grade of the normal integrals. Owing to the source of the quantity y, which clearly is not changed if z be replaced by $z\alpha^s$, s being any integer, it follows that the equation for y must remain substantially unchanged, when this change of variable is made; hence

$$\frac{Q_{N-r}(z\alpha^s)\,\alpha^{-(N-r)s}}{Q_{N-r}(z)} = \lambda,$$

where λ is independent of r.

Now let the variable be changed from z to x, where

$$z^m = mx;$$

then, because

$$\frac{d^\kappa y}{dz^\kappa} = \sum_{a=0} c_{a\kappa} \frac{d^{\kappa-a} y}{dx^{\kappa-a}} z^{(\kappa-a)m-\kappa},$$

for all values of κ, the coefficients $c_{a\kappa}$ being numerical, the equation for y takes the form

$$\sum_{q=0}^{N} R_{N-q} \frac{d^{N-q} y}{dx^{N-q}} = 0,$$

where

$$R_{N-q} = \sum_{r=0}^{q} c_{q-r,\,N-r} z^{(N-q)m-(N-r)} Q_{N-r}.$$

The degree of R_{N-q} in z, as it is determined by the highest terms in Q_{N-q}, is

$$\begin{aligned} &\theta + q(m-1) + (N-q)m - (N-q) \\ &= \theta + N(m-1), \end{aligned}$$

which is independent of q; so that the degree* of all the coefficients R is the same. Further, we have

$$\begin{aligned} R_{N-q}(z\alpha^s) &= \sum_{r=0}^{q} c_{q-r,\,N-r} (z\alpha^s)^{(N-q)m-N-r} Q_{N-r}(z\alpha^s) \\ &= \lambda \sum_{r=0}^{q} c_{q-r,\,N-r} z^{(N-q)m-N-r} Q_{N-r}(z), \end{aligned}$$

for the power of α is

$$\begin{aligned} &\alpha^{s(N-q)m} \\ &= (\alpha^m)^{s(N-q)} = 1; \end{aligned}$$

thus

$$R_{N-q}(z\alpha^s) = \lambda R_{N-q}(z).$$

Hence the equation is substantially unaltered, when z is replaced by $z\alpha^s$ in the coefficients R; hence, multiplying by a power of z, say z^κ, where

$$\kappa + \theta + N(m-1) \equiv 0 \pmod{m},$$

R becomes a uniform function of x, when we substitute

$$\frac{1}{m} z^m = x.$$

* Some might have vanishing coefficients in particular cases: the argument deals with the general case.

The new equation is therefore an equation in the independent variable x such that all its coefficients are uniform. They all are of the same degree, so that it is of rank unity; it has normal integrals, and some of its integrals may be subnormal. Laplace's method can be applied to this equation; and we then have a solution in the form of a definite integral.

The way in which this definite integral is used, in order to bring us nearer a solution of the original equation, is as follows. Let

$$w_s = f(z\alpha^s), \qquad (s = 0, 1, \ldots, m-1),$$

and let

$$y = w_0 w_1 \ldots w_{m-1}.$$

This has to be differentiated $N(=n^m)$ times, derivatives of w_0, $w_1, \ldots, w_{m-1}$ of order n being replaced, whenever they occur, by their values in terms of derivatives of lower order, as given by the differential equations which they satisfy; and, from the $N+1$ equations involving $y, \dfrac{dy}{dx}, \ldots, \dfrac{d^N y}{dx^N}$, the N products

$$\frac{d^a w_0}{dz^a} \cdot \frac{d^b w_1}{dz^b} \cdot \ldots \cdot \frac{d^k w_{m-1}}{dz^k}$$

where $a, b, \ldots, k$ each can have the values $0, 1, \ldots, m-1$, are eliminated. The result is the equation for y. The N equations involving $y, \dfrac{dy}{dx}, \ldots, \dfrac{d^{N-1}y}{dx^{N-1}}$ can be regarded as giving these N products of the type

$$\frac{d^a w_0}{dz^a} \cdot \frac{d^b w_1}{dz^b} \cdot \ldots \cdot \frac{d^k w_{m-1}}{dz^k}$$

each in terms of derivatives of y and the variables. Let two such be

$$w_0 w_1 \ldots w_{m-1}, = y,$$

$$\frac{dw_0}{dz} w_1 \ldots w_{m-1}, = \Phi;$$

then

$$\frac{1}{w_0}\frac{dw_0}{dz} = \frac{\Phi}{y}.$$

Assuming y known, as an integral of its own equation, the value of w_0 is derivable by a quadrature. If y, first obtained as a

definite integral, can be evaluated into a functionally valid normal integral, it is of the form

$$y = e^{ax} Y.$$

The function Φ is linear in y and the derivatives of y, so that, when we substitute the value of y, we have

$$\Phi = e^{ax} \Psi,$$

where Ψ is free from exponentials; and then

$$\frac{1}{w_0}\frac{dw_0}{dz} = \frac{\Psi}{Y},$$

which can be expressed as a series in terms of z. The exponent to which it belongs is easily seen to be an integer, owing to the form of Φ; thus

$$\frac{1}{w_0}\frac{dw_0}{dz} = a_0 z^{m-1} + a_1 z^{m-2} + \ldots + a_{m-1} + \frac{a_m}{z} + \frac{a_{m+1}}{z^2} + \ldots.$$

But if y cannot be evaluated into a functionally valid normal integral, there may be insuperable difficulty in dealing with the quantity $\dfrac{\Phi}{y}$.

In instances, where the actual expression of a normal integral (if it exists) is desired, the process is manifestly cumbrous: as it does not lead to explicit tests for the existence of normal integrals, the simpler plan is to adopt the process indicated in §§ 85—88, which gives either a normal integral or an asymptotic expression for an integral in the form of a normal series.

For further consideration of Poincaré's method, reference may be made to his memoir, already quoted, and to a memoir by Horn*, who discusses in some detail the case, when the linear equation is of the second order and of rank p.

Ex. 1. In the case of an equation of the second order which is of rank 2, say

$$\frac{d^2w}{dx^2} + A_0(x)\frac{dw}{dx} + A_1(x)\,w = 0,$$

shew that, if $w = \phi(x)$, and if $w_1 = \phi(-x)$, which will satisfy the equation

$$\frac{d^2w_1}{dx^2} - A_0(-x)\frac{dw_1}{dx} + A_1(-x)\,w_1 = 0,$$

* *Acta Math.*, t. XXIII (1900), pp. 171—201.

say

$$\frac{d^2w_1}{dx^2}+B_0(x)\frac{dw_1}{dx}+B_1(x)w_1=0,$$

then a variable y, where

$$y=ww_1,$$

generally satisfies an equation of the fourth order, and that $\dfrac{1}{w}\dfrac{dw}{dx}$ is expressible uniquely in terms of y.

If, however, the invariants of the two equations are equal, so that

$$B_1-\tfrac{1}{2}\frac{dB_0}{dx}-\tfrac{1}{4}B_0^2=A_1-\tfrac{1}{2}\frac{dA_0}{dx}-\tfrac{1}{4}A_0^2,$$

shew that y satisfies an equation of the third order, and that $\dfrac{1}{w}\dfrac{dw}{dx}$ is the root of a quadratic equation, the coefficients of which are expressible in terms of y. (Horn.)

Ex. 2. Discuss the equation

$$xy''=(x^3+1)y,$$

for large values of x. (Poincaré.)

Ex. 3. Shew that, in the vicinity of $x=\infty$, the equation

$$y''=(x^2+a)y$$

possesses a normal integral of the second grade, when a is an odd positive integer.

Ex. 4. Obtain the normal integrals of the equations

(i) $x^2y''=(x^4+\frac{3}{4})y,$

(ii) $x^2y''=2x(1+bx)y+(x^4-b^2x^2-2bx-\frac{1}{4})y,$

in the vicinity of $x=\infty$.

CHAPTER VIII.

Infinite Determinants, and their Application to the Solution of Linear Equations.

109. In the investigations of the present chapter, infinite determinants occur. These are not discussed, as a rule, in books on determinants; a brief exposition of their properties will therefore be given here, but only to the extent required for the purposes of this chapter. Their first occurrence in connection with linear differential equations is in a memoir* by G. W. Hill: the convergence of Hill's determinant was first established† by Poincaré. Later, von Koch shewed‡ that the characteristic method in Hill's work is applicable to linear differential equations generally; with this aim, he expounded the principal properties of infinite determinants§. The following account is based upon von Koch's memoirs just quoted, and upon a memoir|| by Cazzaniga.

Let a doubly-infinite aggregate of quantities be denoted by

$$a_{i,k},$$

where i, k acquire all integer values between $-\infty$ and $+\infty$; the quantities may be real or complex, and they may be uniform functions of a real or a complex variable. They are set in an

* First published in 1877; republished *Acta Math.*, t. VIII (1886), pp. 1—36.

† *Bull. de la Soc. Math. de France*, t. XIV (1886), pp. 77—90.

‡ *Acta Math.*, t. XV (1891), pp. 53—63; *ib.*, t. XVI (1892—3), pp. 217—295.

§ For further discussion of their properties and their applications to linear differential equations, see a memoir by the same writer, *Acta Math.*, t. XXIV (1901), pp. 89—122.

|| *Annali di Matematica*, Ser. 2ª, t. XXVI (1897), pp. 143—218. Other memoirs by Cazzaniga, dealing with the same subject, are to be found in that journal, Ser. 3ª, t. I (1898), pp. 83—94, Ser. 3ª, t. II (1899), pp. 229—238.

array, so that all the quantities with their first suffix the same occur in a line, the values of k increasing from left to right, and all the quantities with their second suffix the same occur in a column, the values of i increasing from top to bottom. We then have an infinite determinant, which may be represented in the form

$$[a_{i,k}] \qquad \begin{matrix} i \\ k \end{matrix}\Big\} \begin{matrix} \infty \\ -\infty \end{matrix}\Big\}.$$

Construct the determinant $D_{m,n}$, where

$$D_{m,n} = [a_{i,k}] \qquad \begin{matrix} i \\ k \end{matrix}\Big\} \begin{matrix} m \\ -n \end{matrix}\Big\};$$

then if, as m and n increase indefinitely and without limit, $D_{m,n}$ tends to a unique definite value D, we regard the infinite determinant as converging to the value D. In all other cases, the infinite determinant diverges. To secure this convergence to a unique definite value D, it is sufficient that, when any arbitrary small quantity δ has been assigned, positive integers M and N can be found, such that

$$|D_{m+p,n+q} - D_{m,n}| < \delta,$$

for all values of m greater than M, for all values of n greater than N, and for all positive integers p and q.

The aggregate of all the quantities for which $i = k$, that is, of the quantities $\ldots, a_{-1,-1}, a_{0,0}, a_{1,1}$, as they occur in their place in the determinant, is called the *principal diagonal*, sometimes briefly *the diagonal*; and a constituent of reference in the diagonal, naturally chosen in the first instance to be $a_{0,0}$, is called the *origin*.

Let

$$a_{j,j} = 1 + A_{j,j}, \quad a_{j,l} = A_{j,l}, \quad (j \neq l);$$

then the infinite determinant converges, if the doubly-infinite series

$$\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} |A_{i,k}|$$

converges, all values of i and k between $-\infty$ and $+\infty$ occurring in the summation. To prove this, let

$$P_{m,n} = \prod_{-n}^{i=m} \left\{1 + \sum_{-n}^{k=m} A_{i,k}\right\}, \quad \overline{P}_{m,n} = \prod_{-n}^{i=m} \left\{1 + \sum_{-n}^{k=m} |A_{i,k}|\right\},$$

and consider

$$D_{m,n} = [a_{i,k}] \qquad \begin{matrix} i \\ k \end{matrix}\Big\} \begin{matrix} m \\ -n \end{matrix}\Big\}.$$

Let $P_{m,n}$ be expanded; by omitting suitable terms and changing the signs of others, we obtain $D_{m,n}$. Hence, taking $D_{m,n}$, making all the terms positive, and adding certain other positive terms, we obtain $\bar{P}_{m,n}$. Similarly, we can pass from $D_{m+p,n+q}$ to $\bar{P}_{m+p,n+q}$. Now take $D_{m+p,n+q} - D_{m,n}$; make all the terms positive, and add certain other positive terms, and we have $\bar{P}_{m+p,n+q} - \bar{P}_{m,n}$; hence

$$|D_{m+p,n+q} - D_{m,n}| < |\bar{P}_{m+p,n+q} - \bar{P}_{m,n}|.$$

But, because of the convergence of the series

$$\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} |A_{i,k}|,$$

the product $\bar{P}_{m,n}$ converges when m and n increase without limit; hence, assuming any arbitrary positive quantity δ, however small, integers M and N can be determined such that

$$\bar{P}_{m+p,n+q} - \bar{P}_{m,n} < \delta,$$

for all values of m greater than M, for all values of n greater than N, and for all positive integers p and q. Consequently, for the same integers, we have

$$|D_{m+p,n+q} - D_{m,n}| < \delta;$$

and therefore the infinite determinant converges.

Such a determinant is said* to be of the *normal form*. All the determinants with which we have to deal are of this type.

Next, *the origin may be changed in the diagonal without affecting the value of the determinant.* All the conditions for the convergence of the determinant with the new origin are satisfied; let its value be D', and let D be the value with the old origin. Then taking any small positive quantity δ, we can determine integers M and N such that

$$|D - D_{m,n}| < \delta, \quad |D' - D'_{m_1,n_1}| < \delta,$$

* von Koch, *Acta Math.*, t. XVI, p. 221.

for all values of m greater than M and all values of n greater than N, the determinant D'_{m_1, n_1} being the same as $D_{m,n}$, so that, if $a_{\theta,\theta}$ be the new origin, $m_1 = m - \theta$, $n_1 = n + \theta$. Manifestly, $D_{m,n}$ can be chosen so as to include the new origin. Hence

$$\begin{aligned} |D - D'| &= |D - D_{m,n} - (D' - D'_{m_1,n_1})| \\ &< |D - D_{m,n}| + |D' - D'_{m_1,n_1}| \\ &< 2\delta, \end{aligned}$$

so that, in the limit when δ is made infinitesimal,

$$D = D'.$$

Similarly, the value of the determinant changes its sign when two lines are interchanged, and also when two columns are interchanged: so that, if two lines be the same, or if two columns be the same, the determinant vanishes. Further, if the determinant be changed, so that the lines (in their proper order) become columns and the columns (in their proper order) become lines, the principal diagonal being unchanged, the value of the determinant remains unaltered. If, in any line in a determinant of normal form, each of the constituents be multiplied by any quantity μ, the value of the determinant is multiplied by μ; likewise for any column, and for any number of lines and columns, provided that the product of all the factors (when unlimited in number) converges.

Further, if all the constituents in any line of a converging normal determinant be replaced by a set of quantities of modulus not greater than any assigned finite quantity, the new determinant converges. In the determinant D, let the line $a_{0,k}$ (the constituents occurring for values of k) be changed, so that $a_{0,k}$ is replaced by x_k, where

$$|x_k| < A,$$

A being finite; and let D', $D'_{m,n}$ for the new determinant correspond to D, $D_{m,n}$. For comparison with $D'_{m,n}$, construct a product $\overline{P}_{m,n}$, where

$$\overline{P}_{m,n} = \prod_{-n}^{i=m} \left\{1 + \sum_{-n}^{m} |A_{i,k}|\right\},$$

i having all values from $-n$ to $+m$, except $i = 0$. Then, when $D'_{m,n}$ is expanded, there occurs in $\overline{P}_{m,n}$ a term corresponding to

every term in $D'_{m,n}$, the latter having some one factor x_p that does not occur in $\bar{P}_{m,n}$; hence

$$|\text{term in } D'_{m,n}| \leqslant A \,|\text{term in } \bar{P}_{m,n}|.$$

Now some of the terms in $D'_{m,n}$ are negative, while all the terms in $\bar{P}_{m,n}$ are positive; and terms arise in $\bar{P}_{m,n}$, the terms corresponding to which do not occur in $D'_{m,n}$. Hence

$$|D'_{m,n}| \leqslant A\,|\bar{P}_{m,n}|.$$

Similarly,

$$\begin{aligned}|D'_{m+p,n+q} - D'_{m,n}| &\leqslant A\,|\bar{P}_{m+p,n+q} - \bar{P}_{m,n}| \\ &\leqslant A\delta,\end{aligned}$$

where δ can be chosen as small as we please, because

$$\prod_{-\infty}^{\infty} \left\{1 + \sum_{-\infty}^{\infty} |A_{i,k}|\right\}$$

is a converging product.

The result, which is due to Poincaré, is thus established.

Properties of Converging Infinite Determinants.

110. The development of an infinite determinant can be deduced from the preceding properties. We have

$$\begin{aligned} D_{m,n} &= \begin{vmatrix} a_{-n,-n} & , & a_{-n,-n+1} & , & \ldots, & a_{-n,m} \\ a_{-n+1,-n}, & & a_{-n+1,-n+1}, & & \ldots, & a_{-n+1,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m,-n} & , & a_{m,-n+1} & , & \ldots, & a_{m,m} \end{vmatrix} \\ &= \Sigma \pm (a_{-n,-n} a_{-n+1,-n+1} \ldots a_{m,m}) \\ &= \Sigma_{m,n}, \end{aligned}$$

say. In this expanded form, let

$$a_{i,i} = 1 + A_{i,i}, \quad a_{i,k} = A_{i,k}, \quad (i \neq k);$$

and let every term in the new expression be changed, so as to have a positive sign and so that each factor is replaced by its modulus. The resulting expression is greater than $|\Sigma_{m,n}|$; and every term that occurs in it is contained in $\bar{P}_{m,n}$, where

$$\bar{P}_{m,n} = \prod_{-n}^{m} \left\{1 + \sum_{-n}^{m} |A_{i,k}|\right\}.$$

Also, $\overline{P}_{m,n}$ contains other terms, all of which are positive; thus

$$|\Sigma_{m,n}| < \overline{P}_{m,n}.$$

Similarly,

$$|\Sigma_{m+p,n+q} - \Sigma_{m,n}| < \overline{P}_{m+p,n+q} - \overline{P}_{m,n},$$

for all positive integers p and q. But $\overline{P}_{m,n}$, with indefinite increase of m and n, is a converging product; hence $\Sigma_{m,n}$, in the same limiting circumstances, converges absolutely. Thus the usual method of development of a finite determinant holds in the case of an infinite converging determinant of the normal form, and we have

$$D = [a_{i,k}] \qquad \left.\begin{matrix} i \\ k \end{matrix}\right\} \begin{matrix} \infty \\ -\infty \end{matrix}\bigg\}$$

$$= \Sigma \ldots a_{-2,p_2} a_{-1,p_1} a_{0,p_0} a_{1,q_1} a_{2,q_2} \ldots$$
$$(-1)^{\ldots + (p_2-2) + (p_1-1) + (p_0-0) + (q_1-1) + (q_2-2) + \ldots}$$

the sum being extended over all the permutations

$$\ldots, p_2, p_1, p_0, q_1, q_2, \ldots$$

of the integers

$$\ldots, -2, -1, 0, 1, 2, \ldots.$$

Writing

$$a_{i,i} = 1 + A_{i,i}, \quad a_{i,k} = A_{i,k}, \quad (i \neq k),$$

for all values of i and k, we at once have the expansion

$$D = 1 + \Sigma A_{i,i} + \Sigma \begin{vmatrix} A_{i,i}, & A_{i,j} \\ A_{j,i}, & A_{j,j} \end{vmatrix} + \Sigma \begin{vmatrix} A_{i,i}, & A_{i,j}, & A_{i,k} \\ A_{j,i}, & A_{j,j}, & A_{j,k} \\ A_{k,i}, & A_{k,j}, & A_{k,k} \end{vmatrix} + \ldots,$$

the summations being for all integer values from $-\infty$ to $+\infty$ such that

$$i < j < k < \ldots.$$

111. It follows from the preceding expansion of a converging determinant D of normal form that, when a constituent $a_{i,k}$ enters into any term of the expanded form, no other constituent from the line i or from the column k enters into that term. Taking the aggregate of terms (each with its proper sign) into which $a_{i,k}$ enters, their sum may be denoted by $a_{i,k}\alpha_{i,k}$; and the determinant may be represented in the form

$$D = \sum_{k=-\infty}^{\infty} a_{i,k}\alpha_{i,k},$$

or in the form

$$D = \sum_{i=-\infty}^{\infty} a_{i,k}\,\alpha_{i,k}.$$

The quantity $\alpha_{i,k}$ is called the *minor* of $a_{i,k}$, and sometimes it is denoted by

$$\binom{i}{k}.$$

It can be derived from D by suppressing the line i and the column k, or, what is the equivalent in value, by replacing $a_{i,k}$ by 1, and every other constituent in the line i or in the column k or in both by 0, and then multiplying by $(-1)^{i-k}$. Manifestly, we have

$$\alpha_{i,k} = \binom{i}{k} = \frac{\partial D}{\partial a_{i,k}}.$$

It is an immediate corollary that

$$\left.\begin{aligned} 0 &= \sum_{k=-\infty}^{\infty} a_{j,k}\,\alpha_{i,k}, \\ & \qquad\qquad\qquad (i \neq j) \\ 0 &= \sum_{k=-\infty}^{\infty} a_{i,k}\,\alpha_{j,k}, \end{aligned}\right\}:$$

for the right-hand side in the first is equivalent to D with the line i replaced by the line j, so that the latter is duplicated; and in the second, the right-hand side is equivalent to D with the line j replaced by the line i, so that the latter is duplicated.

More generally, if, in the lines

$$\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_r,$$

and in the columns

$$\beta_1,\ \beta_2,\ \ldots,\ \beta_r,$$

we replace all the terms by 0, except $a_{\alpha_1,\beta_1},\ a_{\alpha_2,\beta_2},\ \ldots,\ a_{\alpha_r,\beta_r}$, each of which we replace by 1, and then multiply by

$$(-1)^{(\alpha_1-\beta_1)+(\alpha_2-\beta_2)+\ldots+(\alpha_r-\beta_r)},$$

the result is the coefficient of

$$\begin{vmatrix} a_{\alpha_1,\beta_1}, & \ldots, & a_{\alpha_1,\beta_r} \\ \ldots & \ldots & \ldots \\ a_{\alpha_r,\beta_1}, & \ldots, & a_{\alpha_r,\beta_r} \end{vmatrix}$$

in D. It manifestly is a minor of order r; and it is denoted by

$$\begin{pmatrix} \alpha_1, & \alpha_2, & \dots, & \alpha_r \\ \beta_1, & \beta_2, & \dots, & \beta_r \end{pmatrix}.$$

Clearly all the minors of any finite order are determinants of normal form, converging absolutely.

If D is not zero, some at least of the minors of constituents in any line must be different from zero, and some of the minors of constituents in any column also must be different from zero. Similar results, when D is not zero, hold for the minors of any order r of finite determinants, which are constructed out of r selected lines and any r columns, or out of r selected columns and any r lines.

Further, the minor

$$\begin{pmatrix} -r, & -r+1, & \dots, & 0, & 1, & \dots, & s \\ -r, & -r+1, & \dots, & 0, & 1, & \dots, & s \end{pmatrix}$$

tends to the value unity, as r and s increase. To prove this, let

$$Q_{s,r} = \Pi \,\{1 + \Sigma\, |A_{p,q}|\},$$

where the product is for all the values of p, and the summation is for all the values of q, that are excluded from the ranges

$$p = -r \text{ to } +s, \quad q = -r \text{ to } +s.$$

Expanding the minor, and changing every term so that its sign is positive and each factor in the term is replaced by its modulus, we have a new expression every term of which is contained in the expanded form of $Q_{s,r}$; and $Q_{s,r}$ contains other terms. Further, the expanded minor contains the term $+1$ as does $Q_{s,r}$, and all other terms involve the quantities A; hence

$$\left|\begin{pmatrix} -r, & -r+1, & \dots, & s \\ -r, & -r+1, & \dots, & s \end{pmatrix} - 1\right| < Q_{s,r} - 1.$$

But the product

$$\prod_{-\infty}^{\infty} \left\{1 + \sum_{-\infty}^{\infty} |A_{p,q}|\right\}$$

converges; and therefore, when any small positive quantity δ is assigned, integers $-r$ and s can be determined such that

$$Q_{s,r} - 1 < \delta.$$

Taking these as the integers defining the minor, we have

$$\left|\begin{pmatrix}-r, & -r+1, & \ldots, & s\\ -r, & -r+1, & \ldots, & s\end{pmatrix} - 1\right| < \delta,$$

so that

$$1-\delta < \left|\begin{pmatrix}-r, & -r+1, & \ldots, & s\\ -r, & -r+1, & \ldots, & s\end{pmatrix}\right| < 1+\delta.$$

Moreover, as integers s', r' are chosen, greater than s and r and gradually increasing, the quantity

$$Q_{s',r'} - 1$$

decreases; and thus the minor tends to the value unity as r and s increase.

One or two properties of minors may be noted. We have

$$\begin{pmatrix}i, & j\\ k, & l\end{pmatrix} = -\begin{pmatrix}j, & i\\ k, & l\end{pmatrix} = -\begin{pmatrix}i, & j\\ l, & k\end{pmatrix} = \begin{pmatrix}j, & i\\ l, & k\end{pmatrix};$$

for the changes from one of these expressions to another are equivalent to an interchange of two lines or an interchange of two columns, each of which changes the sign of the determinant. Similarly for minors of any order.

Again, expanding $\alpha_{i,k}$ by reference to constituents of a column, we have

$$\begin{pmatrix}i\\ k\end{pmatrix} = \alpha_{i,k} = \sum_j a_{j,l}\begin{pmatrix}i, & j\\ k, & l\end{pmatrix};$$

and expanding it by reference to constituents of a line, we have

$$\begin{pmatrix}i\\ k\end{pmatrix} = \alpha_{i,k} = \sum_l a_{j,l}\begin{pmatrix}i, & j\\ k, & l\end{pmatrix}.$$

Similarly,

$$\sum_l a_{j,l}\begin{pmatrix}j, & i\\ k, & l\end{pmatrix} = -\sum_l a_{j,l}\begin{pmatrix}i, & j\\ k, & l\end{pmatrix} = -\alpha_{i,k} = -\begin{pmatrix}i\\ k\end{pmatrix}.$$

Further

$$\sum_j a_{j,q}\begin{pmatrix}i, & j\\ k, & l\end{pmatrix} = 0,$$

when q is neither k nor l, because it is a minor of the first order with two columns the same; also

$$\sum_h a_{h,l}\begin{pmatrix}i, & j\\ k, & l\end{pmatrix} = 0,$$

when h is neither i nor j, because it is a minor of the first order with two lines the same; and

$$\sum_{h \text{ or } q} a_{h,q} \begin{pmatrix} i, & j \\ k, & l \end{pmatrix} = 0,$$

where h is neither i nor j, and q is neither k nor l, because it is a minor of the first order with two columns the same and two lines the same. Similarly for minors of higher order.

The similarity in properties between finite determinants and converging infinite determinants of normal form is not exhausted by the preceding set: in particular, infinite determinants can be multiplied, and determinants framed from minors of an infinite determinant are connected with their complementary in the original, exactly as for finite determinants. The simpler of these properties are contained in the following examples.

Ex. 1. If

$$A = [a_{i,k}], \quad B = [b_{i,k}], \qquad \left.\begin{matrix} i \\ k \end{matrix}\right\} \left.\begin{matrix} \infty \\ -\infty \end{matrix}\right\},$$

are converging determinants of normal type, and if

$$c_{i,k} = \sum_{-\infty}^{j=\infty} a_{i,j} b_{k,j},$$

for all values of i and k, then

$$C = [c_{i,k}] \qquad \left.\begin{matrix} i \\ k \end{matrix}\right\} \left.\begin{matrix} \infty \\ -\infty \end{matrix}\right\}$$

is a converging determinant of normal type, and

$$AB = C.$$

Ex. 2. If $\alpha_{i,k}$ denote the minor of $a_{i,k}$ in the determinant

$$A = [a_{i,k}], \qquad \left.\begin{matrix} i \\ k \end{matrix}\right\} \left.\begin{matrix} \infty \\ -\infty \end{matrix}\right\},$$

then

$$\begin{vmatrix} \alpha_{i_1 k_1}, & \ldots, & \alpha_{i_1 k_r} \\ \alpha_{i_2 k_1}, & \ldots, & \alpha_{i_2 k_r} \\ \cdots & \cdots & \cdots \\ \alpha_{i_r k_1}, & \ldots, & \alpha_{i_r k_r} \end{vmatrix} = \begin{pmatrix} i_1, & i_2, & \ldots, & i_r \\ k_1, & k_2, & \ldots, & k_r \end{pmatrix} A^{r-1},$$

with the preceding notation for minors of order r.

Ex. 3. In connection with the determinant

$$A = [a_{i,k}], \qquad \left.\begin{matrix} i \\ k \end{matrix}\right\} \left.\begin{matrix} \infty \\ -\infty \end{matrix}\right\},$$

prove that

$$\binom{i}{k}\binom{i_1,\ i_2}{k_1,\ k_2}+\binom{i}{k_1}\binom{i_1,\ i_2}{k_2,\ k}+\binom{i}{k_2}\binom{i_1,\ i_2}{k,\ k_1}=\binom{i,\ i_1,\ i_2}{k,\ k_1,\ k_2}A\ ;$$

$$\binom{i}{k}\binom{i_1,\ i_2}{k_1,\ k_2}+\binom{i_1}{k}\binom{i_2,\ i}{k_1,\ k_2}+\binom{i_2}{k}\binom{i\ ,\ i_1}{k_1,\ k_2}=\binom{i,\ i_1,\ i_2}{k,\ k_1,\ k_2}A\ ;$$

and, more generally, that

$$\sum_{n=0}^{r}\binom{i_0}{k_n}\binom{i_1\quad,\ i_2\quad,\ \ldots,\ i_r}{k_{n+1},\ k_{n+2},\ \ldots,\ k_{n-1}}=\binom{i_0,\ i_1,\ i_2,\ \ldots,\ i_r}{k_0,\ k_1,\ k_2,\ \ldots,\ k_r}A,$$

where, in the typical term, $k_n, k_{n+1}, k_{n+2}, \ldots, k_{n-1}$ preserve the same cyclical order as $k_0, k_1, k_2, \ldots, k_r$.

In the first of these, the right-hand side vanishes if k is equal to k_1 or k_2; in the second, it vanishes if i is equal to i_1 or i_2; in the third, it vanishes if k_0 is equal to any one of the quantities $k_1, k_2, \ldots, k_r$; and so in other cases.

112. The infinite determinants which arise in the discussion of linear differential equations have, as their constituents, functions of a parameter ρ. The preceding results are still valid, if the condition that

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}A_{i,j}(\rho)$$

is an absolutely converging series is satisfied; in particular, the determinant converges absolutely, and its value may be denoted by $D(\rho)$. The parameter may be made to vary; and then it is important that the convergence of $D(\rho)$ should be not merely absolute, but also uniform, in order that it may be differentiated. Suppose that, in any region in the ρ-plane, all the functions $A_{i,j}(\rho)$ are regular functions of ρ, such that the series

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}A_{i,j}(\rho)$$

converges uniformly and absolutely. For all values of ρ within that region, any small quantity δ can be assigned, and then integers M and N exist, such that for all integers $m \geqslant M$, and integers $-n \leqslant -N$,

$$\left|\sum_{m}^{\infty}\sum_{-\infty}^{-n}A_{i,j}(\rho)\right|<\delta.$$

By analysis that follows the earlier analysis practically step by step, we then infer that, for all integers $m \geqslant M$, $n \geqslant N$, and for all positive integers p and q, and for all values of ρ within the region indicated, we have

$$|D_{m+p,n+q}(\rho)-D_{m,n}(\rho)|<2\delta\ ;$$

so that $D(\rho)$ converges uniformly. Hence, within the domain considered, $D(\rho)$ is a regular analytic function of ρ.

The expansions of $D(\rho)$ in terms of its constituents have been proved to converge absolutely, by comparison with the expansions of $\bar{P}_{m,n}$, where

$$\bar{P}_{m,n} = \prod_{-n}^{m} \left\{1 + \sum_{-n}^{m} |A_{i,k}|\right\}.$$

As the series

$$\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} A_{i,k}(\rho)$$

converges uniformly and absolutely, $\bar{P}_{m,n}$ is a product that converges uniformly with indefinite increase of m and n. The corresponding modifications in the investigation lead to the conclusion, that the expanded form of $D(\rho)$ converges uniformly as well as absolutely.

Moreover*, this expanded form can be differentiated, and its derivatives are the derivatives of $D(\rho)$. In particular, we have

$$\begin{aligned}\frac{\partial D}{\partial \rho} &= \Sigma\Sigma \frac{\partial D}{\partial a_{i,k}} \frac{\partial a_{i,k}}{\partial \rho} \\ &= \Sigma\Sigma \alpha_{i,k} \frac{\partial a_{i,k}}{\partial \rho}.\end{aligned}$$

Thus if D vanish for a value ρ' of ρ, and if all the first minors of D vanish for that value, we have

$$\alpha_{i,k} = 0,$$

while $\dfrac{\partial a_{i,k}}{\partial \rho}$ is not infinite; the first derivative of the uniform function D vanishes, and therefore ρ' is at least a double zero of D. In that case, we have

$$\begin{aligned}\frac{\partial^2 D}{\partial \rho^2} &= \Sigma\Sigma \alpha_{i,k} \frac{\partial^2 a_{i,k}}{\partial \rho^2} + \Sigma\Sigma\Sigma\Sigma \frac{\partial \alpha_{i,k}}{\partial a_{j,l}} \frac{\partial a_{i,k}}{\partial \rho} \frac{\partial a_{j,l}}{\partial \rho} \\ &= \Sigma\Sigma\Sigma\Sigma \begin{pmatrix} i, & j \\ k, & l \end{pmatrix} \frac{\partial a_{i,k}}{\partial \rho} \frac{\partial a_{j,l}}{\partial \rho}.\end{aligned}$$

Hence, if all the second minors of D vanish for that value of ρ, we have

$$\frac{\partial^2 D}{\partial \rho^2} = 0;$$

* The proof is similar to those given for preceding propositions; see von Koch's memoir, *Acta Math.*, t. XVI, p. 243.

and so ρ' is at least a triple zero of D. And generally, if all minors of all orders up to $r-1$ inclusive vanish, but not all minors of order r, when $\rho=\rho'$, then ρ' is a root of D in multiplicity r; and D is then said to be of *characteristic* r. The quantity r cannot increase indefinitely, for we have seen that minors of sufficiently high order tend to the value unity, so that the general vanishing of all minors of the same order is possible only for finite orders.

But it need hardly be pointed out that the converses of these results are not necessarily true: thus $\rho=\rho'$ might be a double root of D, while not all the first minors of D would vanish.

113. The purpose, for which infinite determinants are to be used in this place, is in connection with the solution of an unlimited number of equations, linear in an unlimited number of constants. Let

$$u_i = \sum_{-\infty}^{k=\infty} a_{i,k} x_k,$$

and suppose that the infinite determinant D, where

$$D = [a_{i,k}], \qquad \left.\begin{matrix} i \\ k \end{matrix}\right\} \left.\begin{matrix} \infty \\ -\infty \end{matrix}\right\},$$

converges uniformly; it is required to find the ratios of the quantities x to one another which satisfy the equations

$$u_i = 0, \qquad (i=-\infty \text{ to } +\infty),$$

the quantities x being themselves finite, so that we have

$$|x_k| \leqslant X,$$

where X is finite.

We know that

$$\sum_j \binom{i}{k} a_{i,j}$$

converges absolutely; its value is D when $j=k$, and is 0 when j is different from k. Moreover, the series $\sum\limits_k a_{i,k}$ is an absolutely converging series, and hence for values of x considered, we have

$$\left|\sum_k a_{i,k} x_k\right| \leqslant X \Sigma |a_{i,k}| \leqslant U,$$

where U is finite. Hence, by one of the propositions already established, the quantity S, defined by the equation

$$S = \sum_i \sum_j \binom{i}{k} a_{i,j} x_j,$$

also converges absolutely, so that

$$S = \sum_i \binom{i}{k} u_i = \sum_j x_j \sum_i \binom{i}{k} a_{i,j},$$
$$= x_k D,$$

for all the other terms give a zero coefficient for x. Hence, if $u_i = 0$ for all values of i, and if we are to have values of x_k different from zero, then

$$D = 0,$$

which is a necessary condition. We shall assume this condition to be satisfied.

If some at least of the first minors are different from zero, then the equation

$$\sum_i \binom{i}{k} u_i = 0$$

shews that any one of the quantities u, which it contains, is then linearly expressible in terms of the others, and so the corresponding equation $u = 0$ is not an independent equation. Let u_0 then be omitted on this ground; we have

$$\sum_i \begin{pmatrix} i, & 0 \\ k, & l \end{pmatrix} u_i = \sum_i \sum_q \begin{pmatrix} i, & 0 \\ k, & l \end{pmatrix} a_{i,q} x_q,$$

where on each side the summation is for all values of i except $i = 0$. The coefficient of x_q on the right-hand side is

$$\sum_i \begin{pmatrix} i, & 0 \\ k, & l \end{pmatrix} a_{i,q}.$$

This is zero, if q is different from both k and l; it is

$$= -\sum_i \begin{pmatrix} 0, & i \\ k, & l \end{pmatrix} a_{i,l} = -a_{0,k},$$

if $q = l$; and it is

$$= \sum_i \begin{pmatrix} i, & 0 \\ k, & l \end{pmatrix} a_{i,k}$$

$$= \sum_i \begin{pmatrix} 0, & i \\ l, & k \end{pmatrix} a_{i,k} = a_{0,l},$$

if $q=k$. Thus

$$\sum_i \begin{pmatrix} i, & 0 \\ k, & l \end{pmatrix} u_i = -\alpha_{0,k}\, x_l + \alpha_{0,l}\, x_k.$$

But all the quantities u_i vanish; hence

$$-\alpha_{0,k}\, x_l + \alpha_{0,l}\, x_k = 0.$$

We thus have

$$x_l = \xi \alpha_{0,l},$$

for all values of l; and ξ is any finite quantity, for only the ratios of the quantities x are determinate.

Similarly, if D be of characteristic r, so that the minors of lowest order which do not all vanish are of order r, let

$$\begin{pmatrix} \alpha_1, & \alpha_2, & \ldots, & \alpha_r \\ \beta_1, & \beta_2, & \ldots, & \beta_r \end{pmatrix}$$

be such a minor different from zero. We then have

$$S = \sum_m \begin{pmatrix} \alpha_1, & \ldots, & \alpha_{n-1}, & m, & \alpha_{n+1}, & \ldots, & \alpha_r \\ \beta_1, & \ldots, & \beta_{n-1}, & \beta_n, & \beta_{n+1}, & \ldots, & \beta_r \end{pmatrix} u_m$$

$$= \sum_m \sum_q \begin{pmatrix} \alpha_1, & \ldots, & \alpha_{n-1}, & m, & \alpha_{n+1}, & \ldots, & \alpha_r \\ \beta_1, & \ldots, & \beta_{n-1}, & \beta_n, & \beta_{n+1}, & \ldots, & \beta_r \end{pmatrix} a_{m,q} x_q.$$

Thus the coefficient of x_q is

$$\sum_m \begin{pmatrix} \alpha_1, & \ldots, & \alpha_{n-1}, & m, & \alpha_{n+1}, & \ldots, & \alpha_r \\ \beta_1, & \ldots, & \beta_{n-1}, & \beta_n, & \beta_{n+1}, & \ldots, & \beta_r \end{pmatrix} a_{m,q}.$$

When q is equal to any one of the integers $\beta_1, \beta_2, \ldots, \beta_r$, this coefficient is equal to a minor of order $r-1$ and so vanishes. When q is not equal to any one of those integers, the coefficient is equal to a determinant with two columns the same, and it is therefore evanescent. Hence

$$S = 0,$$

and therefore

$$\begin{pmatrix} \alpha_1, & \ldots, & \alpha_r \\ \beta_1, & \ldots, & \beta_r \end{pmatrix} u_{\alpha_n} = -\sum{}' \begin{pmatrix} \alpha_1, & \ldots, & \alpha_{n-1}, & m, & \alpha_{n+1}, & \ldots, & \alpha_r \\ \beta_1, & \ldots, & \beta_{n-1}, & \beta_n, & \beta_{n+1}, & \ldots, & \beta_r \end{pmatrix} u_m,$$

where, on the right-hand side, m must not be equal to any one of the integers $\alpha_1, \ldots, \alpha_r$. It thus appears that there are r relations among the quantities u; and that, in particular, each of the quantities $u_{\alpha_1}, u_{\alpha_2}, \ldots, u_{\alpha_r}$ is linearly expressible in terms of the

remaining quantities u. Accordingly, we assume these r quantities u omitted from consideration.

Denoting by α any integer other than $\alpha_1, \ldots, \alpha_r$, and by β any integer other than $\beta_1, \ldots, \beta_r$, we have

$$\sum_{\alpha} \begin{pmatrix} \alpha, & \alpha_1, \ldots, & \alpha_r \\ \beta, & \beta_1, \ldots, & \beta_r \end{pmatrix} u_\alpha = \sum_{\alpha}\sum_{q} \begin{pmatrix} \alpha, & \alpha_1, \ldots, & \alpha_r \\ \beta, & \beta_1, \ldots, & \beta_r \end{pmatrix} a_{\alpha, q} x_q$$

$$= \begin{pmatrix} \alpha_1, \alpha_2, \ldots, \alpha_r \\ \beta_1, \beta_2, \ldots, \beta_r \end{pmatrix} x_\beta - \begin{pmatrix} \alpha_1, \alpha_2, \ldots, \alpha_r \\ \beta\ , \beta_2, \ldots, \beta_r \end{pmatrix} x_{\beta_1} - \begin{pmatrix} \alpha_1, \alpha_2, \ldots, \alpha_r \\ \beta_1, \beta\ , \ldots, \beta_r \end{pmatrix} x_{\beta_2} - \ldots$$

$$\ldots - \begin{pmatrix} \alpha_1, \alpha_2, \ldots, \alpha_r \\ \beta_1, \beta_2, \ldots, \beta \end{pmatrix} x_{\beta_r},$$

in the same way as for the simpler case; hence, as all the quantities u_α vanish, we have

$$\begin{pmatrix} \alpha_1, \alpha_2, \ldots, \alpha_r \\ \beta_1, \beta_2, \ldots, \beta_r \end{pmatrix} x_\beta = \sum_{n=1}^{r} \begin{pmatrix} \alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n, \alpha_{n+1}, \ldots, \alpha_r \\ \beta_1, \beta_2, \ldots, \beta_{n-1}, \beta\ , \beta_{n+1}, \ldots, \beta_r \end{pmatrix} x_{\beta_n},$$

so that all the quantities x_β are linearly expressible in terms of r such quantities.

For further properties of infinite determinants, reference may be made to the memoirs quoted at the beginning of § 109.

Application to Differential Equations.

114. When the differential equation is given in the form

$$\frac{d^n W}{dz^n} + Q_1 \frac{d^{n-1} W}{dz^{n-1}} + \ldots + Q_{n-1} \frac{dW}{dz} + Q_n W = 0,$$

the substitution

$$W = we^{-\frac{1}{n}\int Q_1 dz}$$

leads to an equation of order n in w, which is devoid of the term involving $\dfrac{d^{n-1}w}{dz^{n-1}}$. The coefficients of the new equation are linearly expressible in terms of $Q_2, Q_3, \ldots, Q_{n-1}, Q_n$, and the expressions involve derivatives of Q_1 up to order $n-1$ inclusive and integral powers of Q_1. We may therefore take the differential equation in the form

$$P(w) = \frac{d^n w}{dz^n} + P_2 \frac{d^{n-2} w}{dz^{n-2}} + \ldots + P_{n-1} \frac{dw}{dz} + P_n w = 0.$$

We assume that, in the vicinity of $z=0$, it possesses no synectic integral, no regular integral, no normal integral, and no subnormal integral. The point $z=0$ is then a singularity of the coefficients; and, if it be only an accidental singularity (of order higher than s for P_s, in the case of some value or values of s), the conditions for the existence of a normal integral or a subnormal integral are not satisfied. We assume the coefficients P still to be uniform functions of z, and we shall suppose that their singularities are isolated points. Let an annulus, given by

$$R < |z| < R',$$

be such that its area is free from singularities, no assumption being made as to the behaviour of the coefficients P within the circle of radius R; then it is known* that each such coefficient can be expanded in a Laurent series

$$P_r = \sum_{-\infty}^{\infty} c_{r,\mu} z^{\mu}, \qquad (r=2, 3, \ldots, n),$$

which converges uniformly and unconditionally within the annulus. Without loss of generality, it may be assumed that

$$R < 1 < R':$$

for, otherwise, we should take a new variable $Z = z(RR')^{\frac{1}{2}}$, and the limiting radii $\overline{R}$ and $\overline{R}'$ of the annulus for Z then satisfy the conditions

$$\overline{R} < 1 < \overline{R}'.$$

Further, owing to the character of the convergence of P_r, we have

$$z\frac{dP_r}{dz} = \sum_{-\infty}^{\infty} c_{r,\mu}\mu z^{\mu},$$

$$z\frac{d}{dz}\left(z\frac{dP_r}{dz}\right) = \sum_{-\infty}^{\infty} c_{r,\mu}\mu^2 z^{\mu},$$

and so on; all these series converge uniformly and unconditionally within the annulus. Hence also

$$\sum_{-\infty}^{\infty} R(\mu)\, c_{r,\mu} z^{\mu}$$

* *T. F.*, § 28.

similarly converges within the annulus, where $R(\mu)$ is any polynomial in μ; and therefore, taking the circle $|z| = 1$, every point of which lies within the annulus, the series

$$\sum_{-\infty}^{\infty} |R(\mu)\, c_{r,\mu}|$$

converges.

115. From the general investigations in Chapter II, it follows that the equation certainly possesses an integral of the form

$$y = z^{\rho} \phi(z),$$

where ρ is any one of the values of $\frac{1}{2\pi i} \log \omega$, the quantity ω being a root of the fundamental equation associated with an irreducible (but otherwise simple) closed circuit in the annulus; and the quantity ϕ is a uniform function of z. As the integral is not regular, the number of negative powers of z in ϕ is unlimited; and so we may write

$$y = \sum_{-\infty}^{\infty} a_m z^{\rho+m}.$$

In order to have an adequate expression of the integral, the quantity ρ must be obtained; the value of $a_m \div a_0$, for $m = \pm 1$, $\pm 2, \ldots, \pm \infty$, must be constructed; and the resulting series must converge for values of z within the annulus.

We first consider the formal construction of the expression for the integral. Let

$$\begin{aligned}\phi(\rho) = \rho(\rho-1)\ldots(\rho-n+1) + c_{2,-2}\,\rho(\rho-1)\ldots(\rho-n+3) \\ + c_{3,-3}\,\rho(\rho-1)\ldots(\rho-n+4) + \ldots + c_{n-1,-n+1}\,\rho + c_{n,-n};\end{aligned}$$

$$\begin{aligned}C_{r,\mu} = (\rho+\mu)\ldots(\rho+\mu-n+3)\, c_{2,r-\mu-2} \\ +(\rho+\mu)\ldots(\rho+\mu-n+4)\, c_{3,r-\mu-3} + \ldots \\ \ldots + (\rho+\mu)\, c_{n-1,r-\mu-n+1} + c_{n,r-\mu-n};\end{aligned}$$

and write

$$G_m(\rho) = \phi(\rho+m)\, a_m + \sum_{-\infty}^{\infty}{}' C_{m,\mu}\, a_\mu,$$

where, in the last summation, the values of μ are from $-\infty$ to $+\infty$, with $\mu = m$ excepted. Then we have

$$P(y) = \sum_{-\infty}^{\infty} G_m(\rho)\, z^{\rho+m-n},$$

so that y is an integral of the differential equation if

$$G_m(\rho) = 0,$$

for all values of m from $-\infty$ to $+\infty$, there being no assumption that the negative infinity is the same numerically as the positive infinity. Let

$$\frac{C_{m,\mu}}{\phi(\rho + m)} = \psi_{m,\mu},$$

for all values of μ other than $\mu = m$; and introduce a quantity $\psi_{m,m}$ with the convention

$$\psi_{m,m} = 1;$$

then

$$G_m(\rho) = \phi(m + \rho) \sum_{-\infty}^{\infty} \psi_{m,\mu} a_\mu,$$

where the summation now is for all values of μ. We then require the infinite determinant

$$\Omega(\rho) = [\psi_{m,\mu}] \qquad \begin{matrix} m \\ \mu \end{matrix}\Big\} \begin{matrix} \infty \\ -\infty \end{matrix}\Big\}$$

$$= \begin{vmatrix} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \ldots, & 1, & \psi_{-2,-1}, & \psi_{-2,0}, & \psi_{-2,1}, & \psi_{-2,2}, & \ldots \\ \ldots, & \psi_{-1,-2}, & 1, & \psi_{-1,0}, & \psi_{-1,1}, & \psi_{-1,2}, & \ldots \\ \ldots, & \psi_{0,-2}, & \psi_{0,-1}, & 1, & \psi_{0,1}, & \psi_{0,2}, & \ldots \\ \ldots, & \psi_{1,-2}, & \psi_{1,-1}, & \psi_{1,0}, & 1, & \psi_{1,2}, & \ldots \\ \ldots, & \psi_{2,-2}, & \psi_{2,-1}, & \psi_{2,0}, & \psi_{2,1}, & 1, & \ldots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{vmatrix},$$

the necessary and sufficient condition of the convergence of which is the convergence of the double series

$$\Sigma\Sigma\,\psi_{m,\mu}$$

for all values of m and μ between $-\infty$ and $+\infty$ except $m = \mu$.

116. In order to establish the convergence, we first transform the expression of $C_{m,\mu}$. Let

$$\mu = m - \lambda;$$

then we may take

$$\begin{aligned} &(\rho + \mu)(\rho + \mu - 1)\ldots(\rho + \mu - p + 1) \\ &\qquad = (\rho + m - \lambda)(\rho + m - \lambda - 1)\ldots(\rho + m - \lambda - p + 1) \\ &\qquad = (\rho + m)^p + \alpha_{p,1}(\rho + m)^{p-1} + \alpha_{p,2}(\rho + m)^{p-2} + \ldots + \alpha_{p,p}, \end{aligned}$$

where $\alpha_{p,r}$ is a polynomial in λ of degree r. Using this for all the terms in $C_{m,\mu}$, we have

$$C_{m,\mu} = A_2(\lambda)(\rho+m)^{n-2} + A_3(\lambda)(\rho+m)^{n-3} + \ldots + A_n(\lambda),$$

where

$$A_r(\lambda) = \alpha_{n-2,r}\, c_{2,\lambda-2} + \alpha_{n-3,r-1}\, c_{3,\lambda-3} + \ldots + c_{r,\lambda-r}.$$

Accordingly, we have

$$\left|\sum_m \sum_\mu \psi_{m,\mu}\right| = \left|\sum_m \sum_\mu \frac{C_{m,\mu}}{\phi(\rho+m)}\right|$$

$$\leqslant \sum_m \sum_\lambda \left|\frac{(\rho+m)^{n-2}}{\phi(\rho+m)}\right| |A_2(\lambda)| + \sum_m \sum_\lambda \left|\frac{(\rho+m)^{n-3}}{\phi(\rho+m)}\right| |A_3(\lambda)| + \ldots$$

$$\ldots + \sum_m \sum_\lambda \left|\frac{1}{\phi(\rho+m)}\right| |A_n(\lambda)|.$$

Now the series

$$\sum_{\lambda=-\infty}^{\lambda=\infty} |R(\lambda)\, c_{r,\lambda}|$$

converges for every value 2, 3, ..., n of r, where $R(\lambda)$ is any polynomial in λ. Hence

$$\sum_{\lambda=-\infty}^{\lambda=\infty} |A_s(\lambda)| \leqslant \sum_{p=2}^{n-r} \sum_{\lambda=-\infty}^{\lambda=\infty} |\alpha_{n-p,s-p+2}\, c_{p,\lambda-p}|,$$

every term of which (for the various values of p) converges, because $\alpha_{n-p,s-p+2}$ is a polynomial in λ of degree $s-p+2$, and therefore the whole of the right-hand side is a converging series. Accordingly, we may write

$$\sum_{\lambda=-\infty}^{\lambda=\infty} A_s(\lambda) = H_s, \qquad (s = 2, \ldots, n),$$

and then each of the quantities $|H_s|$ is finite.

We thus have

$$\left|\sum_m \sum_\mu \psi_{m,\mu}\right| \leqslant |H_2| \sum_m \left|\frac{(\rho+m)^{n-2}}{\phi(\rho+m)}\right| + |H_3| \sum_m \left|\frac{(\rho+m)^{n-3}}{\phi(\rho+m)}\right| + \ldots$$

$$\ldots + |H_n| \sum_m \left|\frac{1}{\phi(\rho+m)}\right|.$$

Assuming ρ to be any quantity, different from any of the roots of any of the equations

$$\phi(\rho+m) = 0,$$

each of which is of degree n, we know that all the series

$$\sum_m \frac{(\rho+m)^{n-\kappa}}{\phi(\rho+m)}$$

converge absolutely, for the values $\kappa = 2, 3, \ldots, n$. Moreover, the sum of each such series is a function of ρ: and then, if ρ varies in a region no point of which is at an infinitesimal distance from any of the roots of $\phi(\rho+m)$, the convergence of the series is uniform.

Accordingly, the double series

$$\sum_m \sum_\mu \psi_{m,\mu}$$

converges uniformly and unconditionally; and therefore the infinite determinant $\Omega(\rho)$ converges uniformly and unconditionally, provided ρ does not approach infinitesimally near any root of any of the equations $\phi(\rho+m)=0$. Clearly, $\Omega(\rho)$ is a uniform function of ρ, for such values of ρ.

Further, we have

$$\psi_{m,\mu}(\rho)\,\phi(\rho+m) = C_{m,\mu}(\rho),$$

and therefore

$$\begin{aligned}\psi_{m+1,\mu+1}(\rho)\,\phi(\rho+m+1) &= C_{m+1,\mu+1}(\rho)\\ &= C_{m,\mu}(\rho+1)\\ &= \psi_{m,\mu}(\rho+1)\,\phi(\rho+1+m),\end{aligned}$$

so that

$$\psi_{m+1,\mu+1}(\rho) = \psi_{m,\mu}(\rho+1).$$

Construct the infinite determinant $\Omega(\rho+1)$, and then replace each constituent $\psi_{m,\mu}(\rho+1)$ by $\psi_{m+1,\mu+1}(\rho)$; the result is to give the modification of $\Omega(\rho)$, which arises by moving each column one place to the right and by depressing each row one place, in other words, by taking $\psi_{1,1}(\rho)$ in the diagonal as the origin instead of $\psi_{0,0}(\rho)$. But such a change makes no difference in a determinant which converges absolutely; we therefore have

$$\Omega(\rho+1) = \Omega(\rho),$$

or the infinite determinant Ω is a periodic function of ρ.

Lastly, by making ρ infinitely large in such a manner, that it does not approach infinitesimally near any of the roots of any of the equations

$$\phi(\rho+m)=0,$$

(which roots for different values of m differ only by real integers, so that if we take $\rho = u + iv$, where u and v are real, it will be sufficient to take v large), we reduce to zero every constituent that lies off the diagonal of $\Omega(\rho)$. As every constituent in the diagonal is unity, and every constituent off the diagonal is zero, it follows (from the law of expansion of an absolutely converging determinant) that

$$\operatorname*{Lim}_{\rho=\infty} \Omega(\rho) = 1,$$

provided ρ tends to its infinite value in the manner indicated.

Modification of the Infinite Determinant $\Omega(\rho)$.

117. It is convenient also to consider another infinite determinant associated with $\Omega(\rho)$. The equation $G_m(\rho) = 0$ was taken in the form

$$\phi(m+\rho) \sum_{-\infty}^{\infty} \psi_{m,\mu} a_\mu = 0:$$

and the infinite determinant $\Omega(\rho)$ was composed of the constituents $\psi_{m,\mu}$. If an infinite determinant were composed of constituents $\phi(m+\rho)\psi_{m,\mu}$, then the row determined by the integer would have a common factor $\phi(m+\rho)$; and thus there would be an infinitude of factors, the product of which either should converge or should be made to converge. Let $\rho_1, \rho_2, \ldots, \rho_n$ be the roots of $\phi(\rho) = 0$, so that

$$\phi(\rho) = (\rho - \rho_1)(\rho - \rho_2) \ldots (\rho - \rho_n),$$

and therefore

$$\phi(\rho + m) = m^n \left(1 + \frac{\rho - \rho_1}{m}\right)\left(1 + \frac{\rho - \rho_2}{m}\right) \ldots \left(1 + \frac{\rho - \rho_n}{m}\right).$$

To change this into a form suitable for an infinite converging product, we multiply by

$$h_m(\rho) = \frac{1}{m^n} e^{-\frac{\rho - \rho_1}{m}} \cdot e^{-\frac{\rho - \rho_2}{m}} \cdot \ldots \cdot e^{-\frac{\rho - \rho_n}{m}},$$

with the convention

$$h_0(\rho) = 1.$$

As $h_m(\rho)$ remains finite and is not zero for finite values of ρ, we may replace the equation $G_m(\rho) = 0$ by

$$h_m(\rho) G_m(\rho) = 0.$$

Now let

$$\chi_{m,m}(\rho) = h_m(\rho)\,\phi(\rho+m)$$
$$= \prod_{\sigma=1}^{n}\left\{\left(1+\frac{\rho-\rho_\sigma}{m}\right)e^{-\frac{\rho-\rho_\sigma}{m}}\right\},$$

for all values of m except $m=0$, and

$$\chi_{0,0} = \prod_{\sigma=1}^{n}(\rho-\rho_\sigma);$$

also let

$$\chi_{m,\mu}(\rho) = h_m(\rho)\,\phi(m+\rho)\,\psi_{m,\mu} = h_m(\rho)\,C_{m,\mu}.$$

Then the equations between the constants a have the form

$$\sum_{-\infty}^{\infty}\chi_{m,\mu}a_\mu = 0.$$

In association with these equations, we consider the infinite determinant

$$D(\rho) = [\chi_{m,\mu}] \qquad \left.\begin{matrix} m \\ \mu \end{matrix}\right\} \begin{matrix} \infty \\ -\infty \end{matrix}\Bigg\}$$

$$= \begin{vmatrix} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots, & \chi_{-2,-2}, & \chi_{-2,-1}, & \chi_{-2,0}, & \chi_{-2,1}, & \chi_{-2,2}, & \cdots \\ \cdots, & \chi_{-1,-2}, & \chi_{-1,-1}, & \chi_{-1,0}, & \chi_{-1,1}, & \chi_{-1,2}, & \cdots \\ \cdots, & \chi_{0,-2}, & \chi_{0,-1}, & \chi_{0,0}, & \chi_{0,1}, & \chi_{0,2}, & \cdots \\ \cdots, & \chi_{1,-2}, & \chi_{1,-1}, & \chi_{1,0}, & \chi_{1,1}, & \chi_{1,2}, & \cdots \\ \cdots, & \chi_{2,-2}, & \chi_{2,-1}, & \chi_{2,0}, & \chi_{2,1}, & \chi_{2,2}, & \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{vmatrix}.$$

Taking the diagonal to be $\ldots, \chi_{-2,-2}, \chi_{-1,-1}, \chi_{0,0}, \chi_{1,1}, \chi_{2,2}, \ldots,$ we require to establish, (i), the convergence of the series

$$\Sigma\Sigma\,\chi_{m,\mu},$$

summed for all values of m and μ, except $m=\mu$, from $-\infty$ to $+\infty$, and (ii), the convergence of the series

$$\sum_{-\infty}^{\infty}(\chi_{m,m}-1),$$

in order to know that the infinite determinant $D(\rho)$ converges.

We consider first the double series $\Sigma\Sigma\,\chi_{m,\mu}$. Let

$$k_m(\rho) = m^n h_m(\rho) = \prod_{\sigma=1}^{n} e^{-\frac{\rho-\rho_\sigma}{m}}, \qquad (m \neq 0).$$

The quantities $\rho_1, \rho_2, \ldots, \rho_n$ are finite; hence, so long as ρ remains within a finite region that does not lie at infinity, there is a finite quantity K which is larger than any value of $|k_m(\rho)|$ for values of ρ within that region. Hence, as

$$\chi_{m,\mu} = h_m(\rho)\, C_{m,\mu}$$
$$= k_m(\rho)\, \frac{C_{m,\mu}}{m^n},$$

we have

$$|\chi_{m,\mu}| < K \frac{|C_{m,\mu}|}{m^n},$$

when m is not zero. When m is zero, we have

$$\chi_{0,\mu} = h_0(\rho)\, C_{0,\mu} = C_{0,\mu}.$$

Proceeding exactly as with the series $\Sigma\Sigma\psi_{m,\mu}$ in § 116, summing for all values of m other than zero, and for all values of μ other than $m = \mu$, we have

$$\Sigma\Sigma \frac{|C_{m,\mu}|}{m^n} \leqslant |H_2| \sum_m \frac{|\rho + m|^{n-2}}{m^n} + |H_3| \sum_m \frac{|\rho + m|^{n-3}}{m^n} + \ldots + |H_n| \sum_m \frac{1}{m^n},$$

every term of which is finite, and therefore

$$K\Sigma\Sigma \frac{|C_{m,\mu}|}{m^n}$$

is finite. Also

$$\sum_\mu |C_{0,\mu}| \leqslant |H_2|\, |\rho^{n-2}| + |H_3|\, |\rho^{n-3}| + \ldots + |H_n|,$$

which is finite, so that

$$\Sigma\, |C_{0,\mu}|$$

converges. Hence the double series

$$\Sigma\Sigma\chi_{m,\mu},$$

summed for all values of m and μ between $-\infty$ and $+\infty$ except $m = \mu$, converges. Moreover, all the series, which occur in the superior limits in the inequalities, converge uniformly within the region of ρ considered; hence the double series converges uniformly.

The establishment of the convergence of the series

$$\sum_{-\infty}^{\infty} (\chi_{m,m} - 1)$$

is simple. We know, by Weierstrass's theorem*, that the series

$$\prod_{-\infty}^{\infty} \chi_{m,m}$$

converges uniformly and unconditionally; so that, if

$$\theta_{m,m} = \chi_{m,m} - 1,$$

the infinite product

$$\prod_{-\infty}^{\infty} (1 + \theta_{m,m})$$

converges uniformly and unconditionally; and therefore†

$$\prod_{-\infty}^{\infty} (1 + |\theta_{m,m}|)$$

converges. But

$$\Pi (1 + |\theta_{m,m}|) < 1 + \Sigma |\theta_{m,m}|;$$

hence $\Sigma |\theta_{m,m}|$ converges uniformly, that is, the series

$$\sum_{-\infty}^{\infty} (\chi_{m,m} - 1)$$

converges uniformly and unconditionally.

The convergence can also be established as follows. Let

$$u_\sigma = \left(1 + \frac{\rho - \rho_\sigma}{m}\right) e^{-\frac{\rho - \rho_\sigma}{m}},$$

and choose a finite positive integer p, such that, for values of ρ under consideration, we have

$$|\rho - \rho'| < p,$$

where ρ' is any one of quantities $\rho_1, \rho_2, \ldots, \rho_n$. The sum of the terms

$$\sum_{-p}^{p} (\chi_{m,m} - 1)$$

is finite, and may be omitted without affecting the convergence: and we consider the sum of the remaining terms, for which we have

$$|m| > p.$$

We have

$$u_\sigma = e^{T_\sigma},$$

where

$$T_\sigma = -\tfrac{1}{2} \frac{(\rho - \rho_\sigma)^2}{m^2} + \tfrac{1}{3} \frac{(\rho - \rho_\sigma)^3}{m^3} - \ldots,$$

* *T. F.*, § 50.
† *T. F.*, § 49.

and therefore

$$2m^2 \,|\, T_\sigma \,| < |\, \rho - \rho_\sigma \,|^2 \left\{1 + \frac{|\, \rho - \rho_\sigma \,|}{m} + \frac{|\, \rho - \rho_\sigma \,|^2}{m^2} + \dots\right\}$$

$$< \frac{|\, \rho - \rho_\sigma \,|^2}{1 - \dfrac{|\, \rho - \rho_\sigma \,|}{m}}.$$

Now for all the values of m under consideration

$$1 - \frac{|\, \rho - \rho_\sigma \,|}{m} \geqslant 1 - \frac{|\, \rho - \rho_\sigma \,|}{|\, m \,|} > 1 - \frac{p}{p+1} > \frac{1}{p+1},$$

and therefore

$$|\, T_\sigma \,| < \frac{p+1}{2} \frac{|\, \rho - \rho_\sigma \,|^2}{m^2},$$

so that we may take

$$T_\sigma = \frac{p+1}{2m^2} \mu_\sigma (\rho - \rho_\sigma)^2,$$

where

$$|\, \mu_\sigma \,| < 1.$$

Hence

$$\chi_{m,\,m} = \Pi u_\sigma$$
$$= e^{\Sigma T_\sigma}$$
$$= e^{\frac{p+1}{2m^2} \sum\limits_{\sigma=1}^{n} \mu_\sigma (\rho - \rho_\sigma)^2}.$$

Now $\sum\limits_{\sigma=1}^{n} \mu_\sigma (\rho - \rho_\sigma)^2$ is finite for all the values of ρ under consideration, and it is finite for all values of m if μ_σ involves m; let M denote the greatest value of its modulus. Again, for any quantity θ, we have

$$|\, e^\theta - 1 \,| \leqslant |\, \theta \,| + \frac{|\, \theta \,|^2}{2\,!} + \dots \leqslant e^{|\theta|} - 1,$$

so that

$$|\, \chi_{m,\,m} - 1 \,| < e^{\frac{1}{2}(p+1) M \frac{1}{m^2}} - 1,$$

or writing

$$N = \tfrac{1}{2}(p+1) M,$$

we have

$$|\, \chi_{m,\,m} - 1 \,| < e^{\frac{N}{m^2}} - 1$$
$$< \frac{N}{m^2} + \frac{N^2}{2\,!} \frac{1}{m^4} + \frac{N^3}{3\,!} \frac{1}{m^6} + \dots;$$

and therefore

$$\Sigma \,|\, \chi_{m,\,m} - 1 \,| < N \Sigma \frac{1}{m^2} + \frac{N^2}{2\,!} \Sigma \frac{1}{m^4} + \frac{N^3}{3\,!} \Sigma \frac{1}{m^6} + \dots$$
$$< N \Sigma \frac{1}{m^2} + \frac{N^2}{2\,!} \left(\Sigma \frac{1}{m^2}\right)^2 + \frac{N^3}{3\,!} \left(\Sigma \frac{1}{m^2}\right)^3 + \dots$$
$$< N \frac{\pi^2}{3} + \frac{N^2}{2\,!} \left(\frac{\pi^2}{3}\right)^2 + \frac{N^3}{3\,!} \left(\frac{\pi^2}{3}\right)^3 + \dots$$
$$< e^{\frac{1}{3}\pi^2 N} - 1,$$

shewing that the series

$$\Sigma (\chi_{m,\,m} - 1)$$

converges.

The infinite determinant $D(\rho)$ thus converges uniformly and unconditionally for all values of ρ in the finite part of its plane. Its relation to $\Omega(\rho)$, which converges similarly for values of ρ that are not infinitesimally near any of the roots of any of the equations $\phi(\rho + m) = 0$, is at once derivable from its mode of construction from $\Omega(\rho)$. The row of quantities $\chi_{m,\mu}(\rho)$ in $D(\rho)$ for the same value of m is derived from the row of quantities $\psi_{m,\mu}$ in $\Omega(\rho)$ for that value of m, through multiplication of the latter by

$$h_m(\rho)\,\phi(m+\rho).$$

Hence

$$\begin{aligned} D(\rho) &= \Omega(\rho) \prod_{\infty}^{-\infty} h_m(\rho)\,\phi(m+\rho) \\ &= \Omega(\rho)\,\Pi(\rho), \end{aligned}$$

where

$$\begin{aligned} \Pi(\rho) &= \prod_{-\infty}^{\infty} h_m(\rho)\,\phi(m+\rho) \\ &= \prod_{m=-\infty}^{m=\infty}{}' \left[\prod_{\sigma=1}^{n} \left\{ \left(1 + \frac{\rho - \rho_\sigma}{m}\right) e^{-\frac{\rho-\rho_\sigma}{m}} \right\} \right] \prod_{\sigma=1}^{n} (\rho - \rho_\sigma), \end{aligned}$$

and Π' implies multiplication for all values of m between $+\infty$ and $-\infty$ except $m = 0$. Also

$$(\rho - \rho_\sigma) \prod_{m=-\infty}^{m=\infty}{}' \left\{ \left(1 + \frac{\rho - \rho_\sigma}{m}\right) e^{-\frac{\rho-\rho_\sigma}{m}} \right\} = \frac{\sin(\rho - \rho_\sigma)\pi}{\pi};$$

and therefore

$$\Pi(\rho) = \frac{1}{\pi^n} \prod_{\sigma=1}^{n} \{\sin(\rho - \rho_\sigma)\pi\}.$$

Now $D(\rho)$ has been proved to be finite (that is, to be not infinite) for all finite values of ρ; and manifestly, from its form, it is a uniform function of ρ, so that it is a holomorphic function of ρ everywhere in the finite part of the plane. Further, $\Omega(\rho)$ is a uniform function of ρ; and it has been proved to be not infinite for values of ρ, which are not infinitesimally near any one of the roots of any of the equations $\phi(\rho + m) = 0$, the aggregate of all these roots being

$$\rho_1 + m,\ \rho_2 + m,\ \ldots,\ \rho_n + m, \quad (m = -\infty \text{ to } +\infty).$$

Hence, owing to the relation

$$D(\rho) = \Omega(\rho)\,\Pi(\rho),$$

it follows that these roots are poles of $\Omega(\rho)$. Take a line in the ρ-plane inclined at a finite angle to the axis of real quantities, choosing the inclination so that it does not pass through any of the points $\rho_\sigma + m$ for all values of σ and m; let it cut the axis of real quantities in a point f. Take the point $f+1$ on that axis, and through it draw a line parallel to the former, thus selecting an infinite strip in the ρ-plane. Since

$$\Omega(\rho+1) = \Omega(\rho),$$

the uniform function $\Omega(\rho)$ undergoes all its variations in that strip: and within the strip, we have

$$\operatorname*{Lim}_{\rho=\infty} \Omega(\rho) = 1.$$

Owing to the nature of the poles of $\Omega(\rho)$, the strip contains n of them, which may be regarded as the irreducible poles: suppose that they are $\rho_1, \rho_2, \ldots, \rho_n$. Within the strip, $\rho = \infty$ is an ordinary point of the simply-periodic function $\Omega(\rho)$; it follows* that the number of its irreducible zeros is also n, account of possible multiplicity being taken; let these be $\rho_1', \rho_2', \ldots, \rho_n'$. Hence

$$\Omega(\rho) = A\,\frac{\sin\{(\rho-\rho_1')\pi\}\sin\{(\rho-\rho_2')\pi\}\ldots\sin\{(\rho-\rho_n')\pi\}}{\sin\{(\rho-\rho_1)\pi\}\sin\{(\rho-\rho_2)\pi\}\ldots\sin\{(\rho-\rho_n)\pi\}},$$

taking account of the holomorphic character of $D(\rho)$ for finite values of ρ, and of the relation

$$D(\rho) = \Omega(\rho)\,\Pi(\rho).$$

Here, A is independent of ρ. To determine A, we use the property

$$\operatorname*{Lim}_{\rho=\infty} \Omega(\rho) = 1,$$

which holds for

$$\rho = u + iv,$$

in the limit when v is infinite, whether positive or negative. Taking v positive and infinite, we have

$$Ae^{i\pi(\Sigma\rho_\sigma' - \Sigma\rho_\sigma)} = 1;$$

and taking v negative and infinite, we have

$$Ae^{-i\pi(\Sigma\rho_\sigma' - \Sigma\rho_\sigma)} = 1.$$

* *T. F.*, § 113.

Hence $\Sigma\rho_\sigma' - \Sigma\rho_\sigma$ is an integer; if it is not zero, we can make it zero by substituting, for the quantities ρ', values congruent with them. Assuming this done, we have

$$\Sigma\rho_\sigma' = \Sigma\rho_\sigma,$$

$$A = 1,$$

so that

$$\Omega(\rho) = \frac{\sin\{(\rho-\rho_1')\pi\} \ldots \sin\{(\rho-\rho_n')\pi\}}{\sin\{(\rho-\rho_1)\pi\} \ldots \sin\{(\rho-\rho_n)\pi\}},$$

and therefore

$$D(\rho) = \prod_{\sigma=1}^{n} \frac{\sin\{(\rho-\rho_\sigma')\pi\}}{\pi}.$$

Moreover, the quantities $\rho_1, \rho_2, \ldots, \rho_n$ are the roots of $\phi(\rho) = 0$, so that

$$\Sigma\rho_\sigma = \tfrac{1}{2}n(n-1);$$

hence

$$\Sigma\rho_\sigma' = \tfrac{1}{2}n(n-1).$$

118. Next, we consider the expression

$$Y = \sum_{k=\infty}^{-\infty} \binom{0}{k} z^k;$$

we proceed to prove that this series converges for all values of z within the annulus. It manifestly arises from $D(\rho)$, on replacing $\chi_{0,k}$ in $D(\rho)$ by z^k; we shall therefore assume that Y is transformed into this modified shape of $D(\rho)$. When the determinant is in this shape, we multiply the column associated with m by z^{-m}, and the row associated with m by z^m; these operations, combined, do not change $\chi_{m,m}$, and they do not alter the value of the determinant. Let this combined pair of operations be carried out for all the values of m from $-\infty$ to $+\infty$; the result is to give a determinant, which is equal to Y and has

$$\chi_{p,q}\, z^{p-q}$$

for its constituent in the same place that $\chi_{p,q}$ occupies in $D(\rho)$. Hence, as for $D(\rho)$, so Y converges uniformly and unconditionally for values of ρ within the ρ-region selected, and uniformly and unconditionally for values of z within the annulus, if the doubly-infinite series

$$\Sigma\Sigma\chi_{p,q}\, z^{p-q}, \qquad (p \neq q),$$

converges uniformly and unconditionally within those regions, and if

$$\Sigma\,(\chi_{q,q}-1)$$

converges uniformly and unconditionally.

The latter condition is known to be satisfied, owing to the convergence of $D(\rho)$. It remains therefore to consider the convergence of the double series.

With the notation of §§ 115—117, we have

$$\chi_{m,\mu}z^{m-\mu}=h_m(\rho)\,C_{m,\mu}z^{\lambda}$$

$$=h_m(\rho)\,[(\rho+m)^{n-2}A_2(\lambda)\,z^{\lambda}+(\rho+m)^{n-3}A_3(\lambda)\,z^{\lambda}+\ldots+A_n(\lambda)\,z^{\lambda}].$$

Now

$$A_r(\lambda)\,z^{\lambda}=\alpha_{n-2,r}\,c_{2,\lambda-2}\,z^{\lambda}+\alpha_{n-3,r-1}\,c_{3,\lambda-3}\,z^{\lambda}+\ldots+c_{r,\lambda-r}\,z^{\lambda}.$$

Owing to the definition of the coefficients in the original differential equation, the series

$$\sum_{\lambda=-\infty}^{\infty} c_{s,\lambda-s}\,z^{\lambda-s}$$

converges uniformly and unconditionally, for values of z within the annulus

$$R<|z|<R';$$

and therefore the series

$$\sum_{-\infty}^{\lambda=\infty} A_r(\lambda)\,z^{\lambda}$$

converges uniformly and unconditionally for the same range. Denoting this by J_r, we have

$$J_r=\sum_{-\infty}^{\lambda=\infty} A_r(\lambda)\,z^{\lambda};$$

and $|J_r|$ is not infinite for any of the values of z.

Again, as (§ 117)

$$\chi_{0,\mu}=C_{0,\mu},$$

and

$$\chi_{m,\mu}=k_m(\rho)\,\frac{C_{m,\mu}}{m^n},$$

when m is not zero, we have

$$\chi_{m,\mu}\,z^{m-\mu}=k_m(\rho)\,\frac{C_{m,\mu}}{m^n}\,z^{m-\mu}.$$

Proceeding with the double series $\Sigma\Sigma\chi_{m,\mu} z^{m-\mu}$, exactly as with the double series $\Sigma\chi_{m,\mu}$, omitting for the present the terms corresponding to $m=0$, and remembering that the summation is for all values of m other than $m=\mu$, we have

$$\Sigma\Sigma\,|\chi_{m,\mu} z^{m-\mu}|$$
$$\leqslant K\left[|J_2|\,\Sigma\frac{|\rho+m|^{n-2}}{m^n}+|J_3|\,\Sigma\frac{|\rho+m|^{n-3}}{m^n}+\ldots+|J_n|\,\Sigma\frac{1}{m^n}\right],$$

every group of terms in which is finite, so that

$$\Sigma\Sigma\,|\chi_{m,\mu} z^{m-\mu}|$$

is finite. Also, taking account of the terms omitted for the value $m=0$, we have

$$\Sigma\,|C_{0,\mu} z^{-\mu}| \leqslant |J_2|\,|\rho^{n-2}|+|J_3|\,|\rho^{n-3}|+\ldots+|J_n|,$$

which is finite, so that

$$\Sigma\,|C_{0,\mu} z^{-\mu}|$$

converges. Hence

$$\Sigma\Sigma\chi_{m,\mu} z^{m-\mu},$$

summed for all values of m and μ between $-\infty$ and $+\infty$ except $m=\mu$, converges unconditionally. Moreover, all the series which occur in the superior limits in the inequalities converge uniformly, both for the values of z considered and the retained range of ρ; hence the double series converges uniformly and unconditionally. The proposition is therefore established for

$$\sum_{-\infty}^{\infty}\binom{0}{k} z^k.$$

A similar investigation shews that the series

$$\sum_{k=-\infty}^{k=\infty}\begin{pmatrix}\alpha_1, & \alpha_2, & \ldots, & \alpha_r\\ k\ , & \beta_2, & \ldots, & \beta_r\end{pmatrix} z^k,$$

for any value of r, the numbers α and β being any whatever, converges uniformly and unconditionally for values of z within the annulus, and for values of ρ in the range that has been retained.

Construction of Irregular Integrals.

119. These results may now be used, by a generalisation of the method of Frobenius in Chapter III, to construct expressions for the integrals of the equation

$$P(w) = \frac{d^n w}{dz^n} + P_2 \frac{d^{n-2} w}{dz^{n-2}} + \ldots + P_n w = 0.$$

Writing

$$y = \sum_{-\infty}^{\infty} a_m z^{\rho+m},$$

and adopting the notation of § 115, we have

$$P(y) = \sum_{-\infty}^{\infty} G_m(\rho) z^{\rho+m-n}$$

$$= G_i(\rho) z^{\rho+i-n},$$

if

$$G_m(\rho) = 0,$$

for all values of m between $-\infty$ and $+\infty$, except $m = i$. The last equations are equivalent to

$$h_m(\rho) G_m(\rho) = 0,$$

that is, to

$$\sum_{-\infty}^{\infty} \chi_{m,\mu} a_\mu = u_m = 0,$$

for all the values $0, \pm 1, \pm 2, \ldots$ of m, except $m = i$. Let

$$G_i(\rho) = \sum_{-\infty}^{\infty} \chi_{i,\mu} a_\mu = u_i.$$

We have

$$\sum_{-\infty}^{s=\infty} \binom{s}{k} u_s = a_k D(\rho),$$

that is,

$$\binom{i}{k} u_i = a_k D(\rho),$$

for all the values of k. Hence, writing

$$a_0 = A \binom{i}{0},$$

we have

$$u_0 = A D(\rho),$$

and

$$a_k = A \binom{i}{k}.$$

Thus the quantity y, where

$$y = A \sum_{-\infty}^{\infty} \binom{i}{k} z^{\rho+k},$$

satisfies the equation

$$P(y) = A z^{\rho+i-n} D(\rho).$$

The determinant $D(\rho)$ is of normal form; the series for y converges uniformly and unconditionally, alike for values of z within the annulus $R < |z| < R'$, and for values of ρ within the finite region contemplated.

120. Let $\rho = \rho'$ be an irreducible simple root of $D(\rho) = 0$. Then the first minors of constituents in any line cannot vanish simultaneously for $\rho = \rho'$; for

$$\frac{\partial D}{\partial \rho} = \sum_{-\infty}^{k=\infty} \alpha_{s,k} \frac{\partial a_{s,k}}{\partial \rho},$$

and the left-hand side does not vanish for $\rho = \rho'$. Selecting minors of constituents in the line i, we have

$$y_1 = A \sum_{-\infty}^{\infty} \binom{i}{k}_{\rho=\rho'} z^{\rho'+k},$$

and

$$P(y_1) = A z^{\rho'+i-n} D(\rho') = 0:$$

that is, y_1 is an integral of the equation.

Similarly for any other irreducible simple root of $D(\rho) = 0$.

121. Next, let $\rho = \rho'$ be an irreducible multiple root of $D(\rho) = 0$ of multiplicity σ.

Firstly, suppose that some of the first minors of $D(\rho)$ do not vanish for $\rho = \rho'$; let some of these non-vanishing minors be minors of constituents in the line i. Then, in the vicinity of $\rho = \rho'$, we have

$$y = A \sum_{-\infty}^{\infty} \binom{i}{k} z^{\rho+k},$$

as a quantity satisfying the equation

$$P(y) = A z^{\rho+i-n} (\rho - \rho')^{\sigma} R(\rho - \rho'),$$

where $R(\rho - \rho')$ does not vanish when $\rho = \rho'$. It therefore follows that

$$P\left(\frac{\partial^{\mu} y}{\partial \rho^{\mu}}\right) = (\rho - \rho')^{\sigma-\mu} \Phi(z, \rho, \rho'),$$

so that, if $\mu \leqslant \sigma - 1$, we have

$$P\left[\left(\frac{\partial^\mu y}{\partial \rho^\mu}\right)_{\rho=\rho'}\right] = 0,$$

and therefore

$$y_{\mu}, = \left(\frac{\partial^\mu y}{\partial \rho^\mu}\right)_{\rho=\rho'},$$

is an integral of the equation. Hence, corresponding to the irreducible root ρ' of multiplicity σ, there are integrals

$$y_0 = \Sigma \binom{i}{k} z^{\rho+k},$$

$$y_1 = \Sigma \left[\frac{\partial}{\partial \rho}\binom{i}{k}\right] z^{\rho+k} + y_0 \log z = \eta_1 + y_0 \log z,$$

$$y_2 = \Sigma \left[\frac{\partial^2}{\partial \rho^2}\binom{i}{k}\right] z^{\rho+k} + 2\eta_1 \log z + y_0 (\log z)^2$$

$$= \eta_2 + 2\eta_1 \log z + y_0 (\log z)^2,$$

....

$$y_{\sigma-1} = \eta_{\sigma-1} + (\sigma - 1)\,\eta_{\sigma-2} \log z + \frac{(\sigma-1)(\sigma-2)}{1\,.\,2}\,\eta_{\sigma-3} (\log z)^2 + \ldots$$

$$\ldots + (\sigma - 1)\,\eta_1 (\log z)^{\sigma-2} + y_0 (\log z)^{\sigma-1},$$

when, in each of these expressions on the right-hand side, we take $\rho = \rho'$.

122. Next, still taking $\rho = \rho'$ to be an irreducible root of $D(\rho) = 0$ of multiplicity σ, suppose that, of the minors of successive orders, those of order r are the first set which do not all vanish for $\rho = \rho'$. Let the lowest multiplicity of ρ' for first minors be σ_1, for second minors be σ_2, and so on up to minors of order $r - 1$, the lowest multiplicity for which is denoted by σ_{r-1}. Then, owing to the composition of D in relation to first minors, to the composition of first minors in relation to second minors, and so on, we have

$$\sigma > \sigma_1 > \sigma_2 > \ldots > \sigma_{r-1}.$$

There are two ways of proceeding, according as $r < \sigma$, or $r = \sigma$.

First, let $r < \sigma$. With the preceding notation, we have

$$y = A \sum_{-\infty}^{\infty} \binom{i}{k} z^{\rho+k},$$

and

$$P(y) = Az^{\rho+i-n} D(\rho).$$

After the explanations given in the construction of these expressions, we know that $\rho = \rho'$ is a root of multiplicity σ_1 for some of the minors in the expression for y. As before, in § 121, the quantities

$$y, \frac{\partial y}{\partial \rho}, \ldots, \frac{\partial^{\sigma-1} y}{\partial \rho^{\sigma-1}},$$

when in each of these we take $\rho = \rho'$, are such that

$$P\left(\frac{\partial^\lambda y}{\partial \rho^\lambda}\right)_{\rho=\rho'} = 0,$$

for $\lambda = 0, 1, \ldots, \sigma - 1$. But owing to the fact that $\rho = \rho'$ is a root of all the minors $\binom{i}{k}$ of multiplicity σ_1, all the quantities

$$y, \frac{\partial y}{\partial \rho}, \ldots, \frac{\partial^{\sigma_1-1} y}{\partial \rho^{\sigma_1-1}}$$

vanish when $\rho = \rho'$. Hence the non-evanescent integrals which survive are

$$\frac{\partial^{\sigma_1} y}{\partial \rho^{\sigma_1}}, \frac{\partial^{\sigma_1+1} y}{\partial \rho^{\sigma_1+1}}, \ldots, \frac{\partial^{\sigma-1} y}{\partial \rho^{\sigma-1}},$$

when $\rho = \rho'$. They have the form

$$y_{1,1} = A \sum_{-\infty}^{\infty} \frac{\partial^{\sigma_1}}{\partial \rho^{\sigma_1}} \binom{i}{k} z^{\rho+k},$$

$$\begin{aligned} y_{1,2} &= A \sum_{-\infty}^{\infty} \frac{\partial^{\sigma_1+1}}{\partial \rho^{\sigma_1+1}} \binom{i}{k} z^{\rho+k} + (\sigma_1 + 1)\, y_{1,1} \log z \\ &= \eta_{1,2} + (\sigma_1 + 1)\, y_{1,1} \log z, \end{aligned}$$

$$\begin{aligned} y_{1,3} = A \sum_{-\infty}^{\infty} \frac{\partial^{\sigma_1+2}}{\partial \rho^{\sigma_1+2}} \binom{i}{k} z^{\rho+k} &+ (\sigma_1 + 2)\, \eta_{1,2} \log z \\ &+ \frac{(\sigma_1 + 2)(\sigma_1 + 1)}{2}\, y_{1,1} (\log z)^2, \end{aligned}$$

and so on: their number being

$$\sigma - \sigma_1.$$

Next, $\rho = \rho'$ is a root of least multiplicity σ_1 for some of the minors of the constituents of any line i: and there must be at least two such minors. For

$$D(\rho) = \Sigma \binom{i}{k} a_{i,k};$$

if $\rho = \rho'$ is a root of multiplicity $\sigma_1 + 1$ for all the minors but $\binom{i}{k}$, then, as it is of multiplicity $\geqslant \sigma_1 + 1$ for $D(\rho)$, it would be of multiplicity $\sigma_1 + 1$ for $\binom{i}{k}$. Similarly for any other line. Once more substituting

$$y = \sum_{-\infty}^{\infty} a_m z^{\rho+m}$$

in $P(y)$, we have

$$\begin{aligned} P(y) &= \sum_{-\infty}^{\infty} G_m(\rho)\, z^{\rho+m-n} \\ &= G_i(\rho)\, z^{\rho+i-n} + G_j(\rho)\, z^{\rho+j-n}, \end{aligned}$$

provided $$G_p(\rho) = 0,$$

for all integer values of p from $-\infty$ to $+\infty$ except $p = i$, $p = j$. The last equations are equivalent to

$$h_p(\rho)\, G_p(\rho) = 0,$$

that is, to

$$\sum_{-\infty}^{\mu=\infty} \chi_{p,\mu}\, a_\mu = 0,$$

for all integer values of p except i and j.

Consider quantities a_θ of the form

$$a_\theta = A \begin{pmatrix} i & j \\ \theta & l \end{pmatrix} + B \begin{pmatrix} i & j \\ k & \theta \end{pmatrix},$$

for all values of θ, the quantities A and B being arbitrary. With these expressions for a_θ, we have

$$\sum_{-\infty}^{\mu=\infty} \chi_{p,\mu}\, a_\mu = A \sum_{\mu} \begin{pmatrix} i & j \\ \mu & l \end{pmatrix} \chi_{p,\mu} + B \sum_{\mu} \begin{pmatrix} i & j \\ k & \mu \end{pmatrix} \chi_{p,\mu}.$$

Each of the sums on the right-hand sides vanishes, when p is not equal to either i or j: and thus the preceding expressions satisfy the equations

$$h_p(\rho)\, G_p(\rho) = 0,$$

for all integer values of p except i and j. Further,

$$\begin{aligned} h_i(\rho)\, G_i(\rho) &= \sum_{-\infty}^{\mu=\infty} \chi_{i,\mu} a_\mu \\ &= A \sum^{\mu} \begin{pmatrix} i & j \\ \mu & l \end{pmatrix} \chi_{i,\mu} + B \sum^{\mu} \begin{pmatrix} i & j \\ k & \mu \end{pmatrix} \chi_{i,\mu} \\ &= A \binom{j}{l} - B \binom{j}{k}; \end{aligned}$$

and, similarly,

$$h_j(\rho)\, G_j(\rho) = -A\begin{pmatrix} i \\ l \end{pmatrix} + B\begin{pmatrix} i \\ k \end{pmatrix}.$$

Using these values, we have

$$\sum_{-\infty}^{\infty} \left\{A\begin{pmatrix} i & j \\ m & l \end{pmatrix} + B\begin{pmatrix} i & j \\ k & m \end{pmatrix}\right\} z^{\rho+m}$$

as the expression for y; and it satisfies the relation

$$\begin{aligned} P(y) &= G_i(\rho)\, z^{\rho+i-n} + G_j(\rho)\, z^{\rho+j-n} \\ &= \frac{z^{\rho+i-n}}{h_i(\rho)}\left\{A\begin{pmatrix} j \\ l \end{pmatrix} - B\begin{pmatrix} j \\ k \end{pmatrix}\right\} + \frac{z^{\rho+j-n}}{h_j(\rho)}\left\{-A\begin{pmatrix} i \\ l \end{pmatrix} + B\begin{pmatrix} i \\ k \end{pmatrix}\right\}. \end{aligned}$$

As the right-hand side of the last equation has $\rho = \rho'$ as a root of multiplicity σ_1, the quantities $h_i(\rho)$ and $h_j(\rho)$ having no zero for finite values of ρ, it follows that

$$P\left(\frac{\partial^\lambda y}{\partial \rho^\lambda}\right)_{\rho=\rho'} = 0,$$

for $\lambda = 0, 1, \ldots, \sigma_1 - 1$. Therefore all the quantities

$$y,\ \frac{\partial y}{\partial \rho},\ \ldots,\ \frac{\partial^{\sigma_1-1} y}{\partial \rho^{\sigma_1-1}},$$

when $\rho = \rho'$, satisfy the equation $P(w) = 0$. Owing to the form of y above obtained, which has $\rho = \rho'$ as a root of multiplicity σ_2, all the quantities

$$y,\ \frac{\partial y}{\partial \rho},\ \ldots,\ \frac{\partial^{\sigma_2-1} y}{\partial \rho^{\sigma_2-1}}$$

vanish when $\rho = \rho'$. Therefore the surviving integrals are

$$\begin{aligned} y_{2,1} &= \frac{\partial^{\sigma_2} y}{\partial \rho^{\sigma_2}}, \\ y_{2,2} &= \frac{\partial^{\sigma_2+1} y}{\partial \rho^{\sigma_2+1}} = \eta_{2,2} + (\sigma_2+1)\, y_{2,1} \log z, \\ y_{2,3} &= \frac{\partial^{\sigma_2+2} y}{\partial \rho^{\sigma_2+2}} = \eta_{3,2} + (\sigma_2+2)\, \eta_{2,2} \log z + \frac{(\sigma_2+2)(\sigma_2+1)}{2} y_{2,1} (\log z)^2, \end{aligned}$$

and so on: their number being

$$\sigma_1 - \sigma_2.$$

Similarly for the next sub-group. With the same notation as before, we have

$$P(y) = G_i(\rho)\, z^{\rho+i-n} + G_j(\rho)\, z^{\rho+j-n} + G_h(\rho)\, z^{\rho+h-n},$$

provided

$$G_p(\rho) = 0,$$

for all values of p, other than i, j, h, from $-\infty$ to ∞. The analogy of the preceding case suggests

$$a_\theta = A\begin{pmatrix} i, j, h \\ \theta, l, m \end{pmatrix} + B\begin{pmatrix} i, j, h \\ k, \theta, m \end{pmatrix} + C\begin{pmatrix} i, j, h \\ k, l, \theta \end{pmatrix},$$

for all values of θ, where A, B, C are any quantities. With these expressions for a_θ, we have

$$\sum_{-\infty}^{\mu=\infty} \chi_{p,\mu} a_\mu = A\sum^{\mu}\begin{pmatrix} i, j, h \\ \mu, l, m \end{pmatrix}\chi_{p,\mu} + B\sum^{\mu}\begin{pmatrix} i, j, h \\ k, \mu, m \end{pmatrix}\chi_{p,\mu} + C\sum^{\mu}\begin{pmatrix} i, j, h \\ k, l, \mu \end{pmatrix}\chi_{p,\mu}.$$

Each of the three sums on the right-hand side vanishes, when p is not equal to either i or j or h: so that the preceding expressions for a_θ satisfy the equation

$$G_p(\rho) = 0,$$

for all values of p other than i or j or h. Further,

$$\begin{aligned} G_i(\rho) &= \sum_{-\infty}^{\mu=\infty} \chi_{i,\mu} a_\mu \\ &= A\begin{pmatrix} j, h \\ l, m \end{pmatrix} + B\begin{pmatrix} j, h \\ m, k \end{pmatrix} + C\begin{pmatrix} j, h \\ k, l \end{pmatrix}, \\ G_j(\rho) &= A\begin{pmatrix} h, i \\ l, m \end{pmatrix} + B\begin{pmatrix} h, i \\ m, k \end{pmatrix} + C\begin{pmatrix} h, i \\ k, l \end{pmatrix}, \\ G_h(\rho) &= A\begin{pmatrix} i, j \\ l, m \end{pmatrix} + B\begin{pmatrix} i, j \\ m, k \end{pmatrix} + C\begin{pmatrix} i, j \\ k, l \end{pmatrix}. \end{aligned}$$

Thus

$$P(y) = \Phi(z, \rho),$$

where $\Phi(z, \rho)$ is a linear combination of minors of the second order; and

$$y = \sum_{-\infty}^{\theta=\infty} a_\theta z^{\rho+\theta},$$

the coefficients a_θ being linear combinations of minors of the third order.

As $\Phi(z, \rho)$ has $\rho=\rho'$ as a root of multiplicity σ_2, it follows that

$$P\left(\frac{\partial^\lambda y}{\partial\rho^\lambda}\right)_{\rho=\rho'}=0,$$

for $\lambda=0, 1, \ldots, \sigma_2-1$; so that all the quantities

$$y, \ \frac{\partial y}{\partial\rho}, \ \ldots, \ \frac{\partial^{\sigma_2-1}y}{\partial\rho^{\sigma_2-1}},$$

when $\rho=\rho'$, satisfy the equation $P(w)=0$. Owing to the form of the coefficients a_θ in y, each of which has $\rho=\rho'$ as a root of multiplicity σ_3, all the quantities

$$y, \ \frac{\partial y}{\partial\rho}, \ \ldots, \ \frac{\partial^{\sigma_3-1}y}{\partial\rho^{\sigma_3-1}},$$

vanish when $\rho=\rho'$; and we therefore are left with the integrals

$$y_{3,1}=\frac{\partial^{\sigma_3}y}{\partial\rho^{\sigma_3}},$$

$$y_{3,2}=\frac{\partial^{\sigma_3+1}y}{\partial\rho^{\sigma_3+1}}=\eta_{3,2}+(\sigma_3+1)\,y_{3,1}\log z,$$

$$y_{3,3}=\frac{\partial^{\sigma_3+2}y}{\partial\rho^{\sigma_3+2}}=\eta_{3,3}+(\sigma_3+2)\,\eta_{3,2}\log z+\frac{(\sigma_3+2)(\sigma_3+1)}{2}\,y_{3,1}(\log z)^2,$$

and so on: their number being

$$\sigma_2-\sigma_3.$$

Proceeding in this manner, we obtain successive sub-groups of integrals; the total number in the whole group is

$$(\sigma-\sigma_1)+(\sigma_1-\sigma_2)+(\sigma_2-\sigma_3)+\ldots+(\sigma_{r-2}-\sigma_{r-1})+\sigma_{r-1}$$
$$=\sigma,$$

which is the multiplicity of $\rho=\rho'$ as a root of $D(\rho)=0$.

123. Two cases, both limiting, call for special mention.

It is manifest that, if $\sigma-\sigma_1>1$, the first sub-group contains integrals whose expressions involve logarithms; likewise for the second sub-group, if $\sigma_1-\sigma_2>1$; and so on. If, then, all the integrals belonging to the multiple root $\rho=\rho'$ of $D(\rho)=0$ are to be free from logarithms, we must have

$$\sigma-\sigma_1=1, \ \sigma_1-\sigma_2=1, \ \ldots,$$

and therefore

$$\sigma = r,$$

which thus is a limiting case of the preceding investigation.

An intimation was given that, when $r = \sigma$, a different method of proceeding is possible. As a matter of fact, the property of the infinite system of linear relations, established in § 113, leads at once to the result. Let

$$\begin{pmatrix} \alpha_1, \alpha_2, \ldots, \alpha_r \\ \beta_1, \beta_2, \ldots, \beta_r \end{pmatrix}$$

be one of the non-vanishing minors of order r belonging to $D(\rho)$; then

$$\begin{pmatrix} \alpha_1, \alpha_2, \ldots, \alpha_r \\ \beta_1, \beta_2, \ldots, \beta_r \end{pmatrix} a_\mu = \sum_{p=1}^{r} \begin{pmatrix} \alpha_1, \alpha_2, \ldots, \alpha_{p-1}, \alpha_p, \alpha_{p+1}, \ldots, \alpha_r \\ \beta_1, \beta_2, \ldots, \beta_{p-1}, \mu, \beta_{p+1}, \ldots, \beta_r \end{pmatrix} a_{\mu_p},$$

and the quantities $a_{\mu_1}, a_{\mu_2}, \ldots, a_{\mu_r}$ are bound by no relations, so that they are arbitrary constants. The integral determined by these coefficients is

$$\sum_{-\infty}^{\infty} a_\mu z^{\rho' + \mu};$$

it manifestly is a linear combination, with arbitrary coefficients $a_{\mu_1}, \ldots, a_{\mu_r}$, of r integrals which are, in fact, the group of integrals above indicated.

The other limiting case occurs when $r = 1$: all the σ integrals belong to a single sub-group. In that case, there exists at least one minor of the first order which does not vanish when $\rho = \rho'$; the condition is both necessary and sufficient.

124. We thus have a set of σ integrals, belonging to an irreducible root ρ' of $D(\rho) = 0$ which is of multiplicity σ. Similarly for any other irreducible root of $D(\rho) = 0$; hence, when all the irreducible roots are taken, we have a system of n integrals. We proceed to prove that *this system of integrals is fundamental.*

For, in the first place, it follows (from the lemma in § 27) that the integrals in any sub-group are linearly independent, on account of the powers of $\log z$ which they contain.

Next, there can be no relation of the form

$$C_1 y_{1,1} + C_2 y_{2,1} + \ldots + C_r y_{r,1} = 0,$$

with non-vanishing coefficients C. If such an one could exist, the coefficient of every power of z in the aggregate expression on the left-hand side must vanish. Writing

$$\left.\begin{array}{l} i,\ j,\ h,\ \ldots = p_1,\ p_2,\ p_3,\ \ldots,\ p_r \\ k,\ l,\ m,\ \ldots = q_1,\ q_2,\ q_3,\ \ldots,\ q_r \end{array}\right\},$$

we have

$$y_{s,1} = \sum_{-\infty}^{\infty} \left(\frac{\partial^{\sigma_s} a_\theta}{\partial \rho^{\sigma_s}}\right) z^{\rho'+\theta},$$

where

$$a_\theta = A_{s,1}\begin{pmatrix} p_1, p_2, \ldots, p_s \\ \theta,\ q_2, \ldots, q_s \end{pmatrix} + A_{s,2}\begin{pmatrix} p_1, p_2, \ldots, p_s \\ q_1,\ \theta,\ \ldots, q_s \end{pmatrix} + \ldots$$

$$\ldots + A_{s,s}\begin{pmatrix} p_1, p_2, \ldots, p_{s-1}, p_s \\ q_1, q_2, \ldots, q_{s-1}, \theta \end{pmatrix},$$

and the quantities $A_{s,1}$, $A_{s,2}$, $\ldots$, $A_{s,s}$ are at our disposal. Let these last be chosen so that

$$A_{s,1} = A_{s,2} = \ldots = A_{s,s-1} = 0, \quad A_{s,s} = 1.$$

Then the coefficient of $z^{\rho'+q_t}$ in $y_{s,1}$ is zero if $t < s$, and it is different from zero if $t = s$: let it be denoted by $[y_{s,1}]_{q_t}$.

The above relation being supposed to hold, select the coefficients of $z^{\rho'+q_1}$, $z^{\rho'+q_2}$, $\ldots$, $z^{\rho'+q_r}$ in turn. As they vanish, we have

$$C_1 [y_{1,1}]_{q_1} + C_2 [y_{2,1}]_{q_1} + \ldots + C_r [y_{r,1}]_{q_1} = 0,$$

from the coefficient of $z^{\rho'+q_1}$; every term vanishes except the first, and $[y_{1,1}]_{q_1}$ is not zero; hence

$$C_1 = 0.$$

The vanishing of the coefficient of $z^{\rho'+q_2}$ then gives

$$C_2 [y_{2,1}]_{q_2} + C_3 [y_{3,1}]_{q_3} + \ldots + C_r [y_{r,1}]_{q_r} = 0;$$

every term after the first vanishes, and $[y_{2,1}]_{q_2}$ does not vanish; hence

$$C_2 = 0.$$

And so on; every one of the coefficients C vanishes; and thus no relation of the form

$$C_1 y_{1,1} + C_2 y_{2,1} + \ldots + C_r y_{r,1} = 0$$

can exist.

Next, there can be no linear relation among the σ members of a group. For, in any expression

$$\Sigma C_{s,t}\, y_{s,t},$$

the coefficient of the highest power of $\log z$ is of the form

$$\Sigma C_{s,t}\, y_{s,1},$$

and this can vanish, only if the coefficients $C_{s,t}$ are evanescent; hence $\Sigma C_{s,t}\, y_{s,t}$ can vanish, only if the coefficients $C_{s,t}$ are evanescent.

Lastly, there can be no linear relation among the members of different groups. For let $Y(\rho', z)$, $Y(\rho'', z)$, ... denote the most general integrals of the groups belonging to the irreducible roots ρ', ρ'', ... respectively, of $D(\rho) = 0$. Let z describe a contour enclosing the origin; then $Y(\rho', z)$ acquires a factor $e^{2\pi i\rho'}$, $Y(\rho'', z)$ acquires a factor $e^{2\pi i\rho''}$, and so on. Thus, if there were a relation

$$\alpha Y(\rho', z) + \beta Y(\rho'', z) + \ldots = 0,$$

then

$$\alpha e^{2\pi i\rho'}\, Y(\rho', z) + \beta e^{2\pi i\rho''}\, Y(\rho'', z) + \ldots = 0;$$

and similarly, after κ descriptions of the contour,

$$\alpha e^{2\pi i\rho'\kappa}\, Y(\rho', z) + \beta e^{2\pi i\rho''\kappa}\, Y(\rho'', z) + \ldots = 0,$$

for as many values of the integer κ as we please. Now ρ', ρ'', ... are the irreducible roots of $D(\rho) = 0$; no two of them are equal, and no two can differ by an integer. Hence the preceding relations can be satisfied, only if

$$\alpha = 0,\quad \beta = 0,\quad \ldots;$$

in other words, no linear relation among the n integrals can exist. They therefore form a fundamental system.

The Equation $D(\rho) = 0$ is the Fundamental Equation of the Singularity.

125. Consider the effect which the description of a closed contour, round the origin and lying wholly in the annulus, exercises upon this fundamental system. Let

$$\theta' = e^{2\pi i\rho'},\quad \theta'' = e^{2\pi i\rho''},\quad \ldots;$$

and let y' denote, at the completion of the contour, the value of the integral which initially is y. We have

$$\begin{aligned}
y_{11}' &= \theta' y_{11}, \\
y_{12}' &= \theta'\eta_{12} + \sigma_1\theta' y_{11}(\log z + 2\pi i) = \alpha_{21} y_{11} + \theta' y_{12}, \\
y_{13}' &= \alpha_{31} y_{11} + \alpha_{32} y_{12} + \theta' y_{13}, \\
&\;\vdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
y_{21}' &= \qquad\qquad \theta' y_{21}, \\
y_{22}' &= \qquad\qquad \beta_{21} y_{21} + \theta' y_{22}, \\
&\;\vdots
\end{aligned}$$

and so on. Hence the fundamental equation (Chap. II) is $\Theta = 0$, where

$$\Theta = \begin{vmatrix}
\theta'-\theta, & 0, & 0, & \ldots, & 0, & \ldots & & & & \\
\alpha_{21}, & \theta'-\theta, & 0, & \ldots, & 0, & \ldots & & & & \\
\alpha_{31}, & \alpha_{32}, & \theta'-\theta, & \ldots, & 0, & \ldots & & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0, & 0, & 0, & \ldots, & \theta'-\theta, & 0, & 0, & \ldots, & 0, & \ldots \\
\cdots & \cdots & \cdots & \ldots, & \beta_{21}, & \theta'-\theta, & 0, & \ldots, & 0, & \ldots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \ldots, & 0, & 0, & 0, & \ldots, & \theta''-\theta, & \ldots \\
\cdots & \cdots & \cdots & \ldots, & 0, & 0, & 0, & \ldots, & \gamma_{21}, & \theta''-\theta, \ldots
\end{vmatrix}$$

$$= (\theta'-\theta)^{\sigma'}(\theta''-\theta)^{\sigma''}\ldots,$$

where σ' is the number of integrals in the group belonging to the root ρ' of $D(\rho)=0$ of multiplicity σ'; σ'' is the number in the group belonging to the root ρ''; and so on.

Now it was proved that

$$D(\rho) = \frac{1}{\pi^n}\prod_{\sigma=1}^{n} \sin\{(\rho-\rho_\sigma')\pi\}.$$

But

$$\begin{aligned}
\sin(\rho-\rho_\sigma')\pi &= \frac{1}{2i}e^{-(\rho+\rho_\sigma')\pi i}\left(e^{2\pi i\rho} - e^{2\pi i\rho_\sigma'}\right) \\
&= \frac{1}{2i}e^{-(\rho+\rho_\sigma')\pi i}(\theta-\theta_\sigma'),
\end{aligned}$$

if

$$\theta = e^{2\pi i\rho}.$$

Hence

$$D(\rho) = \frac{1}{(2\pi i)^n}e^{-n\rho\pi i - \pi i\Sigma\rho_\sigma'}\prod_{\sigma=1}^{n}(\theta-\theta').$$

Also (§ 117)

$$\Sigma \rho_\sigma' = \tfrac{1}{2} n (n-1),$$

so that

$$e^{-\pi i \Sigma \rho_\sigma'} = e^{-\frac{1}{2} n (n-1) \pi i} = \pm 1;$$

and therefore

$$D(\rho) = \pm \frac{e^{-n\rho\pi i}}{(2\pi i)^n} (\theta - \theta')^{\sigma'} (\theta - \theta'')^{\sigma''} \dots .$$

As the quantity $e^{-n\rho\pi i}$ has no zero for finite values of ρ, it thus appears that, so far as roots are concerned, $D(\rho) = 0$ and $\Theta = 0$ are effectively the same equation, when the relation between ρ and θ is taken into account. Also, so far as roots are concerned, $\Omega(\rho) = 0$ is effectively the same as $D(\rho) = 0$; hence *any one of the three equations*

$$\Theta = 0, \quad D(\rho) = 0, \quad \Omega(\rho) = 0,$$

may be used for the determination of θ *and the associated quantity* ρ.

It is known that $\Theta = 0$ is an equation remaining invariantive for all modifications of the fundamental system: and, for the form of equation adopted in § 114, the term in Θ independent of θ is equal to unity (§ 14). This property in the present case is verified by means of the values of the quantities θ', θ'', ...; for

$$(-\theta')^{\sigma'} (-\theta'')^{\sigma''} \dots = (-1)^n e^{2\pi i \Sigma \rho_\sigma'} = (-1)^n e^{n(n-1)\pi i} = (-1)^n.$$

The remaining coefficients in Θ are known (§ 14) to be the *invariants* of the equation, whatever fundamental system be chosen.

Replacing Θ by $\Omega(\rho)$ for purposes of this discussion, we have

$$\Omega(\rho) = \frac{\sin \{(\rho - \rho_1') \pi\} \dots \sin \{(\rho - \rho_n') \pi\}}{\sin \{(\rho - \rho_1) \pi\} \dots \sin \{(\rho - \rho_n) \pi\}}.$$

Now

$$\frac{\sin \{(\rho - \rho_r') \pi\}}{\sin \{(\rho - \rho_r) \pi\}} = e^{\pi i (\rho_r - \rho_r')} \frac{\theta - \theta_r'}{\theta - \theta_r},$$

where

$$\theta_r = e^{2\pi i \rho_r}, \quad \theta_r' = e^{2\pi i \rho_r'};$$

so that, as

$$\Sigma \rho_r = \Sigma \rho_r',$$

and therefore

$$\prod_{r=1}^{n} \theta_r = \prod_{r=1}^{n} \theta_r' = e^{2\pi i \frac{1}{2} n (n-1)} = 1,$$

we have

$$\Omega(\rho) = \frac{(\theta-\theta_1')(\theta-\theta_2')\dots(\theta-\theta_n')}{(\theta-\theta_1)(\theta-\theta_2)\dots(\theta-\theta_n)}$$

$$= \frac{\theta^n - \dots + (-1)^n}{\theta^n - \dots + (-1)^n}.$$

Hence, when $\Omega(\rho)$ is expanded in descending powers of θ, the term in θ^0 is unity; and when it is expanded in ascending powers of θ, the term in θ^0 is likewise unity.

When the quantities $\theta_1, \theta_2, \dots, \theta_n$ are unequal, then $\Omega(\rho)$ can be expressed in the form

$$\Omega(\rho) = 1 + \sum_{\sigma=1}^{n} \frac{M_\sigma' \theta_\sigma}{\theta - \theta_\sigma}.$$

On account of the character of $\Omega(\rho)$, when expanded in ascending powers of θ, we have

$$\sum_{\sigma=1}^{n} M_\sigma' = 0,$$

so that there are $n-1$ independent quantities M_σ', and these are equivalent to the $n-1$ invariants. The equation may also be expressed in the form

$$\Omega(\rho) = 1 + \sum_{\sigma=1}^{n} M_\sigma \cot\{(\rho-\rho_\sigma)\pi\},$$

where

$$M_\sigma' = 2\pi i M_\sigma,$$

and therefore

$$\sum_{\sigma=1}^{n} M_\sigma = 0.$$

Corresponding expansions occur in the case when equalities occur among the quantities $\rho_1, \rho_2, \dots, \rho_n$.

126. The integrals, which have been obtained, are valid within the annulus represented by $R \leqslant |z| \leqslant R'$; the inner circle may enclose any number of singularities of the equation, and the outer circle may exclude any number of other singularities of the equation. But care must be exercised in particular cases. If for instance, the only singularity within the inner circle is the origin, and the integrals are regular in the vicinity of the origin, then in the expression of any integral, such as

$$\Sigma a_m z^{\rho+m},$$

there can be only a finite number of terms with negative values of m: the method, which is based upon the supposed existence of an unlimited number of such terms, is no longer applicable. If the only singularity outside the outer circle is $z=\infty$, and if the integrals are regular in the vicinity of $z=\infty$, then in the expression of any integral, such as

$$\Sigma a_m z^{\rho+m},$$

there can be only a finite number of terms with positive values of m: the method again ceases to be applicable.

In such cases, the best procedure is to construct a fundamental system which shall include the regular integrals: this is the customary procedure for, e.g., Bessel's equation, the integrals of which have $z=\infty$ for an essential singularity and are regular near $z=0$. The method, which uses infinite determinants, is best reserved for equations which have their integrals non-regular in the vicinity of every singularity: it is nugatory when applied to Bessel's equation.

Ex. 1. Consider the equation

$$\frac{d^2y}{dz^2}+\left(\frac{a}{z^3}+\frac{b}{z^2}+\frac{\gamma}{z}\right)y=0.$$

It is clear that the point $z=0$ is an essential singularity, there being no integral regular in its vicinity, when a is different from 0: and that $z=\infty$ is likewise an essential singularity, when γ is different from 0. We shall assume that both a and γ are non-vanishing quantities.

Let

$$z=x\left(\frac{a}{\gamma}\right)^{\frac{1}{2}},\quad \alpha=(a\gamma)^{\frac{1}{2}};$$

the equation becomes

$$\frac{d^2y}{dx^2}+\left(\frac{\alpha}{x^3}+\frac{b}{x^2}+\frac{\alpha}{x}\right)y=0.$$

With the notation of the preceding paragraphs, we have

$$\phi(\rho)=\rho(\rho-1)+b=(\rho-\rho_1)(\rho-\rho_2);$$

$$C_{m,\,\mu}=c_{2,\,m-\mu-2},$$

so that

$$C_{r,r}=b,\quad C_{r,r+1}=\alpha,\quad C_{r,r-1}=\alpha,$$

$$C_{r,\mu}=0,\text{ when } \mu<r-1, \text{ and when } \mu>r+1;$$

also

$$\psi_{r,r}=1,$$

$$\psi_{r,r-1}=\frac{\alpha}{\phi(\rho+r)},\quad \psi_{r,r+1}=\frac{\alpha}{\phi(\rho+r)},$$

$$\psi_{r,\mu}=0,\text{ when } \mu<r-1, \text{ and when } \mu>r+1.$$

Hence the value of $\Omega(\rho)$ is

$$\left|\begin{array}{ccccccccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \ldots, 0, & \dfrac{a}{\phi(\rho-2)}, & 1, & \dfrac{a}{\phi(\rho-2)}, & 0, & 0, & 0, & 0, & 0, \ldots \\ \ldots, 0, & 0, & \dfrac{a}{\phi(\rho-1)}, & 1, & \dfrac{a}{\phi(\rho-1)}, & 0, & 0, & 0, & 0, \ldots \\ \ldots, 0, & 0, & 0, & \dfrac{a}{\phi(\rho)}, & 1, & \dfrac{a}{\phi(\rho)}, & 0, & 0, & 0, \ldots \\ \ldots, 0, & 0, & 0, & 0, & \dfrac{a}{\phi(\rho+1)}, & 1, & \dfrac{a}{\phi(\rho+1)}, & 0, & 0, \ldots \\ \ldots, 0, & 0, & 0, & 0, & 0, & \dfrac{a}{\phi(\rho+2)}, & 1, & \dfrac{a}{\phi(\rho+2)}, & 0, \ldots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right|$$

The general investigation shews that, when ρ_1 and ρ_2 are unequal (which will be assumed),

$$\Omega(\rho) = 1 + M_1\pi \cot(\rho-\rho_1)\pi + M_2\pi \cot(\rho-\rho_2)\pi,$$

with the condition

$$M_1 + M_2 = 0;$$

that is, we have

$$\Omega(\rho) = 1 + \pi M[\cot\{(\rho-\rho_1)\pi\} - \cot\{(\rho-\rho_2)\pi\}],$$

where M is independent of ρ.

Taking the determinantal form for $\Omega(\rho)$, and expanding according to the law established in § 110, we have

$$\Omega(\rho) = 1 + a^2M_2 + a^4M_4 + a^6M_6 + \ldots,$$

where odd powers of a do not occur because the combinations which they multiply all vanish. Also

$$M_2 = -\sum_{-\infty}^{m=\infty} \frac{1}{\phi(\rho+m)\,\phi(\rho+m+1)},$$

$$M_4 = \sum_{-\infty}^{m=\infty}\sum_{m+2}^{p=\infty} \frac{1}{\phi(\rho+m)\,\phi(\rho+m+1)\,\phi(\rho+p)\,\phi(\rho+p+1)},$$

$$M_6 = -\sum_{-\infty}^{m=\infty}\sum_{m+2}^{p=\infty}\sum_{p+2}^{q=\infty} \frac{1}{\phi(\rho+m)\phi(\rho+m+1)\phi(\rho+p)\phi(\rho+p+1)\phi(\rho+q)\phi(\rho+q+1)},$$

and so on. Hence we have

$$M = a^2N_2 + a^4N_4 + a^6N_6 + \ldots,$$

and N_{2k} is the coefficient of $\dfrac{1}{\rho-\rho_1}$ (that is, the residue of $\rho=\rho_1$) in M_{2k}.

To find N_2, we notice that the only terms in M_2, which have $\rho=\rho_1$ for a pole, are those given by $m=0$, $m=-1$, these being

$$-\frac{1}{\phi(\rho)\,\phi(\rho+1)} - \frac{1}{\phi(\rho-1)\,\phi(\rho)}.$$

Now

$$\phi(\rho)=(\rho-\rho_1)(\rho-\rho_2);$$

hence

$$N_2=-\frac{1}{\rho_1-\rho_2}\left\{\frac{1}{\phi(\rho_1+1)}+\frac{1}{\phi(\rho_1-1)}\right\}$$

$$=-\frac{1}{\rho_1-\rho_2}\,\frac{1}{1-(\rho_1-\rho_2)^2}=\frac{-1}{4b(\rho_1-\rho_2)}.$$

Again, writing

$$\Phi(\rho+m)=\phi(\rho+m)\,\phi(\rho+m+1),$$

we have

$$M_4=\Sigma\Sigma\,\frac{1}{\Phi(\rho+m)\,\Phi(\rho+p)}.$$

Consider

$$\left\{\sum_{n=-\infty}^{n=\infty}\frac{1}{\Phi(\rho+n)}\right\}^2;$$

it contains the terms

$$\sum_{n=-\infty}^{n=\infty}\left\{\frac{1}{\Phi(\rho+n)}\right\}^2,$$

which do not occur in M_4; it contains terms

$$2\sum_{-\infty}^{\infty}\frac{1}{\Phi(\rho+m)\,\Phi(\rho+m+1)},$$

which do not occur in M_4; and it contains the terms

$$\sum_{p>m+1}\!\!\Sigma\,\frac{1}{\Phi(\rho+m)\,\Phi(\rho+p)}$$

twice over, once in the form

$$\sum_{p>m+1}\!\!\Sigma\,\frac{1}{\Phi(\rho+m)\,\Phi(\rho+p)},$$

and once in the form

$$\sum_{m>p+1}\!\!\Sigma\,\frac{1}{\Phi(\rho+m)\,\Phi(\rho+p)}.$$

Hence

$$\left\{\sum_{-\infty}^{n=\infty}\frac{1}{\Phi(\rho+n)}\right\}^2=\sum_{n=-\infty}^{n=\infty}\left\{\frac{1}{\Phi(\rho+n)}\right\}^2+2\sum_{-\infty}^{\infty}\frac{1}{\Phi(\rho+m)\,\Phi(\rho+m+1)}+2M_4,$$

so that

$$M_4=\tfrac{1}{2}\left\{\sum_{-\infty}^{n=\infty}\frac{1}{\Phi(\rho+n)}\right\}^2-\tfrac{1}{2}\sum_{n=-\infty}^{n=\infty}\left\{\frac{1}{\Phi(\rho+n)}\right\}^2-\sum_{-\infty}^{\infty}\frac{1}{\Phi(\rho+m)\,\Phi(\rho+m+1)}.$$

The first term on the right-hand side is

$$=\tfrac{1}{2}M_2^2$$

$$=\tfrac{1}{2}N_2^2\,\pi^2\left[\cot(\rho-\rho_1)\,\pi-\cot(\rho-\rho_2)\,\pi\right]^2;$$

the residue of this function for $\rho=\rho_1$ is

$$=-N_2^2\,\pi\cot(\rho_1-\rho_2)\,\pi$$

$$=\frac{-\pi\cot\{(\rho_1-\rho_2)\,\pi\}}{16b^2(1-4b)}.$$

In the second term on the right-hand side, the residue for $\rho=\rho_1$ can arise only for the values $n=0$, $n=-1$; thus it is

$$\begin{aligned}&\tfrac{1}{2}\frac{\Phi''(\rho_1)}{\Phi'^3(\rho_1)}+\tfrac{1}{2}\frac{\Phi''(\rho_1-1)}{\Phi'^3(\rho_1-1)}\\ &=\tfrac{1}{2}\frac{(1+2\rho_1-2\rho_2)(2+\rho_1-\rho_2)}{(\rho_1-\rho_2)^3(1+\rho_1-\rho_2)^3}+\tfrac{1}{2}\frac{(1-2\rho_1+2\rho_2)(2-\rho_1+\rho_2)}{(\rho_1-\rho_2)^3(1-\rho_1+\rho_2)^3}\\ &=-\frac{1-6b+4b^2}{8b^3(1-4b)(\rho_1-\rho_2)},\end{aligned}$$

after reduction. Similarly, from the third term, the residue is

$$=-\frac{3-8b}{8b^2(3+4b)(\rho_1-\rho_2)}-\frac{1-8b}{8b^2(1-4b)(\rho_1-\rho_2)}.$$

Hence

$$N_4=-\frac{\pi\cot\{(\rho_1-\rho_2)\pi\}}{16b^2(1-4b)}-\frac{3-8b-52b^2+16b^3}{8b^3(3+4b)(1-4b)(\rho_1-\rho_2)},$$

after reduction.

Other coefficients could be calculated in a similar manner: but it is clear that even N_6 would involve considerable numerical calculations, and it is difficult to see how the general term could thus be obtained. But the method of approximation may be effective in particular applications. Thus, in Hill's discussion* of the motion of the lunar perigee, the convergence is very rapid; and comparatively few terms need be taken in order to obtain an approximation of advanced accuracy. When this is the case, the values of ρ' for the integrals are given by

$$\Omega(\rho)=0,$$

that is,

$$\cos 2\rho\pi=-\cos\{(\rho_1-\rho_2)\pi\}-2\pi M\sin\{(\rho_1-\rho_2)\pi\};$$

and two irreducible values of ρ' chosen are to be such that

$$\rho_1'+\rho_2'=\rho_1+\rho_2=1.$$

The expressions for the integrals are to be obtained. Denoting still by ρ either of the quantities ρ_1' and ρ_2', the relations between the coefficients are

$$\frac{a}{\phi(\rho+r)}a_{r-1}+a_r+\frac{a}{\phi(\rho+r)}a_{r+1}=0:$$

and considering in particular the row 0, we know that the constants a are proportional to the minors of the constituents in that row in the determinant $\Omega(\rho)$. Thus

$$a_0:\binom{0}{0}=a_\kappa:\binom{0}{\kappa},$$

for all positive and negative values of κ: so that, if we take

$$a_0=1,$$

* See the memoir already quoted in § 109.

we have

$$a_\kappa = \frac{\binom{0}{\kappa}}{\binom{0}{0}};$$

and our solution is

$$y = \sum_{-\infty}^{\kappa=\infty} a_\kappa x^\kappa,$$

for the effective expression of which it is sufficient to find the first minors, as the series is known to converge within the annulus.

In order to obtain $\binom{0}{0}$ from $\Omega(\rho)$, we replace $\frac{a}{\phi(\rho-1)}, \frac{a}{\phi(\rho)}, \frac{a}{\phi(\rho+1)}$ by zeros; it will therefore be necessary to do this in the expanded form. We thus have

$$\binom{0}{0} = 1 + a^2 M_{0,2} + a^4 M_{0,4} + \dots,$$

where

$$a^2 M_{0,2} = a^2 M_2 + \frac{a^2}{\phi(\rho)\,\phi(\rho+1)} + \frac{a^2}{\phi(\rho-1)\,\phi(\rho)}$$

$$= a^2 M_2 + 2a^2 \frac{\rho^2 - \rho + 1 + \beta}{\phi(\rho-1)\,\phi(\rho)\,\phi(\rho+1)}.$$

Similarly for $M_{0,4}$ from M_4; and so on.

In order to obtain $\binom{0}{-1}$ from $\Omega(\rho)$, we replace $\frac{a}{\phi(\rho)}$ in the -1 column by unity; the quantities 1 and $\frac{a}{\phi(\rho-2)}$ in that column by zeros; and the quantities 1 and $\frac{a}{\phi(\rho)}$ in the 0 line by zero. We then easily find

$$\binom{0}{-1} = -\frac{a}{\phi(\rho-1)} + a^2 M_{-1,2} + a^4 M_{-1,4} + \dots,$$

where

$$a^2 M_{-1,2} = a^2 M_2 + \frac{a^2}{\phi(\rho)\,\phi(\rho+1)} + \frac{a^2}{\phi(\rho-1)\,\phi(\rho)} + \frac{a^2}{\phi(\rho-1)\,\phi(\rho-2)}$$

$$= a^2 M_{0,2} + \frac{a^2}{\phi(\rho-1)\,\phi(\rho-2)};$$

and similarly for the others.

In the same way, we have

$$\binom{0}{1} = -\frac{a}{\phi(\rho+1)} + a^2 M_{1,2} + a^4 M_{1,4} + \dots,$$

where

$$a^2 M_{1,2} = a^2 M_{0,2} + \frac{a^2}{\phi(\rho+1)\,\phi(\rho+2)};$$

and so for the others.

Lastly, for negative values of p less than -1 and for positive values greater than $+1$, we have

$$\binom{0}{p} = a^2 M_{p,2} + a^4 M_{p,4} + \dots,$$

where

$$M_{p,2} = M_{0,2} + \frac{1}{\phi(\rho+p)\,\phi(\rho+p+1)} + \frac{1}{\phi(\rho+p)\,\phi(\rho+p-1)};$$

and so for the others.

After the remarks made in relation to the formal development of $\Omega(\rho)$, it is manifest that these expressions for the integrals are mainly useful for approximate numerical expansions: they cannot at present be held to constitute a complete formulation of the integrals.

Ex. 2. In his classical memoir, already quoted, Hill considers the equation

$$\frac{1}{w}\frac{d^2w}{dt^2} = a_0 + a_1 \cos 2t + a_2 \cos 4t + \dots,$$

the coefficients a_1, a_2, ... being considerably smaller than a_0. The memoir will well repay perusal, both for the analysis (account being taken of the lacuna as to convergence supplied by Poincaré), and for the numerical approximations.

It will be noticed that the effectiveness of the method is largely influenced by the data as to the smallness of a_1, a_2, ..., when compared with a_0.

Ex. 3. Discuss the equation in Ex. 1, when $b=\frac{1}{4}$, so that $\rho_1=\rho_2$.

Ex. 4. Given an infinite system of differential equations of the form

$$\frac{dx_m}{dt} = \sum_{n=1}^{\infty} a_{m,n} x_n, \qquad (m=1, 2, \dots, \infty),$$

where the coefficients $a_{m,n}$ are regular functions of t within a region $|t| \leqslant R$, such that $|a_{m,n}| < S_m A_n$ in this region, where S_m, A_n (for m, $n=1, \dots, \infty$) are such that the series $S_1A_1 + S_2A_2 + \dots + S_nA_n + \dots$ converges. Shew that, if a set of constants c_1, c_2, ... be chosen, so that the series

$$c_1A_1 + c_2A_2 + \dots + c_nA_n + \dots$$

converges absolutely, then a system of integrals of the equations is uniquely determined by the condition that $x_m = c_m$, when $t=0$, for all values of m.

(von Koch.)

Other Modes of constructing the Fundamental Equation for Irregular Integrals.

127. The preceding method, so far as it is completed, leads to the determination of the fundamental equation for a closed circuit round the origin, the circuit lying entirely in the annulus;

and it leads also to the determination of the integrals. Other methods have been proposed by Fuchs*, Hamburger†, Poincaré‡, and Mittag-Leffler§, some of them referring solely to the construction of the fundamental equation. But all of them seem less direct than the preceding method, due to Hill and von Koch; and they are not less devoid of difficulties in the construction of the complete formal expression of the integrals.

Ex. 1. A modification of Hamburger's method, applied to the equation

$$\frac{d^2y}{dx^2}+\left(\frac{a}{x^3}+\frac{b}{x^2}+\frac{a}{x}\right)y=0,$$

already discussed in Ex. 1, § 126, may give some indication of his process. Changing the variable from x to t, where

$$x=e^{it},$$

we have the equation in the form

$$\frac{d^2y}{dt^2}-i\frac{dy}{dt}=y(b+2a\cos t),$$

or writing

$$ye^{-\frac{1}{2}it}=Y,$$

the equation|| for Y is

$$\frac{d^2Y}{dt^2}=Y(c+2a\cos t),$$

where

$$c=b-\tfrac{1}{4}.$$

Let x describe a circle round the origin, say of radius unity; then on the completion of the circle, t has increased its value by 2π.

Let $y=f(x)$, $y=g(x)$ be two linearly independent integrals; and when x describes its circle, let these become $[f(x)]$, $[g(x)]$, respectively, so that we have

$$[f(x)]=a_{11}f(x)+a_{12}g(x),$$
$$[g(x)]=a_{21}f(x)+a_{22}g(x).$$

The fundamental equation for the circuit is

$$\begin{vmatrix} a_{11}-\omega\ , & a_{12} \\ a_{21}, & a_{22}-\omega \end{vmatrix}=0,$$

* *Crelle*, t. LXXV (1873), pp. 177—223.

† *Crelle*, t. LXXXIII (1877), pp. 185—209. In connection with this memoir, reference should be made to two papers by Günther, *Crelle*, t. CVI (1890), pp. 330—336, *ib.*, t. CVII (1891), pp. 298—318.

‡ *Acta Math.*, t. IV (1884), pp. 201—312. In connection with this memoir, reference should be made to Vogt, *Ann. de l'Éc. Norm.*, Sér. 3^e^, t. VI (1889), Suppl., pp. 3—71.

§ *Acta Math.*, t. XV (1891), pp. 1—32.

|| In this form, it is a special case of Hill's equation: see Ex. 2, § 126.

that is, by Poincaré's theorem (§ 14),

$$\omega^2-(a_{11}+a_{22})\,\omega+1=0,$$

so that $a_{11}+a_{22}$ is the one invariant for the circuit.

Let

$$F(t)=e^{-\frac{1}{2}it}f(e^{it}),\quad G(t)=e^{-\frac{1}{2}it}g(e^{it});$$

then

$$\left.\begin{aligned}F(t+2\pi)&=-a_{11}F(t)-a_{12}G(t)\\ G(t+2\pi)&=-a_{21}F(t)-a_{22}G(t)\end{aligned}\right\}.$$

The fundamental equation is independent of the choice of the linearly independent system, and it is unchanged when any particular selection is made. Accordingly, let the integrals be chosen so that

$$F(t)=1,\quad F'(t)=0,\quad G(t)=0,\quad G'(t)=1,$$

when $t=0$; then, using the foregoing equations, we have

$$F(2\pi)=-a_{11},\quad G'(2\pi)=-a_{22};$$

and therefore

$$a_{11}+a_{22}=-F(2\pi)-G'(2\pi),$$

which accordingly gives the value of the invariant, when the values of $F(2\pi)$ and $G'(2\pi)$ are known.

To obtain these, let

$$u=\sin^2\tfrac{1}{4}t,$$

so that u increases from 0 to 1, as t increases from 0 to 2π. The equation becomes

$$u(1-u)\frac{d^2Y}{du^2}+\tfrac{1}{2}(1-2u)\frac{dY}{du}=4Y(c+2a-16au+16au^2);$$

and this remains unaltered when we change u into $1-u$. Two linearly independent integrals, constituting a fundamental system in the vicinity of $u=0$, are given by

$$Y_1=\sum_{n=0}^{\infty}a_nu^n,\quad Y_2=\sum_{n=0}^{\infty}c_nu^{n+\frac{1}{2}},$$

where $a_0=1$, $c_0=1$; also a_n is the value of b_n when $\rho=0$, and c_n is the value of b_n when $\rho=\frac{1}{2}$, the quantities b_n being given by the equations

$$\begin{aligned}b_0&=1,\\ (\rho+1)(\rho+\tfrac{1}{2})\,b_1&=\rho^2+4c+8a,\\ (\rho+2)(\rho+\tfrac{3}{2})\,b_2&=\{(\rho+1)^2+4c+8a\}\,b_1-64a,\end{aligned}$$

and, for values of $n\geqslant 3$,

$$(n+\rho)(n+\rho-\tfrac{1}{2})\,b_n=\{(n+\rho-1)^2+4c+8a\}\,b_{n-1}-64ab_{n-2}+64ab_{n-3}.$$

Similarly, a fundamental system in the vicinity of $u=1$ is given by

$$Z_1=\sum_{n=0}^{\infty}a_n(1-u)^n,\quad Z_2=\sum_{n=0}^{\infty}c_n(1-u)^{n+\frac{1}{2}}.$$

The integral $F(t)$, defined by the initial conditions

$$F(t)=1, \quad F'(t)=0,$$

when $t=0$, is given by

$$F(t)=Y_1.$$

The integral $G(t)$, defined by the initial conditions

$$G(t)=0, \quad G'(t)=1,$$

when $t=0$, is given by

$$G(t)=4Y_2.$$

To obtain expressions for $F(2\pi)$, $G'(2\pi)$, consider values of u, which lie in the vicinity of $u=1$ and are less than 1. By the ordinary theory of linear equations, we have

$$Y_1=AZ_1+BZ_2, \quad Y_2=CZ_1+DZ_2.$$

First, let $u=\tfrac{1}{2}$, so that $1-u=\tfrac{1}{2}$; then we have

$$F(\pi)=AF(\pi)+\tfrac{1}{4}BG(\pi), \quad G(\pi)=4CF(\pi)+DG(\pi).$$

Next, differentiate with regard to u, and then take $u=\tfrac{1}{2}$, $1-u=\tfrac{1}{2}$; we have

$$F'(\pi)=-AF'(\pi)-\tfrac{1}{4}BG'(\pi), \quad G'(\pi)=-4CF'(\pi)-DG'(\pi).$$

Moreover,

$$\begin{aligned} F(t)\,G'(t)-F'(t)\,G(t)&=\text{constant}\\ &=1, \end{aligned}$$

by taking the initial values; hence

$$F(\pi)\,G'(\pi)-F'(\pi)\,G(\pi)=1.$$

These relations give

$$\begin{gathered} A=F(\pi)\,G'(\pi)+F'(\pi)\,G(\pi)=-D,\\ \tfrac{1}{4}B=-2F(\pi)\,F'(\pi), \quad 4C=2G(\pi)\,G'(\pi). \end{gathered}$$

Now

$$\begin{aligned} G(2\pi-\tau)&=4Y_2(2\pi-\tau)\\ &=\{4CZ_1(\tau)+DZ_2(\tau)\}, \end{aligned}$$

so that

$$-G'(2\pi)=4\left[C\frac{dZ_1}{d\tau}+D\frac{dZ_2}{d\tau}\right]_{\tau=0}=D;$$

and

$$\begin{aligned} F(2\pi-\tau)&=Y_1(2\pi-\tau)\\ &=AZ_1(\tau)+BZ_2(\tau), \end{aligned}$$

so that

$$F(2\pi)=A.$$

Hence

$$\begin{aligned} F(2\pi)+G'(2\pi)&=A-D\\ &=2F(\pi)\,G'(\pi)+2F'(\pi)\,G(\pi). \end{aligned}$$

Now, when t is π, the value of u is $\frac{1}{2}$, so that

$$F(\pi)=\sum_{n=0}^{\infty}\frac{a_n}{2^n},$$

$$F'(\pi)=\sum_{n=0}^{\infty}\frac{na_n}{2^{n+1}};$$

$$G(\pi)=2^{-\frac{1}{2}}\sum_{n=0}^{\infty}\frac{c_n}{2^n},$$

$$G'(\pi)=2^{-\frac{1}{2}}\sum_{n=0}^{\infty}\frac{(n+\frac{1}{2})\,c_n}{2^{n+1}};$$

and therefore

$$\begin{aligned} a_{11}+a_{22} &= -F(2\pi)-G'(2\pi) \\ &= -2^{\frac{1}{2}}\left[\sum_{n=0}^{\infty}\frac{a_n}{2^n}\sum_{m=0}^{\infty}\frac{(m+\frac{1}{2})\,c_m}{2^{m+1}}+\sum_{n=0}^{\infty}\frac{na_n}{2^{n+1}}\sum_{m=0}^{\infty}\frac{c_m}{2^m}\right] \\ &= -2^{\frac{1}{2}}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(m+n+\frac{1}{2})\,a_n c_m}{2^{m+n+1}}, \end{aligned}$$

which is the invariant of the fundamental equation. This gives a formal expression, the only operations required being in the direct construction of a_n and c_m, and no one of these operations is inverse; but the result is less suited to numerical approximation than is the method of infinite determinants in the case when a is small.

We shall return later (§§ 137—139) to a different discussion of this equation.

Ex. 2. Apply the preceding method to Hill's equation

$$\frac{1}{w}\frac{d^2w}{dt^2}=a_0+a_1\cos 2t+a_2\cos 4t+\dots,$$

in the case when $a_1, a_2, \dots$ are not small compared with a_0.

Ex. 3. Discuss, in the same manner as in Ex. 1, the equation

$$\frac{d^3y}{dz^3}+\left(\frac{a}{z^4}+\frac{\beta}{z^3}+\frac{\gamma}{z^2}\right)y=0.$$

In particular, obtain expressions for the invariants of the fundamental equation for $z=0$.

CHAPTER IX.

EQUATIONS WITH UNIFORM PERIODIC COEFFICIENTS.

128. ALL the equations which hitherto have been considered have had uniform functions of the variable for the coefficients of the derivatives; and the only particular class of uniform functions, that has been specially adopted with a view to detailed discussion of the properties of the equation, is constituted by those which are rational. Many of the properties, however, which have been established in the preceding chapters, hold for uniform functions whose form, in the vicinity of a singularity, is similar to that of a rational function when expressed as a power-series in such a vicinity. Among the classes of uniform functions, other than rational functions, there are two characterised by a set of specific properties: viz. simply-periodic functions, and doubly-periodic functions; and accordingly, it seems desirable to consider equations having coefficients of this type. The present chapter will be devoted to the discussion of equations the coefficients in which are uniform periodic functions.

EQUATIONS WITH SIMPLY-PERIODIC COEFFICIENTS.

We begin with the case in which the coefficients have only a single period; and we take the equation in the form

$$\frac{d^m w}{dz^m} + p_1 \frac{d^{m-1} w}{dz^{m-1}} + \dots + p_m w = 0,$$

where $p_1, \dots, p_m$ are uniform functions of z, are periodic in ω, and have no essential singularity for finite values of z. Let a

fundamental system of integrals in the domain of any point be denoted by

$$f_1(z),\ f_2(z),\ \ldots,\ f_m(z),$$

which therefore are linearly independent. A change of z into $z+\omega$ leaves the differential equation unaltered: hence

$$f_1(z+\omega),\ f_2(z+\omega),\ \ldots,\ f_m(z+\omega)$$

are integrals of the equation. That they are linearly independent, and therefore constitute a fundamental system (it may be in a new domain), is easily seen; for

$$\Sigma c_r f_r(z+\omega)$$

satisfies the equation for all values of z, and by making z pass from any position $Z+\omega$ to Z without meeting any singularity, the integral changes from $\Sigma c_r f_r(Z+\omega)$ to $\Sigma c_r f_r(Z)$. If, then, values of c could be found such that the equation

$$\Sigma c_r f_r(z+\omega)=0$$

is satisfied identically (and not merely for special zeros of the function on the left-hand side), then we should have

$$\Sigma c_r f_r(z)=0,$$

also identically. The latter is impossible, because the integrals $f_1(z), \ldots, f_m(z)$ constitute a fundamental system; and therefore the former is impossible. Thus $f_1(z+\omega), \ldots, f_m(z+\omega)$ constitute a fundamental system.

Suppose now that the domain, in which the original fundamental system exists, and the domain, in which the deduced fundamental system exists, have some region in common that is not infinitesimal; and consider the integrals within this common region. As $f_1(z+\omega), \ldots, f_m(z+\omega)$ are integrals, and as $f_1(z), \ldots, f_m(z)$ are a fundamental system, we have equations of the form

$$\left.\begin{array}{l} f_1(z+\omega)=a_{11}f_1(z)+\ldots+a_{1m}f_m(z) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_m(z+\omega)=a_{m1}f_1(z)+\ldots+a_{mm}f_m(z) \end{array}\right\},$$

where the coefficients a are constants; their determinant is not zero, because the set of integrals on the left-hand side constitutes a fundamental system.

Consider any other integral in this region: it is of the form

$$F(z) = \kappa_1 f_1(z) + \kappa_2 f_2(z) + \ldots + \kappa_m f_m(z),$$

where $\kappa_1, \kappa_2, \ldots, \kappa_m$ are constants; and so

$$F(z+\omega) = \sum_{r=1}^{m} a_{r1}\kappa_r f_1(z) + \sum_{r=1}^{m} a_{r2}\kappa_r f_2(z) + \ldots + \sum_{r=1}^{m} a_{rm}\kappa_r f_m(z).$$

In order that $F(z)$ may be characterised by the property

$$F(z+\omega) = \theta F(z),$$

where θ is some constant, the coefficients κ must be chosen so that

$$\sum_{r=1}^{n} a_{rp}\kappa_r = \theta\kappa_p, \qquad (p = 1, 2, \ldots, m),$$

a set of n equations linear and homogeneous in the coefficients κ; and therefore θ must satisfy the equation

$$A(\theta) = \begin{vmatrix} a_{11} - \theta, & a_{12}, & \ldots, & a_{1m} \\ a_{21}, & a_{22} - \theta, & \ldots, & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}, & a_{m2}, & \ldots, & a_{mm} - \theta \end{vmatrix} = 0,$$

an equation involving the coefficients a, and so apparently depending upon the choice of the fundamental system $f_1, \ldots, f_m$.

But, as with the corresponding equation for a set of integrals near a singularity (§ 14), we prove that *this equation is independent of the choice of the fundamental system, so that the coefficients of the powers of θ are invariants.* The proof follows the lines of Hamburger's proof for the earlier proposition. Let another fundamental system $g_1(z), \ldots, g_m(z)$, existing in the region under consideration, be such that

$$g_r(z+\omega) = b_{r1}g_1(z) + \ldots + b_{rm}g_m(z), \quad (r = 1, \ldots, m),$$

the determinant of the coefficients b being different from zero. The equation, to be satisfied by the multiplier θ of $F(z)$, is

$$B(\theta) = \begin{vmatrix} b_{11} - \theta, & b_{12}, & \ldots, & b_{1m} \\ b_{21}, & b_{22} - \theta, & \ldots, & b_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m1}, & b_{m2}, & \ldots, & b_{mm} - \theta \end{vmatrix} = 0.$$

As the integrals f are a fundamental system in the region, in which the integrals g exist, we have

$$g_s(z) = c_{s1} f_1(z) + \ldots + c_{sm} f_m(z), \qquad (s = 1, \ldots, m),$$

where the determinant of the coefficients c_{st}, say C, is not zero. Thus, as

$$\begin{aligned} b_{r1} g_1(z) + \ldots + b_{rm} g_m &= g_r(z+\omega) \\ &= c_{r1} f_1(z+\omega) + \ldots + c_{rm} f_m(z+\omega), \end{aligned}$$

we have

$$\sum_{s=1}^{m} \sum_{t=1}^{m} b_{rs} c_{st} f_t(z) = \sum_{s=1}^{m} \sum_{t=1}^{m} c_{rs} a_{st} f_t(z).$$

This homogeneous linear relation among the linearly independent integrals f must be an identity; and therefore

$$\begin{aligned} \sum_{s=1}^{m} b_{rs} c_{st} &= \sum_{s=1}^{m} c_{rs} a_{st} \\ &= \alpha_{rt}, \end{aligned}$$

say. Then

$$CB(\theta) = \begin{vmatrix} \alpha_{11} - c_{11}\theta, & \alpha_{12} - c_{12}\theta, & \ldots \\ \alpha_{21} - c_{21}\theta, & \alpha_{22} - c_{22}\theta, & \ldots \\ \cdots & \cdots & \cdots \end{vmatrix} = A(\theta) C,$$

so that, as C is not zero, we have

$$B(\theta) = A(\theta),$$

and the equation is invariantive. We therefore call it *the fundamental equation for the period* ω.

Let $\Delta(z)$ denote the determinant

$$\Delta(z) = \begin{vmatrix} \dfrac{d^{m-1} f_1}{dz^{m-1}}, & \dfrac{d^{m-1} f_2}{dz^{m-1}}, & \ldots, & \dfrac{d^{m-1} f_m}{dz^{m-1}} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{df_1}{dz}, & \dfrac{df_2}{dz}, & \ldots, & \dfrac{df_m}{dz} \\ f_1, & f_2, & \ldots, & f_m \end{vmatrix};$$

then, as in § 9, we have

$$\Delta(z) = \Delta(z_0)\, e^{\int_{z_0}^{z} p_1(x)\,dx}.$$

Hence

$$\Delta(z+\omega) = \Delta(z_0)\, e^{\int_{z_0}^{z+\omega} p_1(x)\,dx},$$

so that

$$\frac{\Delta(z+\omega)}{\Delta(z)} = e^{\int_z^{z+\omega} p_1(x)\,dx},$$

where we may assume the integration to take place along a path that does not approach infinitesimally near the singularities of p_1, if any. Now, as p_1 is a uniform function, simply-periodic in ω, it is known* that p_1 is expressible in the form

$$p_1(z) = \sum_{-\infty}^{\alpha=\infty} A_\alpha e^{\frac{2\pi zi}{\omega}\alpha},$$

within such a region as encloses the path of integration; and the series is a converging series. But

$$\int_z^{z+\omega} e^{\frac{2\pi xi\alpha}{\omega}}\,dx = 0,$$

if the integer α is distinct from zero; hence

$$\frac{\Delta(z+\omega)}{\Delta(z)} = e^{\omega A_0}.$$

But, substituting in $\Delta(z+\omega)$ the expressions for $f_1(z+\omega), \ldots, f_m(z+\omega)$ and their derivatives, in terms of $f_1(z), \ldots, f_m(z)$ and their derivatives, we have

$$\frac{\Delta(z+\omega)}{\Delta(z)} = \begin{vmatrix} a_{11}, & a_{12}, & \ldots, & a_{1m} \\ a_{21}, & a_{22}, & \ldots, & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}, & a_{m2}, & \ldots, & a_{mm} \end{vmatrix},$$

which is the non-vanishing constant term in $A(\theta)$; and thus

$$A(\theta) = e^{\omega A_0} + \ldots + (-1)^m \theta^m.$$

In particular, when p_1 is zero, so that the differential equation contains no term in $\dfrac{d^{m-1}w}{dz^{m-1}}$, we have $A_0 = 0$; and then

$$A(\theta) = 1 + \ldots + (-1)^m \theta^m.$$

129. The generic character of the integrals depends upon the nature of the roots of the fundamental equation.

* *T. F.*, § 112.

If the m roots of the fundamental equation are different from one another, and if they are denoted by $\theta_1, \theta_2, \ldots, \theta_m$, then a fundamental system of integrals exists, such that

$$F_r(z+\omega) = \theta_r F_r(z), \qquad (r = 1, \ldots, m).$$

Consider any simple root θ_r of the equation $A(\theta) = 0$. Then not all the minors of $A(\theta)$ of the first order can vanish for $\theta = \theta_r$; hence $m-1$ of the equations

$$\sum_{s=1}^{m} a_{sp}\kappa_s = \theta\kappa_p, \qquad (p = 1, 2, \ldots, m),$$

determine ratios of the m quantities κ, and consequently determine a function $F_r(z)$ having a multiplier θ_r. This holds for each of the m different roots: and thus m different functions $F(z)$ are determined.

These m functions are linearly independent of one another. If there were an equation

$$\gamma_1 F_1(z) + \gamma_2 F_2(z) + \ldots + \gamma_m F_m(z) = 0,$$

which is satisfied identically, then also

$$\gamma_1 F_1(z+\omega) + \gamma_2 F_2(z+\omega) + \ldots + \gamma_m F_m(z+\omega) = 0,$$

that is,

$$\theta_1\gamma_1 F_1(z) + \theta_2\gamma_2 F_2(z) + \ldots + \theta_m\gamma_m F_m(z) = 0.$$

Similarly,

$$\theta_1^2\gamma_1 F_1(z) + \theta_2^2\gamma_2 F_2(z) + \ldots + \theta_m^2\gamma_m F_m(z) = 0;$$

and so on, up to

$$\theta_1^{m-1}\gamma_1 F_1(z) + \theta_2^{m-1}\gamma_2 F_2(z) + \ldots + \theta_m^{m-1}\gamma_m F_m(z) = 0.$$

Now the determinant

$$|\theta_1^0, \theta_2^1, \theta_3^2, \ldots, \theta_m^{m-1}|$$

does not vanish, because the quantities θ are unequal: hence

$$\gamma_1 F_1(z) = 0, \quad \gamma_2 F_2(z) = 0, \quad \ldots, \quad \gamma_m F_m(z) = 0,$$

so that the constants γ all vanish. The m functions F therefore constitute a fundamental system.

130. Next, let ϑ be a root of $A(\theta) = 0$ of multiplicity μ, where $\mu > 1$. The equations

$$\sum_{s=1}^{m} a_{sp}\kappa_s = \theta\kappa_p, \qquad (p = 1, \ldots, m),$$

are consistent with one another, though not necessarily independent of one another: any $m-1$ of them are satisfied by ratios of the quantities κ, which are finite and may contain arbitrary elements. Giving any particular values to the last, we have an integral, say $\Phi_1(z)$, defined by means of these quantities: it is such that

$$\Phi_1(z+\omega) = \vartheta \Phi_1(z),$$

and it is a linear combination of $f_1(z), \ldots, f_m(z)$. Taking any one of the integrals which occur in the expression of this linear combination, say $f_1(z)$, we modify the fundamental system so as to replace $f_1(z)$ by $\Phi_1(z)$. Let the equations for the increase of the argument by ω in the modified fundamental system be

$$\begin{aligned} &\Phi_1(z+\omega) = \vartheta \Phi_1(z), \\ &f_r(z+\omega) = c_{r1}\Phi_1(z) + c_{r2} f_2(z) + \ldots + c_{rm} f_m(z), \quad (r = 2, \ldots, m); \end{aligned}$$

then the fundamental equation is

$$\begin{vmatrix} \vartheta - \theta, & 0, & 0, & \ldots \\ c_{21}, & c_{22}-\theta, & c_{23}, & \ldots \\ c_{31}, & c_{32}, & c_{33}-\theta, & \ldots \\ \multicolumn{4}{c}{\dotfill} \end{vmatrix} = 0,$$

which, owing to its invariantive character, is $A(\theta) = 0$, and therefore has ϑ for a root of multiplicity μ. Consequently, the equation

$$A_1(\theta) = \begin{vmatrix} c_{22}-\theta, & c_{23}, & \ldots \\ c_{32}, & c_{33}-\theta, & \ldots \\ \multicolumn{3}{c}{\dotfill} \end{vmatrix} = 0$$

has ϑ for a root of multiplicity $\mu - 1$; and therefore the equations

$$\begin{aligned} (c_{22} - \vartheta)\kappa_2' + c_{23}\kappa_3' + \ldots + c_{2m}\kappa_m' &= 0, \\ &\ldots\ldots\ldots\ldots \\ c_{m2}\kappa_2' + c_{m3}\kappa_3' + \ldots + (c_{mm} - \vartheta)\kappa_m' &= 0, \end{aligned}$$

are consistent with one another, and any $m-2$ are satisfied by ratios of the quantities κ', which are finite and may contain arbitrary elements. Giving any particular values to the latter, and writing

$$\Phi_2(z) = \kappa_2' f_2(z) + \kappa_3' f_3(z) + \ldots + \kappa_m' f_m(z),$$

we have

$$\Phi_2(z+\omega) = \lambda_{21}\Phi_1(z) + \vartheta\Phi_2(z),$$

where

$$\lambda_{21} = \sum_{r=2}^{m} \kappa_r' c_{r1},$$

so that λ_{21} is a constant, which may be zero. The quantity $\Phi_2(z)$ is an integral of the differential equation: we use it to replace some one of the integrals in its expression, say $f_2(z)$, in the fundamental system, so that the latter then is constituted by $\Phi_1(z)$, $\Phi_2(z)$, $f_3(z)$, ..., $f_m(z)$.

Proceeding similarly from stage to stage, we infer that, associated with a root ϑ of multiplicity μ of the fundamental equation, there exists a set of μ integrals such that

$$\begin{aligned}
&\Phi_1(z+\omega) = \vartheta\Phi_1(z),\\
&\Phi_2(z+\omega) = \lambda_{21}\Phi_1(z) + \vartheta\Phi_2(z),\\
&\Phi_3(z+\omega) = \lambda_{31}\Phi_1(z) + \lambda_{32}\Phi_2(z) + \vartheta\Phi_3(z),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\Phi_\mu(z+\omega) = \lambda_{\mu1}\Phi_1(z) + \lambda_{\mu2}\Phi_2(z) + \ldots + \lambda_{\mu,\mu-1}\Phi_{\mu-1}(z) + \vartheta\Phi_\mu(z),
\end{aligned}$$

where the coefficients λ are constants.

Similarly, if the roots of the equation $A(\theta) = 0$ are $\vartheta_1, \ldots, \vartheta_n$ of multiplicities $\mu_1, \ldots, \mu_n$ respectively, so that $\mu_1 + \ldots + \mu_n = m$, the fundamental system can be chosen so that it arranges itself in n sets, each set being associated with one root of the fundamental equation and having properties of the same nature as the set associated with the preceding root of multiplicity ϑ.

A function, characterised by the property

$$F(z+\omega) = F(z),$$

is strictly periodic, and sometimes it is said to be periodic of the first kind. A function, characterised by the property

$$F(z+\omega) = \theta F(z),$$

where θ is a constant different from unity, is pseudo-periodic, and sometimes it is said to be periodic of the second kind, θ being called its multiplier. A function, characterised by the property

$$F(z+\omega) = e^{\lambda z+\mu} F(z),$$

where λ and μ are constants, is also pseudo-periodic, and sometimes it is said to be periodic of the third kind.

With these definitions, the preceding result can be enunciated as follows*:—

A linear differential equation, the coefficients of which are simply-periodic in a period ω, possesses integrals which are periodic of the second kind: and the number of such integrals is at least as great as the number of distinct roots of the fundamental equation for the period.

Ex. 1. Prove that, if the equation

$$\frac{d^2w}{dz^2}+p_1(z)\frac{dw}{dz}+p_2(z)\,w=0$$

possesses an integral which is periodic of the third kind with a multiplier $e^{\lambda z+\mu}$, then

$$p_1(z+\omega)=p_1(z)-2\lambda,$$
$$p_2(z+\omega)=p_2(z)-\lambda p_1(z)+\lambda^2.$$

Hence integrate the equation

$$\frac{d^2w}{dz^2}-\frac{8\pi^2z}{\omega^2}\frac{dw}{dz}+\frac{16\pi^4}{\omega^4}z^2w=0,$$

shewing that $\lambda\omega=4\pi^2$. (Craig.)

Ex. 2. Shew that, if the coefficients in the equation

$$\frac{d^2w}{dz^2}+p_1(z)\frac{dw}{dz}+p_2(z)\,w=0$$

have the form

$$p_1(z)=\phi(z)+\frac{2\lambda z}{\omega},$$

$$p_2(z)=\psi(z)+\frac{\lambda z}{\omega}\phi(z)+\frac{\lambda^2}{\omega^2}z^2,$$

where ϕ and ψ are periodic of the first kind, then the equation certainly possesses one integral that is periodic of the third kind. (Craig.)

131. On the basis of these properties, we can take one step towards the analytical expression of the integrals.

The integral $\Phi_1(z)$ is a periodic function of the second kind.

As regards the integral $\Phi_2(z)$, we have

$$\frac{\Phi_2(z+\omega)}{\Phi_1(z+\omega)}=\frac{\Phi_2(z)}{\Phi_1(z)}+\frac{\lambda_{21}}{\vartheta},$$

so that

$$\frac{\Phi_2(z+\omega)}{\Phi_1(z+\omega)}-\frac{\lambda_{21}}{\vartheta\omega}(z+\omega)=\frac{\Phi_2(z)}{\Phi_1(z)}-\frac{\lambda_{21}}{\vartheta\omega}z,$$

* Floquet, *Ann. de l'Éc. Norm.*, Sér. 2ᵉ, t. XII (1883), p. 55.

so that the function on the right-hand side is a periodic function of the first kind, say $\psi(z)$. Therefore

$$\Phi_2(z) = \Phi_{21}(z) + z\Phi_{22}(z),$$

where $\Phi_{22}(z)$ is a constant multiple of $\Phi_1(z)$, and the constant factor may be zero; and $\Phi_{21}(z)$, $=\psi(z)\,\Phi_1(z)$, is a periodic function of the second kind, with the same multiplier as $\Phi_1(z)$.

As regards the integral $\Phi_3(z)$, we have

$$\begin{aligned}\frac{\Phi_3(z+\omega)}{\Phi_1(z+\omega)} &= \frac{\Phi_3(z)}{\Phi_1(z)} + \frac{\lambda_{32}}{\vartheta}\frac{\Phi_2(z)}{\Phi_1(z)} + \frac{\lambda_{31}}{\vartheta} \\ &= \frac{\Phi_3(z)}{\Phi_1(z)} + \frac{\lambda_{32}}{\vartheta}\left\{\frac{\lambda_{21}}{\vartheta\omega}z + \psi(z)\right\} + \frac{\lambda_{31}}{\vartheta}.\end{aligned}$$

Now, if

$$\theta(z) = \frac{\Phi_3(z)}{\Phi_1(z)} - \frac{\lambda_{32}}{\vartheta\omega}z\psi(z) - \frac{\lambda_{32}\lambda_{21}}{2\vartheta^2\omega^2}z^2 + \frac{\lambda_{32}\lambda_{21} - 2\vartheta\lambda_{31}}{2\vartheta^2\omega}z,$$

we have

$$\theta(z+\omega) = \theta(z),$$

so that $\theta(z)$ is periodic of the first kind. Hence

$$\Phi_3(z) = \Phi_{31}(z) + z\Phi_{32}(z) + z^2\Phi_{33},$$

where $\Phi_{31}(z) = \theta(z)\,\Phi_1(z)$, and therefore is a periodic function of the second kind with the same multiplier as Φ_1; where $\Phi_{32}(z)$ is a linear combination of $\Phi_{21}(z)$ and $\Phi_1(z)$, and thus is periodic of the second kind with the same multiplier as $\Phi_1(z)$; and $\Phi_{33}(z)$ is a constant multiple of $\Phi_1(z)$, in which the constant factor, viz.

$$-\frac{\lambda_{32}\lambda_{21}}{2\vartheta^2\omega^2},$$

may be zero, and certainly is zero if $\Phi_{22}(z)$ disappears from $\Phi_2(z)$ owing to the vanishing of its constant factor.

Proceeding in this way stage by stage, we obtain expressions for the integrals in succession; and we find

$$\Phi_r(z) = \Phi_{r1}(z) + z\Phi_{r2}(z) + z^2\Phi_{r3}(z) + \ldots + z^{r-1}\Phi_{rr}(z),$$

where

$$\Phi_{rr}(z) = \frac{(-1)^r\lambda_{r,r-1}\lambda_{r-1,r-2}\ldots\lambda_{32}\lambda_{21}}{(r-1)!\,\vartheta^r\omega^r}\Phi_1(z),$$

so that it is a constant multiple of $\Phi_1(z)$, the constant factor being capable of vanishing; and all the functions $\Phi_{r1}(z)$, $\Phi_{r2}(z)$, ..., $\Phi_{r,r-1}(z)$ are periodic functions of the second kind with the

same multiplier as $\Phi_1(z)$, and are expressible as linear combinations of $\Phi_1, \Phi_{21}, \Phi_{31}, \ldots, \Phi_{r-1,1}$. This holds for the values $r = 1, 2, \ldots, \mu$.

Similarly for any other set of integrals, associated with any multiple root of the fundamental equation of the period.

It may, however, happen that some one of the coefficients $\lambda_{s,s-1}$ vanishes, so that, for all values of $r \geqslant s$, the term in $\Phi_{rr}(z)$ disappears. The alternative result is that a linear combination of the functions $\Phi_s(z), \Phi_{s-1}(z), \ldots, \Phi_1(z)$ can be constructed which is periodic of the second kind. This linear combination can be used to replace $\Phi_s(z)$, and thus may be the initial member of another set of integrals in the group associated with the multiplier ϑ. The proof of this statement is simple. Assume that $\lambda_{s,s-1}$ vanishes, and that no one of the coefficients $\lambda_{r,r-1}$ for values of $r \leqslant s$ vanishes; and construct the linear combination

$$\kappa_s\Phi_s(z+\omega) + \kappa_{s-1}\Phi_{s-1}(z+\omega) + \ldots + \kappa_2\Phi_2(z+\omega),$$

choosing the coefficients κ so that the term in Φ_1 disappears and that the remaining terms are

$$\vartheta\left\{\kappa_s\Phi_s(z) + \kappa_{s-1}\Phi_{s-1}(z) + \ldots + \kappa_2\Phi_2(z)\right\}.$$

To satisfy these conditions, we must have

$$\begin{aligned}
0 &= \kappa_s\lambda_{s1} + \kappa_{s-1}\lambda_{s-1,1} + \ldots + \kappa_4\lambda_{41} + \kappa_3\lambda_{31} + \kappa_2\lambda_{21},\\
0 &= \kappa_s\lambda_{s2} + \kappa_{s-1}\lambda_{s-1,2} + \ldots + \kappa_4\lambda_{42} + \kappa_3\lambda_{32},\\
0 &= \kappa_s\lambda_{s3} + \kappa_{s-1}\lambda_{s-1,3} + \ldots + \kappa_4\lambda_{43},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
0 &= \kappa_s\lambda_{s,s-2} + \kappa_{s-1}\lambda_{s-1,s-2}.
\end{aligned}$$

Transfer the terms in κ_s to the left-hand side: the determinant of the coefficients κ on the remaining right-hand side is

$$\pm\lambda_{s-1,s-2}\lambda_{s-2,s-3}\ldots\lambda_{43}\lambda_{32}\lambda_{21},$$

which by the initial hypothesis does not vanish. Some of the coefficients $\lambda_{s1}, \lambda_{s2}, \ldots, \lambda_{s,s-2}$ are different from zero, for $\Phi_s(z)$ is not a periodic function of the second kind; hence there are finite non-zero values for the ratios of $\kappa_{s-1}, \ldots, \kappa_2$ to κ_s. When these values are inserted, let

$$\Psi_s(z) = \kappa_s\Phi_s(z) + \ldots + \kappa_2\Phi_2(z);$$

then

$$\Psi_s(z+\omega) = \vartheta\Psi_s(z),$$

so that $\Psi_s(z)$ is periodic of the second kind, with the common multiplier ϑ; it can replace $\Phi_s(z)$ in the fundamental system, and then can, like $\Phi_1(z)$, be the initial member of another set within the group of the same type as $\Phi_1(z), \ldots, \Phi_{s-1}(z)$. The statement is thus proved.

132. Any set, such as $\Phi_1(z), \ldots, \Phi_{s-1}(z)$ in the preceding group of integrals, whether $s=\mu$ or be less than μ, can be replaced by an equivalent set of simpler form.

Let the equation be written

$$0 = Pw = \left(\frac{d^m}{dz^m} + p_1 \frac{d^{m-1}}{dz^{m-1}} + \ldots + p_m\right) w$$
$$= (D^m + p_1 D^{m-1} + \ldots + p_m) w,$$

so that

$$P = D^m + p_1 D^{m-1} + \ldots + p_{m-1} D + p_m.$$

Also let

$$P_1 = mD^{m-1} + (m-1) p_1 D^{m-2} + \ldots + p_{m-1}$$
$$= \frac{\partial P}{\partial D},$$
$$P_2 = \frac{\partial^2 P}{\partial D^2};$$

and, generally, let

$$P_s = \frac{\partial^s P}{\partial D^s}.$$

Let the integral of the set containing the highest power of z, say z^{r-1}, be expressed in the form

$$w = z^{r-1}\phi_1 + (r-1) z^{r-2}\phi_2 + \ldots + \frac{(r-1)!}{(\mu-1)!\,(r-\mu)!} z^{r-\mu}\phi_\mu + \ldots$$
$$\ldots + (r-1) z\phi_{r-1} + \phi_r,$$

the binomial factors being inserted for simplicity. Then, as

$$P(z^\kappa\psi) = z^\kappa P(\psi) + \kappa z^{\kappa-1} P_1(\psi) + \kappa(\kappa-1) z^{\kappa-2} P_2(\psi) + \ldots,$$

we have

$$\begin{aligned}0 &= P(w)\\ &= z^{r-1}P(\phi_1) + (r-1) z^{r-2} P_1(\phi_1) + \tfrac{1}{2}(r-1)(r-2) z^{r-3} P_2(\phi_1) + \ldots\\ &\qquad + (r-1)\{z^{r-2}P(\phi_2) + (r-2) z^{r-3} P_1(\phi_2) + \ldots\\ &\qquad + \tfrac{1}{2}(r-1)(r-2)\{z^{r-3}P(\phi_3) + \ldots\\ &\qquad + \ldots\ldots\ldots\ldots\ldots\ldots,\end{aligned}$$

which can be satisfied identically, only if

$$\begin{aligned} 0 &= P(\phi_1), \\ 0 &= P(\phi_2) + P_1(\phi_1), \\ 0 &= P(\phi_3) + 2P_1(\phi_2) + P_2(\phi_1), \\ &\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

The first of these conditions shews that

$$w = \phi_1$$

is an integral of the equation. The second shews that

$$w = \phi_2 + z\phi_1$$

is an integral; the third that

$$w = \phi_3 + 2z\phi_2 + z^2\phi_1$$

is an integral. And generally, if $\overline{w}$ denote

$$\zeta^{r-1}\phi_1 + (r-1)\zeta^{r-2}\phi_2 + \dots + \frac{(r-1)!}{(\mu-1)!\,(r-\mu)!}\zeta^{r-\mu}\phi_\mu + \dots$$
$$\dots + (r-1)\zeta\phi_{r-1} + \phi_r,$$

$\overline{w}$ being an integral of the equation, then each of the quantities

$$\overline{w},\quad \frac{1}{r-1}\frac{\partial\overline{w}}{\partial\zeta},\quad \frac{2!\,(r-3)!}{(r-1)!}\frac{\partial^2\overline{w}}{\partial\zeta^2},\quad \dots,\quad \frac{(\mu-1)!\,(r-\mu)!}{(r-1)!}\frac{\partial^{r-\mu}\overline{w}}{\partial\zeta^{r-\mu}},\ \dots$$

is an integral of the equation, when ζ is replaced by z after differentiation. Accordingly, the group of r integrals in the set are linearly equivalent to

$$\begin{aligned} u_1 &= \phi_1, \\ u_2 &= \phi_2 + z\phi_1, \\ u_3 &= \phi_3 + 2z\phi_2 + z^2\phi_1, \\ u_4 &= \phi_4 + 3z\phi_3 + 3z^2\phi_2 + z^3\phi_1, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ u_r &= \phi_r + (r-1)z\phi_{r-1} + \dots + (r-1)z^{r-2}\phi_2 + z^{r-1}\phi_1, \end{aligned}$$

and any linear combination of these is an integral of the differential equation; all the quantities ϕ which occur in them are periodic of the second kind, having the same multiplier.

Similarly for any other set; and thus *the m integrals of the equation will be constituted by sets of $r_1, r_2, \dots, r_n$ integrals of the preceding types, where $r_1 + r_2 + \dots + r_n = m$, and the system contains n periodic functions of the second kind.*

Group of Integrals associated with a Multiple Root of the Fundamental Equation of the Period.

133. These results can also be obtained by using the properties of the elementary divisors of the quantity $A(\theta)$, when it is expressed in its determinantal form. Let the elementary divisors associated with the root ϑ be

$$(\theta-\vartheta)^{\mu-\mu_1},\ (\theta-\vartheta)^{\mu_1-\mu_2},\ \ldots,\ (\theta-\vartheta)^{\mu_{\tau-2}-\mu_{\tau-1}},\ (\theta-\vartheta)^{\mu_{\tau-1}},$$

so that, as in § 15, the highest power of $\theta-\vartheta_1$ common to all the first minors of $A(\theta)$ is $(\theta-\vartheta)^{\mu_1}$, the highest power common to all the second minors of $A(\theta)$ is $(\theta-\vartheta)^{\mu_2}$, and so on; and the minors of order τ (and therefore of degree $m-\tau$ in the coefficients) of $A(\theta)$ are the earliest in successively increasing orders not to vanish simultaneously when $\theta=\vartheta$. As in the earlier case discussed in §§ 15, 16, we have

$$\mu-\mu_1 \geqslant \mu_1-\mu_2 \geqslant \mu_2-\mu_3 \geqslant \ldots \geqslant \mu_{\tau-1}.$$

Proceeding on lines precisely similar to those followed in § 23 for the arrangement, in sub-groups, of the group of integrals associated with a multiple root of the fundamental equation belonging to the singularity, we obtain a corresponding result in the present case, as follows:—

The group of μ integrals associated with the root ϑ of multiplicity μ, belonging to the fundamental equation for the period ω, can be arranged in τ sub-groups, where τ is the number of elementary divisors of $A(\theta)$ which are powers of $\theta-\vartheta$. If the λ members of any one of these sub-groups be denoted by $g_1(z)$, $g_2(z)$, …, $g_\lambda(z)$, these integrals of the differential equation satisfy the characteristic equations

$$\left.\begin{array}{l} g_1(z+\omega)=\vartheta g_1(z) \\ g_2(z+\omega)=\vartheta g_2(z)+g_1(z) \\ g_3(z+\omega)=\vartheta g_3(z)+g_2(z) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ g_\lambda(z+\omega)=\vartheta g_\lambda(z)+g_{\lambda-1}(z) \end{array}\right\}.$$

Taking all these sub-groups together, the number of first equations which occur in them is equal to the number of the sub-groups, that is, the number of the elementary divisors of $A(\theta)$ connected

with $\theta - \vartheta$; the number of second equations which occur is the same as the number of those indices of the elementary divisors connected with $\theta - \vartheta$ that are not less than 2; the number of third equations is the same as the number of those indices that are not less than 3; and so on, the number of equations in the first sub-group being $\mu - \mu_1$.

The analogy with the Hamburger sub-groups in Chapter II is complete.

COROLLARY. *The total number of integrals of the second kind, defined as satisfying a relation of the form*

$$g(z+\omega) = \theta g(z),$$

where θ is a constant, is the total number of elementary divisors of $A(\theta)$ associated with all the roots of $A(\theta)=0$; a theorem more exact than Floquet's (§ 130). For the total number of such integrals, in the group associated with a multiple root of $A(\theta)=0$, is equal to the number of elementary divisors of $A(\theta)$ associated with that root: and the total number of groups is equal to the number of distinct roots of $A(\theta)=0$.

134. Some approach to the analytical expressions of the functions, satisfying the equations characteristic of the sub-group, can be made, as in § 23. Let

$$\zeta = \frac{z}{\omega};$$

and introduce a difference-symbol ∇, such that*

$$\nabla F(z) = F(z+\omega) - F(z),$$

for any function F; also let

$$G(z) = \chi_\lambda + \binom{\lambda-1}{1}\zeta\chi_{\lambda-1} + \binom{\lambda-1}{2}\zeta^2\chi_{\lambda-2} + \dots$$
$$\dots + \binom{\lambda-1}{1}\zeta^{\lambda-2}\chi_2 + \zeta^{\lambda-1}\chi_1,$$

where the functions $\chi_1, \chi_2, \dots, \chi_\lambda$ are periodic functions of z, with a period ω, and

$$\binom{\lambda-1}{r} = \frac{(\lambda-1)!}{r!\,(\lambda-1-r)!}.$$

* For these difference-symbols in general, see a memoir by Casorati, *Ann. di Mat.*, Ser. 2ª, t. x (1882), pp. 10—45.

Then if we take

$$h_{\lambda-n}(z) = \vartheta^{\frac{z}{\omega}} \vartheta^n \nabla^n G(z),$$

for all values of n, we have

$$\begin{aligned} h_{\lambda-n}(z+\omega) &= \vartheta^{\frac{z}{\omega}} \vartheta^{n+1} \nabla^n G(z+\omega) \\ &= \vartheta^{\frac{z}{\omega}} \vartheta^{n+1} \nabla^n \{\nabla F(z) + F(z)\} \\ &= h_{\lambda-n-1}(z) + \vartheta h_{\lambda-n}(z), \end{aligned}$$

holding for all values of n. These are the characteristic equations of the sub-group; and we therefore can write

$$g_{\lambda-n}(z) = \vartheta^{\frac{z}{\omega}} \vartheta^n \nabla^n G(z),$$

with the above notations, for $n = 0, 1, \ldots, \lambda - 1$.

These λ integrals are a linearly independent set out of the fundamental system; the system will remain fundamental, if $g_1, g_2, \ldots, g_\lambda$ are replaced by λ other functions, linearly equivalent to them and linearly independent of one another. This modification can be effected in the same way as the corresponding modification was effected in § 24, viz. by introducing a set of functions associated with G and defined by the relations

$$\begin{aligned} G_1(z) &= \chi_1, \\ G_2(z) &= \chi_2 + \chi_1\zeta, \\ G_3(z) &= \chi_3 + 2\chi_2\zeta + \chi_1\zeta^2, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

$$G = G_\lambda(z) = \chi_\lambda + \binom{\lambda-1}{1}\chi_{\lambda-1}\zeta + \ldots + \binom{\lambda-1}{1}\chi_2\zeta^{\lambda-2} + \chi_1\zeta^{\lambda-1},$$

the functions χ being periodic functions of z, with the period ω. Constructing the expressions $\nabla G, \nabla^2 G, \ldots, \nabla^{\lambda-1}G$, we find

$$\begin{aligned} \nabla G &= c_{1,1}G_{\lambda-1} + c_{1,2}G_{\lambda-2} + \ldots + c_{1,\lambda-2}G_2 + c_{1,\lambda-1}G_1, \\ \nabla^2 G &= c_{2,1}G_{\lambda-2} + c_{2,2}G_{\lambda-3} + \ldots + c_{2,\lambda-2}G_1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \nabla^{\lambda-2}G &= c_{\lambda-2,1}G_2 + c_{\lambda-2,2}G_1, \\ \nabla^{\lambda-1}G &= c_{\lambda-1,1}G_1, \end{aligned}$$

where the constants c are non-vanishing numbers, the exact values of which are not needed for the present purpose.

It follows, from the last of the equations, that G_1 is a constant multiple of $\nabla^{\lambda-1}G$, and therefore that $\vartheta^{\frac{z}{\omega}} G_1$ is a constant multiple of $g_1(z)$; we replace $g_1(z)$ in the fundamental system by $\vartheta^{\frac{z}{\omega}} G_1$.

It follows, from the last two of the equations, that G_2 is a linear combination of $\nabla^{\lambda-2}G$ and $\nabla^{\lambda-1}G$, and therefore that $\vartheta^{\frac{z}{\omega}} G_2$ is a linear combination of $g_2(z)$ and $g_1(z)$. As $g_1(z)$ has been replaced in the fundamental system, we now replace $g_2(z)$ by $\vartheta^{\frac{z}{\omega}} G_2$; and the system remains fundamental.

And so on, for the integrals in succession. Proceeding thus, we obtain λ integrals of the form

$$\vartheta^{\frac{z}{\omega}} G_1(z),\ \vartheta^{\frac{z}{\omega}} G_2(z), \ldots, \vartheta^{\frac{z}{\omega}} G_\lambda(z).$$

Further, these integrals are linearly independent, and so they are linearly equivalent to $g_1(z), g_2(z), \ldots, g_\lambda(z)$. For if any relation, linear and homogeneous among these quantities, were to exist with non-vanishing coefficients, we should, on substitution for $G_1, G_2, \ldots, G_\lambda$ in terms of $G, \nabla G, \nabla^2 G, \ldots, \nabla^{\lambda-1}G$, obtain a relation, linear and homogeneous among the quantities $g_1(z), \ldots, g_\lambda(z)$ with non-vanishing coefficients. Such a relation does not exist. Accordingly, the λ integrals

$$\vartheta^{\frac{z}{\omega}} G_1(z),\ \vartheta^{\frac{z}{\omega}} G_2(z),\ \ldots,\ \vartheta^{\frac{z}{\omega}} G_\lambda(z)$$

can be taken as constituting the required sub-group of integrals.

We now are in a position to enunciate the following result, defining the group of integrals associated with a multiple root ϑ of the fundamental equation of the period :—

When a root ϑ of the fundamental equation $A(\theta)=0$ is of multiplicity μ, there is a group of μ integrals associated with that root; the group can be arranged in a number of sub-groups, their number being equal to the number of elementary divisors of $A(\theta)$ which are powers of $\vartheta-\theta$; the number of integrals in the first sub-group is equal to the number of those elementary divisors; the number in the second sub-group is equal to the number of the exponents of those divisors which are equal to or greater than 2; the number in the third sub-group is equal to the number of the

exponents of those divisors which are equal to or greater than 3; and a sub-group, which contains λ integrals, is equivalent to the λ linearly independent quantities

$$\vartheta^{\frac{z}{\omega}} G_1(z),\ \vartheta^{\frac{z}{\omega}} G_2(z),\ \ldots,\ \vartheta^{\frac{z}{\omega}} G_\lambda(z),$$

where

$$G_r(z) = \chi_r + \binom{r-1}{1}\chi_{r-1}\zeta + \binom{r-1}{2}\chi_{r-2}\zeta^2 + \ldots$$
$$\ldots + \binom{r-1}{1}\chi_2\zeta^{r-2} + \chi_1\zeta^{r-1},$$

for $r = 1, 2, \ldots, \lambda$: the quantities $\chi_1, \ldots, \chi_\lambda$ are periodic functions of z, but they are not necessarily uniform: ζ denotes $\dfrac{z}{\omega}$, and

$$\binom{r-1}{\kappa} = \frac{(r-1)!}{\kappa!\,(r-1-\kappa)!}.$$

NOTE. By taking $\chi_n = \omega^{-n}\phi_n$, for $n = 1, \ldots, \lambda$, and writing

$$\omega^r G_r(z) = \overline{G}_r(z),$$

the integrals become

$$\vartheta^{\frac{z}{\omega}} \overline{G}_1(z),\ \vartheta^{\frac{z}{\omega}} \overline{G}_2(z),\ \ldots,\ \vartheta^{\frac{z}{\omega}} \overline{G}_n(z),$$

where

$$\overline{G}_r(z) = \phi_r + \binom{r-1}{1}'\phi_{r-1}z + \binom{r-1}{2}\phi_{r-2}z^2 + \ldots$$
$$\ldots + \binom{r-1}{1}\phi_2 z^{r-2} + \phi_1 z^{r-1},$$

the functions ϕ having the same character as the functions χ.

135. There is a theorem of the nature of a converse to the foregoing proposition, which is analogous to Fuchs's theorem proved in §§ 25—28. The theorem, which manifestly is important as regards the reducibility of a given equation, is as follows:—

If an expression for a quantity u is given in the form

$$u = \vartheta^{\frac{z}{\omega}}\{\phi_n + \phi_{n-1}\zeta + \phi_{n-2}\zeta^2 + \ldots + \phi_2\zeta^{n-2} + \phi_1\zeta^{n-1}\},$$

where ϑ is a constant, all the functions $\phi_1, \ldots, \phi_n$ are periodic in ω, and ζ denotes $\dfrac{z}{\omega}$, then u satisfies a homogeneous linear differential

equation of order n, the coefficients of which are uniform periodic functions of z, having the period ω ; moreover,

$$\frac{\partial u}{\partial \zeta}, \frac{\partial^2 u}{\partial \zeta^2}, \ldots, \frac{\partial^{n-1} u}{\partial \zeta^{n-1}}$$

are integrals of the same equation and, taken together with u, they constitute a fundamental system for the equation of order n.

The course of the proof is so similar to the proof of the corresponding theorem as established in §§ 26—28 that it need not be set out here*. It can be divided into three sections; in the first, it is proved that $\frac{\partial u}{\partial \zeta}, \ldots, \frac{\partial^{n-1} u}{\partial \zeta^{n-1}}$ satisfy such an equation, if u satisfies it; in the second, it is proved that these must form a fundamental system, for no homogeneous linear relation with non-evanescent coefficients can exist among them; in the third, it is shewn that the linear equation, which has these quantities for its fundamental system, has uniform periodic functions of z with period ω for its coefficients. The details of the proof are left to the student.

Mode of obtaining Integrals that are Uniform.

136. The further determination of the analytical expressions of the integrals, on the basis of the properties already established, is not possible in the general case. Thus the functions $\chi_1, \ldots, \chi_\lambda$, occurring in the sub-group specially considered in § 134, are periodic functions of the second kind with a multiplier ϑ. If we take new functions $\psi_1(z), \ldots, \psi_\lambda(z)$, such that

$$\begin{aligned} \chi_r(z) &= \vartheta^{\frac{z}{\omega}} \psi_r(z) \\ &= e^{\frac{z}{\omega}\log\vartheta} \psi_r(z), \qquad (r = 1, \ldots, \lambda), \end{aligned}$$

these new functions are periodic of the first kind. But further properties of the functions must be given if there is to be any further determination of their form.

When we limit ourselves to the consideration of those equations whose integrals are uniform functions, (criteria are determined

* Some of the analysis of § 132 is useful in establishing the theorem.

independently by considering the integrals in the vicinity of the singularities), some further progress can be made; but, of course, the assumption that the integrals are of this character must be justified by appropriate limitations upon the forms of the coefficients $p_1, \ldots, p_m$ in the original differential equation. In such cases, every quantity such as $\psi_r(z)$ is a uniform simply-periodic function of the first kind; it can therefore* be expressed in the form of a Laurent-Fourier series such as

$$\sum_{\kappa=-\infty}^{\kappa=\infty} A_\kappa e^{\frac{2\pi i\kappa z}{\omega}}.$$

Such a form of expression does not lead, however, towards the determination of the criteria for securing such a result or any other result of a corresponding kind for any other assumption. In particular examples, we adopt a different method of practical procedure.

In order to determine some of the functional properties of the integrals, it frequently is expedient to change the variable so that, if possible, the transformed equation belongs to one or other of the classes of equations considered in preceding chapters.

Thus if the coefficients $p_1, \ldots, p_m$, which are uniform periodic functions of z, occur as rational functions of $\sin \frac{2\pi z}{\omega}$ and $\cos \frac{2\pi z}{\omega}$, then, introducing a new variable t, where

$$t = e^{\frac{2\pi z i}{\omega}},$$

we obtain a linear equation, the coefficients of which are rational functions of t. Some characteristic properties of the integrals of the equation in the latter form can be obtained by earlier processes; it may even be possible to determine the fundamental system of integrals.

The preceding transformation is, however, not the only one that can be used with advantage; and it often happens that the special form of a particular equation suggests a special transformation which is effective. In particular, if the coefficients in the equation are alternately odd and even functions,

* *T. F.*, § 112.

such that $p_1, p_3, p_5, \ldots$ are odd, and $p_2, p_4, p_6, \ldots$ are even, then we may take

$$t = \cos\frac{\pi z}{\omega}, \text{ or } \sin\frac{\pi z}{\omega},$$

as a new independent variable: it is easy to prove that the transformed equation has uniform functions of t for its coefficients. Also, some indication is occasionally given as to a choice between these two transformations; for example, if an irreducible pole of the original equation is $z = 0$, we should choose

$$t = \sin\frac{\pi z}{\omega}$$

as the transformation, and consider the integrals in the vicinity of $t = 0$; whereas the other would be chosen, if an irreducible pole of the equation is $z = \frac{1}{2}\omega$.

Another transformation, that sometimes can prove effective, is

$$t = \tan\frac{\pi z}{2\omega};$$

any uniform function of z, periodic in ω, can be expressed as a uniform function of t; and the differential equation is transformed into one which has uniform functions of t for its coefficients.

Ex. 1. Consider the equation

$$\frac{d^2w}{dz^2} + 2a\frac{dw}{dz}\cot z + (b + c\cot^2 z)\, w = 0,$$

where a, b, c are constants. Writing

$$w \sin^a z = y,$$

we have the equation

$$\frac{d^2y}{dz^2} + (\beta + \gamma\cot^2 z)\, y = 0,$$

where

$$\beta = a + b, \quad \gamma = c + a - a^2.$$

As the equation is periodic in π, and as $z = 0$ is a singularity, we take a transformation

$$t = \sin z;$$

and the equation is

$$(1 - t^2)\frac{d^2y}{dt^2} - t\frac{dy}{dt} + y\left(\frac{\gamma}{t^2} + \beta - \gamma\right) = 0.$$

The indicial equation for $t = 0$ is

$$\rho(\rho - 1) + \gamma = 0.$$

If

$$y = \Sigma a_n t^{\rho + n}$$

satisfies the equation, we have

$$\{(n+\rho)(n+\rho-1)+\gamma\}\,a_n=\{(n+\rho-2)^2-\beta+\gamma\}\,a_{n-2},$$

so that

$$a_n=\frac{n^2+2n(\rho-2)+4-\beta-3\rho}{n^2+n(2\rho-1)}\,a_{n-2}.$$

The form of a_n, in terms of a_{n-2}, shews that the series for y converges for values of $t \leqslant 1$. If the two roots of the indicial equation are ρ_1 and ρ_2, and $f(t, \rho_1)$, $f(t, \rho_2)$ be the two values of y, the primitive of the original equation is

$$w=\sin^{-a} z\,\{Af(\sin z, \rho_1)+Bf(\sin z, \rho_2)\}.$$

Ex. 2. Consider the equation

$$\frac{d^2y}{dz^2}\sin^2 z=ay,$$

where a is a constant. Taking

$$i\mu=\cot z,$$

we find the transformed equation for z to be

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dy}{d\mu}\right\}+ay=0,$$

which is Legendre's equation and so its primitive is known.

Ex. 3. Obtain integrals of the equations

(i) $$\frac{d^2w}{dz^2}+\frac{dw}{dz}\cot z-w\,\mathrm{cosec}^2 z=0\,;$$

(ii) $$\frac{d^2w}{dz^2}+4\frac{dw}{dz}\,\mathrm{cosec}\,2z+2w\sec^2 z=0\,;$$

(iii) $$\frac{d^2w}{dz^2}+\left(\frac{2}{\sin z\cos z}-1\right)\frac{dw}{dz}+\left(\frac{2}{\cos^2 z}-\frac{1}{\sin z\cos z}\right)w=0.$$

Ex. 4. One integral, $f(z)$, of the equation

$$4(2-\sin z)\frac{d^2w}{dz^2}+2(3\sin z+2\cos z-6)\frac{dw}{dz}+(5-3\cos z-\sin z)\,w=0$$

satisfies the relation

$$f(z+2\pi)+e^{2\pi}f(z)=0\,;$$

find the general solution. (Math. Tripos, Part II, 1896.)

Ex. 5. Shew that the equation

$$\frac{1}{w}\frac{d^2w}{dz^2}=\frac{2}{\sin^2 z}+h$$

has an integral

$$w=\frac{\sin(z-z_1)}{\sin z}\,e^{z\cot z_1},$$

where z_1 has an appropriate constant value; and obtain the primitive.

(M. Elliott.)

Ex. 6. Obtain an integral of the equation

$$\frac{1}{w}\frac{d^2w}{dz^2}=\frac{6}{\sin^2 z}+h,$$

where h is a constant, in the form

$$w=\frac{\sin(z-z_1)\sin(z-z_2)}{\sin^2 z}\,e^{z(\cot z_1+\cot z_2)},$$

where z_1 and z_2 are appropriate determinate constants: and obtain the primitive. (M. Elliott.)

Ex. 7. Integrate the equation

$$\frac{d^2w}{dz^2}=\left\{\frac{n(n+1)}{\sin^2 z}+h\right\}w,$$

where n is an integer, and h is a constant. (M. Elliott.)

137. A somewhat different form of the theory is developed by Liapounoff*, whose investigation deals with a more general equation, given by

$$\frac{d^2w}{dz^2}+\mu w p(z)=0,$$

where μ is a parameter, and $p(z)$ is a uniform periodic function of period ω.

Let $f(z)$ and $\phi(z)$ be two integrals of the differential equation, respectively determined by the initial conditions

$$\left.\begin{aligned}f\,(0)&=1\\ f'(0)&=0\end{aligned}\right\},\qquad \left.\begin{aligned}\phi\,(0)&=0\\ \phi'(0)&=1\end{aligned}\right\}.$$

Then we have relations of the form

$$\left.\begin{aligned}f\,(z+\omega)&=\alpha f(z)+\beta\phi(z)\\ \phi(z+\omega)&=\gamma f(z)+\delta\phi(z)\end{aligned}\right\};$$

and the equation for determining the multipliers is

$$(\Omega-\alpha)(\Omega-\delta)-\beta\gamma=0,$$

that is,

$$\Omega^2-(\alpha+\delta)\,\Omega+1=0,$$

as in § 127, Ex. 1. Clearly, we have

$$f(\omega)=\alpha,\quad \phi'(\omega)=\delta;$$

* *Comptes Rendus*, t. CXXIII (1896), pp. 1248—1252; *ib.*, t. CXXVIII (1899), pp. 910—913, 1085—1088.

so that, if we write

$$A = \tfrac{1}{2}\{f(\omega) + \phi'(\omega)\},$$

the equation is

$$\Omega^2 - 2A\Omega + 1 = 0.$$

Writing

$$\rho = A + (A^2 - 1)^{\frac{1}{2}},$$

and assuming that $A^2 - 1$ does not vanish, we obtain two integrals in the form

$$\rho^{\frac{z}{\omega}} F_1(z), \quad \rho^{-\frac{z}{\omega}} F_2(z),$$

where $F_1(z)$, $F_2(z)$ are functions of z, periodic in ω; and thus the complete primitive of the equation can be obtained. The actual expressions for $F_1(z)$ and $F_2(z)$ can be constructed as in the preceding sections; and the value of ρ depends upon that of A.

When $\mu = 0$, the primitive of the original equation is

$$w = C + Dz,$$

shewing that the equation for determining the multipliers is

$$(\Omega - 1)^2 = 0;$$

and then $A = 1$. Hence, when μ is not zero, and when A is expanded in powers of μ, it is inferred that A is of the form

$$A = 1 - \mu A_1 + \mu^2 A_2 - \mu^3 A_3 + \dots.$$

When A_1, A_2, A_3, ... are known, the two values of Ω, which satisfy the equation

$$\Omega^2 - 2A\Omega + 1 = 0,$$

can be regarded as known, and the primitive of the differential equation can be obtained.

For the purpose of obtaining the value of A, which is

$$A = \tfrac{1}{2}\{f(\omega) + \phi'(\omega)\},$$

where the integrals $f(z)$ and $\phi(z)$ are defined by the initial conditions, we assume both $f(z)$ and $\phi(z)$ expanded in powers of μ. Let

$$f(z) = u_0 + \mu u_1 + \mu^2 u_2 + \dots;$$

then, in order that it may satisfy the equation

$$\frac{d^2 w}{dz^2} + \mu w p(z) = 0,$$

we have

$$\frac{d^2u_0}{dz^2}=0,$$

$$\frac{d^2u_1}{dz^2}+u_0p(z)=0,$$

$$\frac{d^2u_2}{dz^2}+u_1p(z)=0,$$

and so on. From the first, we have

$$u_0=a_0+b_0z;$$

from the second, we have

$$u_1=a_1+b_1z-\int_0^z dy\int_0^y u_0(x)\,p(x)\,dx;$$

from the third, we have

$$u_2=a_2+b_2z-\int_0^z dy\int_0^y u_1(x)\,p(x)\,dx;$$

and so on. Now

$$f(0)=1,\quad f'(0)=0;$$

accordingly,

$$a_0+\mu a_1+\mu^2a_2+\ldots=1,$$

$$b_0+\mu b_1+\mu^2b_2+\ldots=0.$$

Taking account of the fact that μ is parametric, we have

$$a_0=1,\quad a_s=0 \text{ for } s=1,2,\ldots,\quad b_s=0 \text{ for } s=0,1,2,\ldots;$$

and thus we have

$$u_0=1,$$

$$u_1=-\int_0^z dy\int_0^y p(x)\,dx,$$

$$u_2=-\int_0^z dy\int_0^y u_1(x)\,p(x)\,dx,$$

and so on. The value of $f(z)$ is given by

$$f(z)=1+\mu u_1+\mu^2u_2+\ldots.$$

Similarly for $\phi(z)$, which is determined by the conditions

$$\phi(0)=0,\quad \phi'(0)=1;$$

its value is given by

$$\phi(z)=z+\mu v_1+\mu^2v_2+\ldots,$$

where

$$v_1 = -\int_0^z dy \int_0^y xp(x)\,dx,$$

$$v_2 = -\int_0^z dy \int_0^y p(x)\,v_1(x)\,dx,$$

and so on.

We require the quantities $f(\omega)$ and $\phi'(\omega)$: let them be denoted by

$$f(\omega) = 1 + \mu U_1 + \mu^2 U_2 + \dots,$$

$$\phi'(\omega) = 1 + \mu V_1 + \mu^2 V_2 + \dots,$$

where

$$U_1 = -\int_0^\omega dy \int_0^y p(x)\,dx,$$

$$V_1 = -\int_0^\omega xp(x)\,dx,$$

and so for the others. Substituting the value of A in the form

$$A = 1 - \mu A_1 + \mu^2 A_2 - \dots,$$

we have

$$\begin{aligned} 2A_1 &= -U_1 - V_1 \\ &= \int_0^\omega dy \int_0^y p(x)\,dx + \int_0^\omega xp(x)\,dx \\ &= \int_0^\omega dy \int_0^y p(x)\,dx + \int_0^\omega yp(y)\,dy. \end{aligned}$$

But

$$\frac{d}{dy}\left\{y \int_0^y p(x)\,dx\right\} = \int_0^y p(x)\,dx + yp(y);$$

integrating between the limits 0 and ω, we have

$$2A_1 = \omega \int_0^\omega p(x)\,dx.$$

Next, we have

$$\begin{aligned} 2A_2 &= -U_2 - V_2 \\ &= -\int_0^\omega dy \int_0^y u_1(x)\,p(x)\,dx - \int_0^\omega p(x)\,v_1(x)\,dx. \end{aligned}$$

To transform these definite integrals, we write

$$\int_0^x p(x)\,dx = P(x), \quad P(\omega) = \Omega,$$

so that

$$u_1(x) = -\int_0^x dt \int_0^t p(\theta)\, d\theta = -\int_0^x P(t)\, dt,$$

$$v_1(x) = -\int_0^x dt \int_0^t \theta p(\theta)\, d\theta$$

$$= -\int_0^x tP(t)\, dt + \int_0^x dt \int_0^t P(\theta)\, d\theta$$

$$= \int_0^x dt \int_0^t \{P(\theta) - P(t)\}\, d\theta.$$

We have

$$\frac{d}{dy}\left\{u_1(y)\int_0^y p(y)\, dy\right\} = u_1(y)\, p(y) - P^2(y);$$

therefore

$$\int_0^y u_1(x)\, p(x)\, dx = u_1(y)\, P(y) + \int_0^y P^2(x)\, dx$$

$$= \int_0^y P(x)\, \{P(x) - P(y)\}\, dx;$$

and thus the first integral in the expression for $2A_2$ is equal to

$$\int_0^\omega dy \int_0^y \{P(y) - P(x)\}\, P(x)\, dx.$$

Similarly, we have

$$\frac{d}{dy}\left\{v_1(y)\int_0^y p(y)\, dy\right\} = v_1(y)\, p(y) - yP^2(y) + P(y)\int_0^y P(x)\, dx;$$

therefore

$$\int_0^\omega v_1(x)\, p(x)\, dx = v_1(\omega)\,\Omega + \int_0^\omega yP^2(y)\, dy - \int_0^\omega dy \int_0^y P(y)\, P(x)\, dx$$

$$= -\Omega \int_0^\omega yP(y)\, dy + \Omega \int_0^\omega dy \int_0^y P(x)\, dx$$

$$+ \int_0^\omega yP^2(y)\, dy - \int_0^\omega dy \int_0^y P(y)\, P(x)\, dx.$$

The first and third terms on the right-hand side together are

$$= -\int_0^\omega \{\Omega - P(y)\}\, P(y)\, y\, dy$$

$$= -\int_0^\omega dy \int_0^y \{\Omega - P(y)\}\, P(y)\, dx,$$

so that

$$\int_0^\omega v_1(x)\,p(x)\,dx = -\int_0^\omega dy \int_0^y \{\Omega - P(y)\}\{P(y) - P(x)\}\,dx,$$

which gives a transformation of the second integral in $2A_2$. Combining the results of transformation for the two integrals in $2A_2$, we have

$$2A_2 = \int_0^\omega dy \int_0^y \{\Omega - P(y) + P(x)\}\{P(y) - P(x)\}\,dx.$$

Similarly, it may be proved that the value of $2A_3$ is

$$\int_0^\omega dz \int_0^z dy \int_0 \{\Omega - P(z) + P(x)\}\{P(z) - P(y)\}\{P(y) - P(x)\}\,dx,$$

and so on: so that the value of A, and therefore the value of P, is known.

The investigation is continued by Liapounoff, especially for the purpose of discussing the values of μ which satisfy the equation

$$A^2 - 1 = 0;$$

and the results appear to be of importance in the discussion of the stability of motion. The reader is referred to the notes by Liapounoff already cited (p. 425, note); other references to more detailed investigations are there given.

Ex. 1. Establish by induction, or otherwise, the general law for the coefficients A, viz.

$$2A_n = \int_0^\omega dx_1 \int_0^{x_1} dx_2 \ldots \int_0^{x_{n-1}} \Theta\,dx_n,$$

where

$$\Theta = \{\Omega - P(x_1) + P(x_n)\}\{P(x_1) - P(x_2)\}\{P(x_2) - P(x_3)\}\ldots\{P(x_{n-1}) - P(x_n)\}.$$

Ex. 2. Shew that, if the periodic function $p(x)$ always is positive, then all the coefficients A are positive; and prove that

$$A_{m+n} < \frac{m!\,n!}{(m+n)!} A_m A_n.$$

Hence shew that, when $p(x)$ is positive and satisfies the inequality

$$\mu\omega \int_0^\omega p(x)\,dx \leqslant 4,$$

then $A^2 < 1$, so that $|\rho| = 1$.

Ex. 3. Prove that, if the periodic function $p(x)$ be real and odd, so that the series for A contains only even powers of μ, then

$$A_2 = -2\int_0^{\frac{1}{2}\omega} dx_1 \int_0^{x_1} (P_1 - P_2)^2\, dx_2,$$

$$A_4 = \ \ 4\int_0^{\frac{1}{2}\omega} dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 \int_0^{x_3} (P_1 - P_2)^2 (P_3 - P_4)^2\, dx_4,$$

and so on, where P_r denotes $P(x_r)$. Prove also that, if

$$\int_a^x p(x)\, dx = P,$$

the constant a being determined so that $\int_0^\omega P dx = 0$, and if

$$\mu^2\omega \int_0^\omega P^2 dx \leqslant 4,$$

then $A^2 < 1$.

Ex. 4. Discuss the values of μ which are roots of the equation

$$A^2 = 1.$$

(All these results are due to Liapounoff.)

Discussion of the Equation of the Elliptic Cylinder.

138. One of the most important equations of the class, which has been considered in § 137, is the equation

$$\frac{d^2w}{dz^2} + (a + c\cos 2z)\, w = 0,$$

commonly called the equation of the elliptic cylinder; it is of frequent occurrence in mathematical physics and astronomical dynamics. It forms the subject of many investigations*. It is known (§ 55) to be a transformation of the limiting form of an equation of Fuchsian type. Moreover, it has already (§ 127, Ex. 1) been partially discussed in connection with another equation and for another purpose. In this place, it will be brought into relation with the preceding general theory.

Let new independent variables u or v be introduced, such that

$$u = \cos^2 z, \quad v = 1 - u = \sin^2 z.$$

* Heine, *Handbuch der Kugelfunctionen*, t. I, pp. 404—415; Lindemann, *Math. Ann.*, t. XXII (1883), pp. 117—123; Tisserand, *Mécanique Céleste*, t. III, ch. I, at the end of which other references are given.

The equation becomes

$$u(1-u)\frac{d^2w}{du^2}+\tfrac{1}{2}(1-2u)\frac{dw}{du}+\tfrac{1}{4}(a-c+2cu)\,w=0,$$

when u is the independent variable; and it becomes

$$v(1-v)\frac{d^2w}{dv^2}+\tfrac{1}{2}(1-2v)\frac{dw}{dv}+\tfrac{1}{4}(a+c-2cv)\,w=0,$$

when v is the independent variable. Accordingly, if

$$w=f(u, c)$$

is an integral of the equation, another integral is provided by

$$w=f(v, -c).$$

The indicial equation for $u=0$ is

$$\rho(\rho-\tfrac{1}{2})=0;$$

if

$$w=\Sigma a_p u^{\rho+p}$$

be the integral, the scale of relation between the coefficients a_p is

$$(p+\rho)(p+\rho-\tfrac{1}{2})\,a_p=\{(p+\rho-1)^2-\tfrac{1}{4}(a-c)\}\,a_{p-1}-\tfrac{1}{2}ca_{p-2},$$

with the relations

$$a_0=1,$$
$$(\rho+1)(\rho+\tfrac{1}{2})\,a_1=\rho^2-\tfrac{1}{4}(a-c).$$

When $\rho=0$, let $a_p=\theta(p, c)$; when $\rho=\frac{1}{2}$, let $a_p=\vartheta(p, c)$. Then two integrals of the equation are

$$x_0=\sum_{p=0}^{\infty}u^p\theta(p, c),\quad x_1=\sum_{p=0}^{\infty}u^{p+\frac{1}{2}}\vartheta(p, c),$$

with the convention

$$\theta(0, c)=1=\vartheta(0, c).$$

It is clear that, when z is $\frac{1}{2}\pi$, so that u is 0,

$$x_0=1,\quad \frac{dx_0}{dz}=0,\quad x_1=0,\quad \frac{dx_1}{dz}=-1.$$

Moreover, as the equation in w is satisfied by x_0 and x_1, we have

$$x_1\frac{dx_0}{du}-x_0\frac{dx_1}{du}=\frac{C}{\{u(1-u)\}^{\frac{1}{2}}}.$$

But $du = -2\sin z \cos z\, dz = -2\{u(1-u)\}^{\frac{1}{2}} dz$, so that

$$x_1 \frac{dx_0}{dz} - x_0 \frac{dx_1}{dz} = -2C.$$

When $z = \frac{1}{2}\pi$, the left-hand side is equal to 1: hence

$$C = -\tfrac{1}{2},$$

and therefore, for all values of z, we have

$$x_1 \frac{dx_0}{dz} - x_0 \frac{dx_1}{dz} = 1.$$

Two other integrals of the equation are given by

$$y_0 = \sum_{p=0}^{\infty} v^p \theta(p, -c), \quad y_1 = \sum_{p=0}^{\infty} v^{p+\frac{1}{2}} \vartheta(p, -c);$$

they are such that, when z is 0, and therefore v is 0,

$$y_0 = 1, \quad \frac{dy_0}{dz} = 0, \quad y_1 = 0, \quad \frac{dy_1}{dz} = 1,$$

and, for all values of z,

$$y_0 \frac{dy_1}{dz} - y_1 \frac{dy_0}{dz} = 1.$$

Now when z is real, both u and v are real and lie between 0 and 1; and, in particular, when $z = \frac{1}{4}\pi$, then $u = v = \frac{1}{2}$. For such values, x_0, x_1, y_0, y_1, coexist; and so we have relations of the form

$$\left.\begin{aligned} y_0 &= \alpha x_0 + \beta x_1 \\ y_1 &= \gamma x_0 + \delta x_1 \end{aligned}\right\},$$

where α, β, γ, δ are constants. Hence

$$y_0(\tfrac{1}{2}) = \alpha x_0(\tfrac{1}{2}) + \beta x_1(\tfrac{1}{2}), \quad -y_0'(\tfrac{1}{2}) = \alpha x_0'(\tfrac{1}{2}) + \beta x_1'(\tfrac{1}{2}),$$

where

$$x_0'(\tfrac{1}{2}) = \left(\frac{dx_0}{dz}\right)_{z=\frac{1}{4}\pi},$$

and so for the others. Hence

$$\begin{aligned} \alpha &= -y_0(\tfrac{1}{2})\, x_1'(\tfrac{1}{2}) - y_0'(\tfrac{1}{2})\, x_1(\tfrac{1}{2}), \\ \beta &= \quad y_0(\tfrac{1}{2})\, x_0'(\tfrac{1}{2}) + y_0'(\tfrac{1}{2})\, x_0(\tfrac{1}{2}). \end{aligned}$$

Similarly

$$\left.\begin{aligned} \gamma &= -y_1(\tfrac{1}{2})\, x_1'(\tfrac{1}{2}) - y_1'(\tfrac{1}{2})\, x_1(\tfrac{1}{2}) \\ \delta &= \quad y_1(\tfrac{1}{2})\, x_0'(\tfrac{1}{2}) + y_1'(\tfrac{1}{2})\, x_0(\tfrac{1}{2}) \end{aligned}\right\},$$

and it is easy to verify that

$$\alpha\delta - \beta\gamma = 1.$$

Moreover, we have

$$\left.\begin{aligned} x_0 &= \quad \delta y_0 - \beta y_1 \\ x_1 &= -\gamma y_0 + \alpha y_1 \end{aligned}\right\}.$$

139. The integrals x_0 and x_1 are valid in the domain of $u=0$; the integrals y_0 and y_1 are valid in the domain of $v=0$, that is, of $u=1$. Lindemann* proceeds, as follows, to obtain uniform integrals valid over the whole of the finite part of the plane.

After a small closed circuit of u round its origin, x_0 returns to its initial value and x_1 changes its sign; hence y_0 becomes $\alpha x_0 - \beta x_1$, and y_1 becomes $\gamma x_0 - \delta x_1$. After a small closed circuit of u round the point 1, the integral y_0 returns to its initial value and y_1 changes its sign. Consider a quantity η, where

$$\eta = Ay_0^2 + By_1^2,$$

as a function of u. It remains unchanged when u moves round the point 1. Its two values in the vicinity of $u=0$ are

$$(A\alpha^2 + B\gamma^2)\, x_0^2 + (A\beta^2 + B\delta^2)\, x_1^2 + 2\,(A\alpha\beta + B\gamma\delta)\, x_0 x_1,$$
$$(A\alpha^2 + B\gamma^2)\, x_0^2 + (A\beta^2 + B\delta^2)\, x_1^2 - 2\,(A\alpha\beta + B\gamma\delta)\, x_0 x_1,$$

which are the same if

$$A\alpha\beta + B\gamma\delta = 0:$$

hence the function is uniform in the vicinity of $u=0$ if this condition is satisfied, that is, the function is uniform over the whole plane.

The condition is satisfied if we take

$$A = -\gamma\delta, \quad B = \alpha\beta;$$

and then

$$\eta = \alpha\beta y_1^2 - \gamma\delta y_0^2.$$

Moreover, in the region of existence common to y_0, y_1, x_0, x_1, we have

$$\alpha\beta y_1^2 - \gamma\delta y_0^2 = \beta\delta x_1^2 - \alpha\gamma x_0^2.$$

Hence defining the function η in the domain of $z=0$ by its value in terms of y_0 and y_1, and defining it in the domain of $z=1$ by its value in terms of x_0 and x_1, we have a function

$$\eta = F(u) = F(\cos^2 z) = \Phi(z),$$

* *Math. Ann.*, t. XXII (1883), pp. 117—123.

say, which is regular in the vicinity of $u=0$, regular in the vicinity of $u=1$, and therefore is regular over the whole finite part of the z-plane. Now let

$$\left.\begin{aligned} Y_1 &= y_1(\alpha\beta)^{\frac{1}{2}} + y_0(\gamma\delta)^{\frac{1}{2}} \\ Y_0 &= y_1(\alpha\beta)^{\frac{1}{2}} - y_0(\gamma\delta)^{\frac{1}{2}} \end{aligned}\right\};$$

then

$$Y_0\frac{dY_1}{dz} - Y_1\frac{dY_0}{dz} = -2(\alpha\beta\gamma\delta)^{\frac{1}{2}}\left(y_0\frac{dy_1}{dz} - y_1\frac{dy_0}{dz}\right)$$

$$= -2(\alpha\beta\gamma\delta)^{\frac{1}{2}}.$$

Also

$$Y_1 Y_0 = \Phi(z),$$

and therefore

$$Y_0\frac{dY_1}{dz} + Y_1\frac{dY_0}{dz} = \Phi'(z).$$

Hence

$$\frac{1}{Y_1}\frac{dY_1}{dz} = \frac{1}{2}\frac{\Phi'(z)}{\Phi(z)} - \frac{(\alpha\beta\gamma\delta)^{\frac{1}{2}}}{\Phi(z)},$$

$$\frac{1}{Y_0}\frac{dY_0}{dz} = \frac{1}{2}\frac{\Phi'(z)}{\Phi(z)} + \frac{(\alpha\beta\gamma\delta)^{\frac{1}{2}}}{\Phi(z)};$$

and therefore

$$Y_1 = K\,\{\Phi(z)\}^{\frac{1}{2}}\, e^{-\int\frac{(\alpha\beta\gamma\delta)^{\frac{1}{2}}}{\Phi(z)}dz} = K\,\{\Phi(z)\}^{\frac{1}{2}}\, e^{-M\int\frac{dz}{\Phi(z)}},$$

$$Y_0 = K'\,\{\Phi(z)\}^{\frac{1}{2}}\, e^{M\int\frac{dz}{\Phi(z)}},$$

where

$$M = (\alpha\beta\gamma\delta)^{\frac{1}{2}}, \quad KK' = 1.$$

These integrals of the original differential equation are valid over the whole of the finite part of the plane. Accordingly, we may take two integrals

$$\left.\begin{aligned} G(z) &= \{\Phi(z)\}^{\frac{1}{2}}\, e^{-M\int\frac{dz}{\Phi(z)}} \\ G_1(z) &= \{\Phi(z)\}^{\frac{1}{2}}\, e^{M\int\frac{dz}{\Phi(z)}} \end{aligned}\right\},$$

as integrals, which are valid over the plane and have $z=\infty$ for their sole essential singularity. We now proceed to shew that they are uniform over the plane.

Substituting in the original differential equation, we have

$$(a + c\cos 2z)\,\Phi^2 - \tfrac{1}{4}\Phi'^2 + \tfrac{1}{2}\Phi\Phi'' + M^2 = 0;$$

28—2

so that, as M in general is not zero, any root of $\Phi = 0$ is a simple root. Let k denote such a root: then

$$M = \tfrac{1}{2}\Phi'(k).$$

Now let z describe a simple closed contour, including k and no other root of $\Phi = 0$, and passing through no root of $\Phi = 0$. Then, at the end of the contour, $\{\Phi(z)\}^{\frac{1}{2}}$ has changed its sign. As for the exponential factors in $G(z)$ and $G_1(z)$, they are multiplied by

$$e^{\pm M\int \frac{dz}{\Phi(z)}}$$

respectively, the integral being taken round the contour, that is, they are multiplied by

$$e^{\pm \frac{1}{2}\Phi'(k)\frac{2\pi i}{\Phi'(k)}},$$

that is, by -1. Thus $G(z)$ and $G_1(z)$ are unaffected by the contour; they are therefore uniform in the vicinity. Moreover, in the immediate vicinity of k, we have

$$\Phi(z) = (z-k)\,\Phi'(k) + \ldots,$$

so that

$$G(z) = [\{\Phi'(k)\}^{\frac{1}{2}}(z-k)^{\frac{1}{2}} + \ldots]\,e^{-\frac{1}{2}\log(z-k) + P(z-k)}$$

$$= \{\Phi'(k)\}^{\frac{1}{2}}\,e^{P(z-k)}\,Q(z-k),$$

$$G_1(z) = \{\Phi'(k)\}^{\frac{1}{2}}(z-k)\,e^{-P(z-k)}\,Q(z-k),$$

so that k is a simple root for one of the integrals and it is not a root for the other. Similarly, in the vicinity of any other root of $\Phi = 0$; hence G and G_1 are uniform over the whole plane.

Now take any path from z to $z+\pi$, for π is the period for the original equation. We have

$$\Phi(z) = F(\cos^2 z),$$

where F is uniform; hence

$$\Phi(z+\pi) = \Phi(z), \quad \{\Phi(z+\pi)\}^{\frac{1}{2}} = (-1)^r\{\Phi(z)\}^{\frac{1}{2}},$$

where r is 0 or 1, depending upon the path from 0 to π. The effect upon the exponential factor of $G(z)$ is to multiply it by

$$e^{-M\int_z^{z+\pi}\frac{dz}{\Phi(z)}}.$$

We know that $\Phi(z)$ is regular over the whole plane, that it is periodic in π, and that it has only simple roots; hence, taking a path between z and $z+\pi$, that nowhere is near a root, we can expand $\dfrac{1}{\Phi(z)}$ in the form

$$\frac{1}{\Phi(z)}=\sum_{-\infty}^{n=\infty} C_n e^{2nzi},$$

valid everywhere in the range of integration. Then

$$e^{-M\int_z^{z+\pi}\frac{dz}{\Phi(z)}}=e^{-MC_0\pi};$$

and, consequently, if

$$\mu=(-1)^r e^{-MC_0\pi},$$

then

$$G(z+\pi)=\mu G(z).$$

Similarly

$$G_1(z+\pi)=\frac{1}{\mu}G(z).$$

Hence G and G_1 are the two periodic functions of the second kind, which are integrals of the original equation*; and they have been proved to be uniform functions, regular everywhere in the finite part of the plane.

Ex. Shew that the equation

$$z(1-z)\frac{d^2w}{dz^2}+\tfrac{1}{2}(1-2z)\frac{dw}{dz}+(az+b)w=0$$

has two particular integrals the product of which is a single-valued transcendental function $F(z)$; and shew that the integrals are

$$y_1=\{F(z)\}^{\frac{1}{2}}.\exp.\left[\quad C\int\frac{dz}{\{z(1-z)\}^{\frac{1}{2}}F(z)}\right],$$

$$y_2=\{F(z)\}^{\frac{1}{2}}.\exp.\left[-C\int\frac{dz}{\{z(1-z)\}^{\frac{1}{2}}F(z)}\right],$$

where C is a determinate constant. In what circumstances are these two particular integrals coincident? (Math. Tripos, Part II, 1898.)

140. The multipliers μ and $\dfrac{1}{\mu}$ are thus the roots of the equation

$$\Omega^2-I\Omega+1=0,$$

* This inclusion of Lindemann's special result within the general theory is due to Stieltjes, *Astr. Nachr.*, t. CIX (1884), pp. 147, 148.

where the invariant I of the period ω is

$$(-1)^r \left(e^{MC_0\pi} + e^{-MC_0\pi}\right).$$

Another expression for this invariant, consequently leading to another mode of obtaining these multipliers, has already been given in Ex. 1, § 127. Both processes are dependent upon the determination of simple special solutions of the original differential equation.

Another method of proceeding is as follows. Let

$$\mu = e^{\pi i h},$$

so that

$$e^{ih(z+\pi)} = \mu e^{ihz};$$

so that, if

$$G(z) = e^{ihz}\,\Theta(z),$$

then, as $G(z)$ is a uniform function of z, regular over the whole plane, $\Theta(z)$ is a uniform periodic function of the first kind, regular over the whole plane; and π is the period. Hence we have

$$\Theta(z) = \sum_{-\infty}^{n=\infty} \kappa_n e^{2nzi},$$

and therefore

$$G(z) = \sum_{-\infty}^{n=\infty} \kappa_n e^{(2n+h)zi}.$$

Now in the vicinity of $z = 0$, the integral y_0 is even and y_1 is odd: hence $G(z)$ contains both odd and even parts. The form of the differential equation shews that, if $f(z)$ is an integral, then $f(-z)$ also is an integral; hence, as $G(z)$ exists over the finite part of the plane, $G(-z)$ also is an integral. Hence, taking

$$\begin{aligned} H(z) &= \tfrac{1}{2}\{G(z) + G(-z)\}\cos\alpha + \tfrac{1}{2}i\{G(z) - G(-z)\}\sin\alpha \\ &= \sum_{-\infty}^{n=\infty} \kappa_n \cos\{(2n+h)z + \alpha\}, \end{aligned}$$

where α is an arbitrary constant, it follows that $H(z)$ is an integral of the original equation, which exists for all finite values of z. Substituting in the differential equation, and noting that

$$\begin{aligned} &\cos 2z \cos\{(2n+h)z + \alpha\} \\ &\qquad = \tfrac{1}{2}\cos\{(2n-2+h)z + \alpha\} + \tfrac{1}{2}\cos\{(2n+2+h)z+\alpha\}, \end{aligned}$$

we have

$$\sum_{-\infty}^{n=\infty} \left[\{a - (2n+h)^2\}\kappa_n + \tfrac{1}{2}c\,(\kappa_{n-1} + \kappa_{n+1})\right] \cos\{(2n+h)z + \alpha\} = 0,$$

as an equation which must be identically satisfied; hence

$$\{a-(2n+h)^2\}\,\kappa_n+\tfrac{1}{2}c\,(\kappa_{n-1}+\kappa_{n+1})=0,$$

for all values of n from $-\infty$ to $+\infty$.

The mode of dealing with this infinite set of equations by means of infinite determinants has been indicated in a preceding chapter, and much of the analysis of the first example in § 126 is directly applicable here: so we shall not further discuss this mode of obtaining h and the ratios of the coefficients κ. There is, however, another method of obtaining these quantities: it is due to Lindstedt* and is specially adapted to the differential equation under consideration, for purposes of approximation when c is conveniently small. Writing

$$\alpha_n=2\,(2n+h)^2-2a,$$

we have

$$\frac{\kappa_n}{\kappa_{n-1}}=\frac{\kappa_n}{\dfrac{\alpha_n}{c}\kappa_n-\kappa_{n+1}}$$

$$=\frac{\dfrac{c}{\alpha_n}}{1-\dfrac{c}{\alpha_n}\dfrac{\kappa_{n+1}}{\kappa_n}}$$

$$=\frac{\dfrac{c}{\alpha_n}}{1-}\,\frac{\dfrac{c^2}{\alpha_n\alpha_{n+1}}}{1-}\,\frac{\dfrac{c^2}{\alpha_{n+1}\alpha_{n+2}}}{1-}\ldots\text{ ad inf.}$$

Owing to the form of $\dfrac{c}{\alpha_r}$ for increasing values of r, it is easy to prove that this infinite continued fraction converges, for all values of n. We therefore have

$$\frac{\kappa_1}{\kappa_0}=\frac{\dfrac{c}{\alpha_1}}{1-}\,\frac{\dfrac{c^2}{\alpha_1\alpha_2}}{1-}\,\frac{\dfrac{c^2}{\alpha_2\alpha_3}}{1-}\ldots\text{ ad inf.}$$

Similarly

$$\frac{\kappa_{-n}}{\kappa_{-n+1}}=\frac{\dfrac{c}{\alpha_{-n}}}{1-}\,\frac{\dfrac{c^2}{\alpha_{-n}\alpha_{-n-1}}}{1-}\,\frac{\dfrac{c^2}{\alpha_{-n-1}\alpha_{-n-2}}}{1-}\ldots\text{ ad inf.},$$

* *Mém. de l'Acad. St Pétersbourg*, t. XXXI (1883), No. 4.

which is a converging continued fraction; and, in particular,

$$\frac{\kappa_{-1}}{\kappa_0}=\frac{\frac{c}{\alpha_{-1}}}{1-}\,\frac{\frac{c^2}{\alpha_{-1}\alpha_{-2}}}{1-}\,\frac{\frac{c^2}{\alpha_{-2}\alpha_{-3}}}{1-}\dots \text{ ad inf.}$$

But, from the fundamental difference-equation,

$$\frac{\kappa_1+\kappa_{-1}}{\kappa_0}=2\,\frac{h^2-a}{c}=\frac{\alpha_0}{c};$$

therefore

$$1=\frac{\frac{c^2}{\alpha_0\alpha_1}}{1-}\,\frac{\frac{c^2}{\alpha_1\alpha_2}}{1-}\,\frac{\frac{c^2}{\alpha_2\alpha_3}}{1-}\dots+\frac{\frac{c^2}{\alpha_0\alpha_{-1}}}{1-}\,\frac{\frac{c^2}{\alpha_{-1}\alpha_{-2}}}{1-}\,\frac{\frac{c^2}{\alpha_{-2}\alpha_{-3}}}{1-}\dots,$$

a transcendental equation to determine h, which of course is equivalent to the corresponding equation arising out of the vanishing of the infinite determinant $D(\rho)$.

Denoting the first continued fraction by $\frac{p}{q}$ and the second by $\frac{p'}{q'}$, so that these values may be regarded as convergents of infinite order, we easily find

$$\begin{aligned}p=\frac{c^2}{\alpha_0\alpha_1}\Bigg[1-\sum_{r=2}^{\infty}\frac{c^2}{\alpha_r\alpha_{r+1}}+\sum_{r=2}^{\infty}\sum_{s=r+2}^{\infty}\frac{c^2}{\alpha_r\alpha_{r+1}}\frac{c^2}{\alpha_s\alpha_{s+1}}\\-\sum_{r=2}^{\infty}\sum_{s=r+2}^{\infty}\sum_{t=s+2}^{\infty}\frac{c^2}{\alpha_r\alpha_{r+1}}\frac{c^2}{\alpha_s\alpha_{s+1}}\frac{c^2}{\alpha_t\alpha_{t+1}}+\dots\Bigg],\\ q=1-\sum_{r=1}^{\infty}\frac{c^2}{\alpha_r\alpha_{r+1}}+\sum_{r=1}^{\infty}\sum_{s=r+2}^{\infty}\frac{c^2}{\alpha_r\alpha_{r+1}}\frac{c^2}{\alpha_s\alpha_{s+1}}\\-\sum_{r=1}^{\infty}\sum_{s=r+2}^{\infty}\sum_{t=s+2}^{\infty}\frac{c^2}{\alpha_r\alpha_{r+1}}\frac{c^2}{\alpha_s\alpha_{s+1}}\frac{c^2}{\alpha_t\alpha_{t+1}}+\dots;\end{aligned}$$

the values of p' and q' are derivable from the expressions in p and q respectively, by changing α_μ into $\alpha_{-\mu}$ (for all values of μ) wherever α_μ occurs.

The equation manifestly lends itself easily to successive approximations. Thus, if we neglect c^4 and higher powers, we have

$$1=\frac{c^2}{\alpha_0\alpha_1}+\frac{c^2}{\alpha_0\alpha_{-1}},$$

which, to this order of approximation, gives

$$h^2=a+\tfrac{1}{8}\frac{c^2}{1-a}.$$

The calculation of the coefficients can similarly be effected.

Ex. 1. Prove that, up to sixth powers of c inclusive,

$$ha^{-\frac{1}{2}}=1+\frac{1}{16}\frac{c^2}{a(1-a)}-\frac{c^4}{1024}\frac{15a^2-35a+8}{a^2(1-a)^3(4-a)}$$
$$-\frac{c^6}{16384}\frac{105a^5-1155a^4+3815a^3-4705a^2+1652a-288}{a^3(1-a)^5(4-a)^2(9-a)}.$$

(In astronomical applications, a is usually not an integer, and c is small compared with a.) (Poincaré, Tisserand.)

Ex. 2. Taking $\kappa_0=1$, and writing

$$hz+a=Z,$$

prove that, up to c^3 inclusive,

$$H(z)=\cos Z+\left\{\frac{\frac{1}{8}c}{1+q}-\frac{c^3}{1024}\frac{q^3+4q^2+15q+16}{q(1+q)^3(2+q)(1-q)}\right\}\cos(Z+2z)$$
$$+\left\{\frac{\frac{1}{8}c}{1-q}-\frac{c^3}{1024}\frac{q^3-4q^2+15q-16}{q(1-q)^3(2-q)(1+q)}\right\}\cos(Z-2z)$$
$$+\frac{(\frac{1}{8}c)^2}{2!}\left\{\frac{\cos(Z+4z)}{(1+q)(2+q)}+\frac{\cos(Z-4z)}{(1-q)(2-q)}\right\}$$
$$+\frac{(\frac{1}{8}c)^3}{3!}\left\{\frac{\cos(Z+6z)}{(1+q)(2+q)(3+q)}+\frac{\cos(Z-6z)}{(1-q)(2-q)(3-q)}\right\},$$

where $q^2=a$. (Poincaré, Tisserand.)

Ex. 3. In the investigation of § 138, the quantity M is supposed to be different from zero. When M is zero, the integrals $G(z)$ and $G_1(z)$ are effectively the same; and neither of them is uniform, so that the remainder of the investigation does not apply.

Discuss the case when $M=0$. (Heine.)

Equations with Uniform Doubly-periodic Coefficients.

141. We proceed now to the consideration of linear equations, the coefficients in which are uniform doubly-periodic functions of the independent variable. Let the equation be

$$\frac{d^m w}{dz^m}+p_1\frac{d^{m-1}w}{dz^{m-1}}+\ldots+p_m w=0,$$

where $p_1, \ldots, p_m$ are uniform functions of z, which have no essential singularity in the finite part of the plane and are doubly-periodic in periods ω and ω', such that the ratio of ω' to ω

is not purely real. A fundamental system of integrals exists in the domain of any finite value of z, and may be denoted by

$$f_1(z),\ f_2(z),\ \ldots,\ f_m(z),$$

which accordingly are linearly independent of one another.

The differential equation is unaltered when $z+\omega$ is written for z: hence

$$f_1(z+\omega),\ f_2(z+\omega),\ \ldots,\ f_m(z+\omega)$$

are integrals of the equation and, as in § 128, they constitute a fundamental system of integrals. Similarly, as the differential equation is unaltered when $z+\omega'$ is written for z,

$$f_1(z+\omega'),\ f_2(z+\omega'),\ \ldots,\ f_m(z+\omega')$$

constitute a fundamental system of integrals. Choosing therefore a region common to the domains of these three fundamental systems (a choice that always can be made because the singularities of the integrals are isolated points, finite in number within any limited portion of the plane), we have relations of the form

$$\left.\begin{array}{l} f_1\,(z+\omega\,) = a_{11}\,f_1(z) + \ldots + a_{1m}\,f_m(z) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_m(z+\omega\,) = a_{m1}f_1(z) + \ldots + a_{mm}f_m(z) \end{array}\right\},$$

and

$$\left.\begin{array}{l} f_1\,(z+\omega') = b_{11}\,f_1(z) + \ldots + b_{1m}\,f_m(z) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_m(z+\omega') = b_{m1}f_1(z) + \ldots + b_{mm}f_m(z) \end{array}\right\},$$

valid within the region chosen. The coefficients a are constant, and their determinant is not zero; the coefficients b also are constant, and their determinant also is not zero. The two sets of relations may be represented in the form

$$f(z+\omega) = Sf(z),\quad f(z+\omega') = S'f(z),$$

where S and S' denote the linear substitutions in the relations.

The coefficients in the two substitutions are not entirely independent of one another. We manifestly have

$$f_r\{(z+\omega\,)+\omega'\} = f_r\{(z+\omega')+\omega\},$$

for all values of r. The symbolic expression of this property is

$$f\{(z+\omega\,)+\omega'\} = S'f(z+\omega\,) = S'Sf(z),$$
$$f\{(z+\omega')+\omega\,\} = Sf(z+\omega') = SS'f(z),$$

so that

$$SS' = S'S,$$

or the linear substitutions are interchangeable. The explicit expression of relations between the constants is obtainable from the equation

$$\begin{aligned} b_{r1} f_1(z+\omega) + b_{r2} f_2(z+\omega) + \ldots + b_{rm} f_m(z+\omega) \\ = f_r(z+\omega+\omega') \\ = a_{r1} f_1(z+\omega') + a_{r2} f_2(z+\omega') + \ldots + a_{rm} f_m(z+\omega'), \end{aligned}$$

by substituting for the functions $f(z+\omega)$ in the left-hand side and the functions $f(z+\omega')$ in the right-hand side. The result must be an identity, for otherwise there would be a linear relation between the members of the fundamental system $f_1(z), \ldots, f_m(z)$; hence, comparing the coefficients of $f_s(z)$ on the two sides after substitution, we have

$$\begin{aligned} b_{r1} a_{1s} + b_{r2} a_{2s} + \ldots + b_{rm} a_{ms} = a_{r1} b_{1s} + a_{r2} b_{2s} + \ldots + a_{rm} b_{ms} \\ = c_{rs}, \end{aligned}$$

say. This holds for the m^2 equations that arise from the values $r, s = 1, \ldots, m$. Of the m^2 equations, only $m^2 - m$ are independent of one another, a statement the verification of which (alike in general, and for the special values $m=2$, $m=3$) is left to the reader: it can also be inferred from some equations which will be obtained immediately. The number of the relations is less important, than their existence and their form, for the establishment of Picard's theorem relating to integrals with the characteristic property of doubly-periodic functions of the second kind.

Consider a linear combination of the members of the fundamental system in the form

$$F(z) = \lambda_1 f_1(z) + \lambda_2 f_2(z) + \ldots + \lambda_m f_m(z),$$

where λ_1 will be taken as equal to unity when it is not bound to be zero; and let the constants $\lambda_2, \ldots, \lambda_m$ be chosen so that, if possible, the relation

$$F(z+\omega) = \theta F(z)$$

is satisfied, θ being some constant. To this end, we must have

$$\begin{aligned} \lambda_1 \theta &= \lambda_1 a_{11} + \lambda_2 a_{21} + \lambda_3 a_{31} + \ldots + \lambda_m a_{m1}, \\ \lambda_2 \theta &= \lambda_1 a_{12} + \lambda_2 a_{22} + \lambda_3 a_{32} + \ldots + \lambda_m a_{m2}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \lambda_m \theta &= \lambda_1 a_{1m} + \lambda_2 a_{2m} + \lambda_3 a_{3m} + \ldots + \lambda_m a_{mm}, \end{aligned}$$

and therefore

$$\Omega(\theta) = \begin{vmatrix} a_{11}-\theta, & a_{21}\ , & \dots, & a_{m1} \\ a_{12}\ , & a_{22}-\theta, & \dots, & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1m}\ , & a_{2m}\ , & \dots, & a_{mm}-\theta \end{vmatrix} = 0,$$

the equation satisfied by θ.

As in the case of the single period in § 128, it may be proved that this equation is independent of the original choice of the fundamental system of integrals $f_1(z), \dots, f_m(z)$. *The coefficients of the various powers of θ are therefore invariants,* and the equation is called *the fundamental equation for the period ω.*

Now let

$$\begin{aligned} \lambda_1\theta' &= \lambda_1 b_{11} + \lambda_2 b_{21} + \lambda_3 b_{31} + \dots + \lambda_m b_{m1}, \\ \mu_2 &= \lambda_1 b_{12} + \lambda_2 b_{22} + \lambda_3 b_{32} + \dots + \lambda_m b_{m2}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \mu_m &= \lambda_1 b_{1m} + \lambda_2 b_{2m} + \lambda_3 b_{3m} + \dots + \lambda_m b_{mm}. \end{aligned}$$

Multiply the earlier equations, which define the quantities λ and lead to the equation $\Omega(\theta) = 0$, by $b_{1r}, b_{2r}, \dots, b_{mr}$ respectively, and add: then

$$\begin{aligned} \theta\mu_r = &\quad \lambda_1\ (a_{11}\ b_{1r} + a_{12}\ b_{2r} + \dots + a_{1m}\ b_{mr}) \\ &+ \lambda_2\ (a_{21}\ b_{1r} + a_{22}\ b_{2r} + \dots + a_{2m}\ b_{mr}) \\ &+ \dots\dots\dots\dots\dots\dots\dots\dots \\ &+ \lambda_m\ (a_{m1} b_{1r} + a_{m2} b_{2r} + \dots + a_{mm} b_{mr}) \\ = &\quad \lambda_1\ (b_{11}\ a_{1r} + b_{12}\ a_{2r} + \dots + b_{1m}\ a_{mr}) \\ &+ \lambda_2\ (b_{21}\ a_{1r} + b_{22}\ a_{2r} + \dots + b_{2m}\ a_{mr}) \\ &+ \dots\dots\dots\dots\dots\dots\dots\dots \\ &+ \lambda_m\ (b_{m1} a_{1r} + b_{m2} a_{2r} + \dots + b_{mm} a_{mr}) \\ = &\quad \lambda_1 a_{1r}\theta' + a_{2r}\mu_2 + a_{3r}\mu_3 + \dots + a_{mr}\mu_m. \end{aligned}$$

This holds for all values of r; and thus we have

$$\begin{aligned} \lambda_1\theta &= \lambda_1 a_{11} + \frac{\mu_2}{\theta'} a_{21} + \frac{\mu_3}{\theta'} a_{31} + \dots + \frac{\mu_m}{\theta'} a_{m1}, \\ \frac{\mu_2}{\theta'}\theta &= \lambda_1 a_{12} + \frac{\mu_2}{\theta'} a_{22} + \frac{\mu_3}{\theta'} a_{32} + \dots + \frac{\mu_m}{\theta'} a_{m2}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{\mu_m}{\theta'}\theta &= \lambda_1 a_{1m} + \frac{\mu_2}{\theta'} a_{2m} + \frac{\mu_3}{\theta'} a_{3m} + \dots + \frac{\mu_m}{\theta'} a_{mm}. \end{aligned}$$

When these are compared with the earlier equations, we have uniquely

$$\frac{\mu_r}{\theta'} = \lambda_r,$$

for all values of r; and therefore the same values of $\lambda_2, \ldots, \lambda_m$, that enable the equations connected with the period ω to be satisfied, lead to the equations

$$\begin{aligned}
\lambda_1 \theta' &= \lambda_1 b_{11} + \lambda_2 b_{21} + \lambda_3 b_{31} + \ldots + \lambda_m b_{m1}, \\
\lambda_2 \theta' &= \lambda_1 b_{12} + \lambda_2 b_{22} + \lambda_3 b_{32} + \ldots + \lambda_m b_{m2}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\lambda_m \theta' &= \lambda_1 b_{1m} + \lambda_2 b_{2m} + \lambda_3 b_{3m} + \ldots + \lambda_m b_{mm}.
\end{aligned}$$

Hence

$$\begin{aligned}
F(z+\omega') &= \lambda_1 f_1(z+\omega') + \lambda_2 f_2(z+\omega') + \ldots + \lambda_m f_m(z+\omega') \\
&= \theta' \{\lambda_1 f_1(z) + \lambda_2 f_2(z) + \ldots + \lambda_m f_m(z)\} \\
&= \theta' F(z),
\end{aligned}$$

on using the preceding equations. Moreover, this multiplier θ' satisfies the equation

$$\Omega'(\theta') = \begin{vmatrix} b_{11} - \theta', & b_{21}, & \ldots, & b_{m1} \\ b_{12}, & b_{22} - \theta', & \ldots, & b_{m2} \\ \ldots & \ldots & \ldots & \ldots \\ b_{1m}, & b_{2m}, & \ldots, & b_{mm} - \theta' \end{vmatrix} = 0.$$

This equation, like $\Omega(\theta) = 0$, is independent of the initial choice of the fundamental system of integrals $f_1(z), \ldots, f_m(z)$, the proof being similar to that in § 128. *The coefficients of the various powers of θ' are therefore invariants*; and the equation is called *the fundamental equation for the period ω'.*

The term independent of θ in $\Omega(\theta)$, and the term independent of θ' in $\Omega'(\theta')$, can be obtained simply. Let $\Delta(z)$ denote the determinant

$$\Delta(z) = \begin{vmatrix} \dfrac{d^{m-1} f_1}{dz^{m-1}}, & \dfrac{d^{m-1} f_2}{dz^{m-1}}, & \ldots, & \dfrac{d^{m-1} f_m}{dz^{m-1}} \\ \ldots & \ldots & \ldots & \ldots \\ \dfrac{df_1}{dz}, & \dfrac{df_2}{dz}, & \ldots, & \dfrac{df_m}{dz} \\ f_1, & f_2, & \ldots, & f_m \end{vmatrix};$$

then, as in § 9, we have

$$\Delta(z) = \Delta(z_0)\, e^{\int_{z_0}^{z} p_1(x)\,dx}.$$

Hence

$$\Delta(z+\omega) = \Delta(z_0)\, e^{\int_{z_0}^{z+\omega} p_1(x)\,dx},$$

so that

$$\frac{\Delta(z+\omega)}{\Delta(z)} = e^{\int_{z}^{z+\omega} p_1(x)\,dx};$$

and similarly

$$\frac{\Delta(z+\omega')}{\Delta(z)} = e^{\int_{z}^{z+\omega'} p_1(x)\,dx},$$

where we manifestly may assume that the path of integration does not approach infinitesimally near the singularities of p_1. Now p_1 is a uniform doubly-periodic function with no essential singularity in the finite part of the plane; if, therefore, $a_1, \ldots, a_n$ denote its irreducible poles, and if $\zeta(z)$ denote the usual Weierstrassian function in the same periods ω and ω' as p_1, we have*

$$p_1 = C + \sum_{r=1}^{n} A_r \zeta(z-a_r) + \sum_{r=1}^{n} B_r \frac{d\zeta(z-a_r)}{dz} + \sum_{r=1}^{n} C_r \frac{d^2\zeta(z-a_r)}{dz^2} + \ldots,$$

with the condition

$$\Sigma A_r = 0.$$

Now

$$\begin{aligned}\int_z^{z+\omega} p_1(x)\,dx &= C\omega + \sum_{r=1}^{n} A_r \log \frac{\sigma(z+\omega-a_r)}{\sigma(z-a_r)} \\ &\quad + \sum_{r=1}^{n} B_r \{\zeta(z+\omega-a_r) - \zeta(z-a_r)\} \\ &\quad - \sum_{r=1}^{n} C_r \{\wp(z+\omega-a_r) - \wp(z-a_r)\} - \ldots \\ &= C\omega + \sum_{r=1}^{n} A_r \{i\pi + \eta\omega + 2\eta(z-a_r)\} + \sum_{r=1}^{n} 2\eta B_r \\ &= C\omega - 2\eta \sum_{r=1}^{n} A_r a_r + 2\eta \sum_{r=1}^{n} B_r = D,\end{aligned}$$

* *T. F.*, § 129.

say; and similarly

$$\int_z^{z+\omega'} p_1(x)\,dx = C\omega' - 2\eta' \sum_{r=1}^{n} A_r a_r + 2\eta' \sum_{r=1}^{n} B_r = D',$$

say. But, substituting in $\Delta(z+\omega)$ the expressions for $f_1(z+\omega)$, ..., $f_m(z+\omega)$ and their derivatives in terms of $f_1(z)$, ..., $f_m(z)$ and their derivatives, we have

$$\frac{\Delta(z+\omega)}{\Delta(z)} = \begin{vmatrix} a_{11}, & a_{12}, & \ldots, & a_{1m} \\ a_{21}, & a_{22}, & \ldots, & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}, & a_{m2}, & \ldots, & a_{mm} \end{vmatrix},$$

which is the non-vanishing constant term in $\Omega(\theta)$. Thus

$$\Omega(\theta) = e^D + \ldots + (-1)^m \theta^m;$$

and similarly

$$\Omega'(\theta') = e^{D'} + \ldots + (-1)^m \theta'^m.$$

In particular, when p_1 is zero, so that the differential equation has no term in $\dfrac{d^{m-1}w}{dz^{m-1}}$, we have $D = 0$, $D' = 0$; and then

$$\Omega(\theta) = 1 + \ldots + (-1)^m \theta^m, \quad \Omega'(\theta') = 1 + \ldots + (-1)^m \theta'^m.$$

Integrals which are Doubly-periodic Functions of the Second Kind.

142. Let θ be a root of the equation $\Omega(\theta) = 0$. Then quantities $\lambda_2, \ldots, \lambda_m$ exist such that the equations leading to $\Omega(\theta) = 0$ are satisfied; and a quantity θ' is obtained, when the values of $\lambda_2, \ldots, \lambda_m$ are substituted in its expression. It thus follows that there is an integral $F(z)$ of the differential equation such that

$$F(z+\omega) = \theta F(z), \quad F(z+\omega') = \theta' F(z),$$

where θ and θ' are constants. Such a function is called* doubly-periodic of the second kind: and therefore it follows that *a linear differential equation, which has uniform doubly-periodic functions for its coefficients, possesses an integral which is a doubly-periodic function of the second kind:* a result first given by Picard.

* *T. F.*, § 136.

When θ is a simple root of the equation $\Omega(\theta)=0$, then $\lambda_2, \ldots, \lambda_m$ are uniquely determinate: and θ' is uniquely determinate. When θ is a multiple root of its equation, quantities $\lambda_2, \ldots, \lambda_m$ exist satisfying the associated equations but they are not uniquely determinate: and assigned values of $\lambda_2, \ldots, \lambda_m$ determine θ'. Similarly for θ' as a root of the equation $\Omega'(\theta')=0$. Combining these results, we have the theorem*:

A linear differential equation, having doubly-periodic functions for its coefficients, possesses at least as many integrals which are doubly-periodic functions of the second kind as either of the equations $\Omega(\theta)=0$, $\Omega'(\theta')=0$ *has distinct roots.*

By using the elementary divisors of $\Omega(\theta)=0$, we can obtain a more exact estimate of the number of integrals which are periodic functions of the second kind, associated with a multiple root.

Let θ_1 be a root of $\Omega(\theta)=0$ of multiplicity λ_1, and let n_1 be the number of different elementary divisors of $\Omega(\theta)$ which are powers of $\theta-\theta_1$, so that the minors of $\Omega(\theta)$ of order n_1 are the first in successively increasing order which do not vanish simultaneously when $\theta=\theta_1$. Then (§ 133) the number of integrals, which satisfy the equation

$$I(z+\omega)=\theta_1 I(z),$$

is precisely equal to n_1.

* These equations appear to have been considered first by Picard in general; see *Comptes Rendus*, t. XC (1880), pp. 128—131, 293—295; *Crelle*, t. XC (1880), pp. 281—302.

Their properties were further developed by Floquet, *Comptes Rendus*, t. XCVIII (1884), pp. 82—85, *Ann. de l'Éc. Norm. Sup.*, 3me Sér., t. I (1884), pp. 181—238, which should be consulted in connection with many of the following investigations.

A proof of Picard's theorem, different from that in the text, is given by Barnes, *Messenger of Mathematics*, t. XXVII (1897), pp. 16, 17.

Investigations of a different kind, leading to equations the primitives of which are expressible in terms of doubly-periodic functions, are carried out in Halphen's memoir "Sur la réduction des équations différentielles linéaires aux formes intégrables," *Mém. des Sav. Étrang.*, t. XXVIII (1882), No. 1, 301 pp.; particularly, chapters II and IX.

The most important equation of the type under consideration is the general form of Lamé's equation. It had been considered by Hermite, previous to Picard's investigations; and it has formed the subject of many memoirs, references to some of which will be found in my *Theory of Functions*, §§ 137—141.

Moreover, in that case, n_1 of the equations in § 141 for determining the quantities λ are dependent upon the remaining $m-n_1$. Let the last $m-n_1$ be a set of independent equations, determining $\lambda_{n_1+1}, \ldots, \lambda_m$ in terms of $\lambda_1, \lambda_2, \ldots, \lambda_{n_1}$; and suppose that the expressions are

$$\lambda_s = k_{s1}\lambda_1 + k_{s2}\lambda_2 + k_{s3}\lambda_3 + \ldots + k_{sn_1}\lambda_{n_1},$$

for $s = n_1+1, n_1+2, \ldots, m$. Then

$$\begin{aligned} F(z) &= \lambda_1 f_1(z) + \lambda_2 f_2(z) + \ldots + \lambda_m f_m(z) \\ &= \lambda_1 g_1(z) + \lambda_2 g_2(z) + \ldots + \lambda_{n_1} g_{n_1}(z), \end{aligned}$$

where

$$g_r(z) = f_r(z) + \sum_{s=n_1+1}^{m} k_{sr} f_s(z),$$

for $r = 1, 2, \ldots, n_1$; and each of the functions $g_1, \ldots, g_{n_1}$ is such that

$$g_r(z+\omega) = \theta_1 g_r(z).$$

As regards the possible multiplier θ_1' for the other period, we have

$$\begin{aligned} \theta_1' &= \lambda_1 b_{11} + \lambda_2 b_{21} + \lambda_3 b_{31} + \ldots + \lambda_m b_{m1} \\ &= \lambda_1 B_1 + \lambda_2 B_2 + \ldots + \lambda_{n_1} B_{n_1}, \end{aligned}$$

say, where

$$B_r = b_{r1} + \sum_{s=n_1+1}^{m} k_{sr} b_{s1};$$

and the effect upon $F(z)$ of the increase of argument by the period ω' is given by

$$F(z+\omega') = \theta_1' F(z).$$

Now θ_1' is not zero, for it is a root of $\Omega'(\theta') = 0$ which has no zero root; and therefore not all the quantities $B_1, B_2, \ldots, B_{n_1}$ can vanish. Let $B_1, B_2, \ldots, B_s$ be those which do not vanish; then we have

$$\begin{aligned} &\lambda_1 g_1(z+\omega) + \lambda_2 g_2(z+\omega) + \ldots + \lambda_{n_1} g_{n_1}(z+\omega) \\ &\quad = (\lambda_1 B_1 + \lambda_2 B_2 + \ldots + \lambda_s B_s)\{\lambda_1 g_1(z) + \lambda_2 g_2(z) + \ldots + \lambda_{n_1} g_{n_1}(z)\}. \end{aligned}$$

As some one of the quantities $\lambda_1, \lambda_2, \ldots, \lambda_{n_1}$ is not zero (for, thus far, all these quantities are arbitrary), we shall take $\lambda_1 = 1$. In order that this equation may hold, we assign definite values to $\lambda_2, \ldots, \lambda_s$; we write

$$\begin{aligned} B_1 + \lambda_2 B_2 + \ldots + \lambda_s B_s &= \theta_1', \\ g_1(z) + \lambda_2 g_2(z) + \ldots + \lambda_s g_s(z) &= G(z), \end{aligned}$$

and then, as $\lambda_{s+1}, \ldots, \lambda_{n_1}$ can remain arbitrary, we have

$$G(z+\omega') = \theta_1' G(z),$$
$$g_r(z+\omega') = \theta_1' g_r(z),$$

for $r = s+1, \ldots, n_1$. Moreover, on account of the composition of $G(z)$, we have

$$G(z+\omega) = \theta G(z),$$

and we had

$$g_r(z+\omega) = \theta g_r(z).$$

Accordingly, *the number of integrals, which are doubly-periodic functions of the second kind and are associated with the multiple root* θ_1 *of the fundamental equation* $\Omega(\theta) = 0$, *is*

$$1 + n_1 - s,$$

where n_1 *is the number of elementary divisors of* $\Omega(\theta)$ *which are powers of* $\theta - \theta_1$, *and* s *is the number of quantities*

$$b_{r1} + \sum_{s=n_1+1}^{m} k_{sr} b_{s1}$$

which do not vanish, so that $0 < s \leqslant n_1$.

143. We now can indicate the total number of integrals, which are doubly-periodic functions of the second kind.

Let θ_1 be a root of multiplicity λ_1 of $\Omega(\theta) = 0$, and let it give rise to n_1 elementary divisors of $\Omega(\theta)$ which are powers of $\theta - \theta_1$; and let s_1 be the number of quantities

$$b_{r1} + \sum_{s=n_1+1}^{m} k_{sr} b_{s1}$$

in the preceding investigation which do not vanish, so that

$$0 < s_1 \leqslant n_1 \leqslant \lambda_1.$$

Let $\theta_2, \theta_3, \ldots$ be other multiple roots; and let λ_2, n_2, s_2; λ_3, n_3, s_3; … be the numbers for them, corresponding to λ_1, n_1, s_1 for θ_1; so that

$$\lambda_1 + \lambda_2 + \lambda_3 + \ldots = m.$$

Then the number of integrals, which are doubly-periodic functions of the second kind, is

$$\sum_{r=1} (1 + n_r - s_r).$$

In particular, if the roots of $\Omega(\theta)=0$ be all distinct from one another, a fundamental system can be composed of m integrals, each of which is a doubly-periodic function of the second kind; the constant multipliers are the m roots of $\Omega(\theta)=0$, and the corresponding quantities θ' derived from them, these quantities θ' themselves satisfying the equation $\Omega'(\theta')=0$.

Moreover, the relation between the equations satisfied by θ and $\lambda_1, \ldots, \lambda_m$, and the equations satisfied by θ' and $\lambda_1, \ldots, \lambda_m$, is reciprocal; for each set can be constructed from the other as in § 141. Hence, if either of the equations $\Omega(\theta)=0$ and $\Omega'(\theta')=0$ has all its roots distinct from one another, there is no necessity to take account of possible multiplicity of the roots of the other, so far as the present purpose is concerned: the implication merely is that one of the two multipliers has the same value for several of the integrals.

Further, if θ and θ' are two associated multipliers, each of them arising as repeated roots of their respective equations, we shall suppose, for the same reason as in the preceding case, that the construction of the doubly-periodic functions of the second kind is initially associated with that one of the two equations which has the repeated root in the smaller multiplicity.

Multiple Roots of the Fundamental Equations and Associated Integrals.

144. We have now to consider the form of the integrals associated with a multiple root of $\Omega(\theta)=0$, the fundamental equation for the period ω; and we assume that the corresponding root of $\Omega'(\theta')=0$ is also multiple, to at least as great an order of multiplicity. Denoting this root by θ, and the corresponding root of $\Omega'(\theta')=0$ by θ', we know that there certainly is one integral, which is doubly-periodic of the second kind and has multipliers θ and θ'; let it be denoted by ϕ_1, so that

$$\phi_1(z+\omega)=\theta\phi_1(z), \quad \phi_1(z+\omega')=\theta'\phi_1(z).$$

Considering the integrals first in relation to the period ω, we know (§ 134) that the number of them associated with the multiple root θ is equal to the order of multiplicity of θ: and

further, that this group of integrals is linearly equivalent to sub-groups of integrals of the form

$$\begin{aligned}
u_1 &= \phi_1, \\
u_2 &= \phi_2 + z\phi_1, \\
u_3 &= \phi_3 + 2z\phi_2 + z^2\phi_1, \\
u_4 &= \phi_4 + 3z\phi_3 + 3z^2\phi_2 + z^3\phi_1, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots;
\end{aligned}$$

the aggregate number of integrals in the various sub-groups is equal to the order of the multiplicity of θ, and each of the functions ϕ is such that

$$\phi(z+\omega) = \theta\phi(z).$$

In these integrals, ϕ_2 can have any added constant multiple of ϕ_1; also ϕ_3 can have any linear combination of constant multiples of ϕ_2 and ϕ_1; and so on. All the functions ϕ, so changed, still have the multiplier θ for the period ω.

Now u_1 has the multiplier θ' for the period ω'. The simplest case arises when some other integral of the group, say u_r, also has this multiplier θ' for the period ω': for then all the intervening integrals have this multiplier for the period ω'. What is necessary to secure this result is that, first,

$$\phi_2(z+\omega') + (z+\omega')\,\phi_1(z+\omega') = \theta'\,\{\phi_2(z) + z\phi_1(z)\},$$

that is,

$$\phi_2(z+\omega') + \omega'\phi_1(z+\omega') = \theta'\phi_2(z),$$

and therefore

$$\frac{\phi_2(z+\omega')}{\phi_1(z+\omega')} + \omega' = \frac{\phi_2(z)}{\phi_1(z)}.$$

Secondly, we must have

$$\begin{aligned}
\phi_3(z+\omega') + 2(z+\omega')\,\phi_2(z+\omega') &+ (z+\omega')^2\,\phi_1(z+\omega') \\
&= \theta'\,\{\phi_3(z) + 2z\phi_2(z) + z^2\phi_1(z)\},
\end{aligned}$$

which, in connection with the preceding equations, is satisfied if

$$\phi_3(z+\omega') + 2\omega'\phi_2(z+\omega') + \omega'^2\phi_1(z+\omega') = \theta'\phi_3(z),$$

that is, if

$$\frac{\phi_3(z+\omega')}{\phi_1(z+\omega')} + 2\omega'\frac{\phi_2(z+\omega')}{\phi_1(z+\omega')} + \omega'^2 = \frac{\phi_3(z)}{\phi_1(z)}.$$

Similarly, we must have

$$\frac{\phi_4(z+\omega')}{\phi_1(z+\omega')}+3\omega'\frac{\phi_3(z+\omega')}{\phi_1(z+\omega')}+3\omega'^2\frac{\phi_2(z+\omega')}{\phi_1(z+\omega')}+\omega'^3=\frac{\phi_4(z)}{\phi_1(z)},$$

and so on.

Let $\zeta(z)$ denote the usual Weierstrassian ζ-function, with periods ω and ω'; and let η, η' denote the increments of $\zeta(z)$ for an increase of z by the respective periods, so that we have

$$\eta\omega'-\eta'\omega=\pm 2\pi i,$$

the sign being the same as that of the real part of $\omega'\div i\omega$. Then, if a function $u(z)$ be defined by the equation

$$u(z)=\pm\frac{\omega\omega'}{2\pi i}\left\{\frac{\eta}{\omega}z-\zeta(z)\right\},$$

we have

$$u(z+\omega)=u(z),$$

$$u(z+\omega')=u(z)+\omega'.$$

Then we have

$$\frac{\phi_2(z+\omega')}{\phi_1(z+\omega')}+u(z+\omega')=\frac{\phi_2(z)}{\phi_1(z)}+u(z),$$

that is, the function on the right-hand side is periodic in ω'. Moreover, ϕ_2 and ϕ_1 have the same multiplier for ω, and $u(z)$ is periodic in ω; hence the function on the right-hand side is periodic in ω also. It thus is a doubly-periodic function of the first kind; denoting it by ψ_2, we have

$$\phi_2=\phi_1\psi_2-u\phi_1,$$

so that $\phi_1\psi_2$ is a doubly-periodic function of the second kind, with the same multipliers as ϕ_1, viz. θ and θ'.

Similarly, we have

$$\frac{\phi_3(z+\omega')}{\phi_1(z+\omega')}+2u(z+\omega')\frac{\phi_2(z+\omega')}{\phi_1(z+\omega')}+\{u(z+\omega')\}^2$$

$$=\frac{\phi_3(z)}{\phi_1(z)}+2u(z)\frac{\phi_2(z)}{\phi_1(z)}+\{u(z)\}^2,$$

so that the function on the right-hand side manifestly is periodic in ω'; and it is periodic in ω on account of the properties of u and

the functions ϕ. Denoting this doubly-periodic function of the first kind by ψ_3, we have

$$\phi_3 = \phi_1\psi_3 - 2u\phi_1\psi_2 + u^2\phi_1.$$

And so on in succession. The group of integrals, in the case suggested, can be represented in the form

$$\begin{aligned}
&\phi_1,\\
&\chi_2 - u\phi_1,\\
&\chi_3 - 2u\chi_2 + u^2\phi_1,\\
&\chi_4 - 3u\chi_3 + 3u^2\chi_2 - u^3\phi_1,\\
&\vdots
\end{aligned}$$

where the functions $\phi_1, \chi_2, \chi_3, \chi_4, \ldots$ are doubly-periodic functions of the second kind with the multipliers θ and θ', and

$$u(z) = \pm \frac{\omega\omega'}{2\pi i}\left\{\frac{\eta}{\omega} z - \zeta(z)\right\}.$$

145. Returning now to the less simple case, when not more than one of the integrals associated with the corresponding multiple roots can be assumed to be doubly-periodic of the second kind, we know that one integral certainly exists in the form of a doubly-periodic function of the second kind with the multipliers θ and θ'. Denoting it by $\phi_1(z)$, we use it to replace some one of the integrals, say $f_1(z)$, in the fundamental system, which then becomes

$$\phi_1(z),\ f_2(z),\ \ldots,\ f_m(z).$$

We have

$$\begin{aligned}
\phi_1(z+\omega) &= \theta\phi_1(z),\\
f_2(z+\omega) &= c_{21}\phi_1(z) + c_{22}f_2(z) + \ldots + c_{2m}f_m(z),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
f_m(z+\omega) &= c_{m1}\phi_1(z) + c_{m2}f_2(z) + \ldots + c_{mm}f_m(z).
\end{aligned}$$

The fundamental equation for the period ω is

$$\Omega(x) = \begin{vmatrix} \theta - x, & 0, & 0, & \ldots, & 0 \\ c_{21}, & c_{22}-x, & c_{23}, & \ldots, & c_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{m1}, & c_{m2}, & c_{m3}, & \ldots, & c_{mm}-x \end{vmatrix} = 0;$$

and so θ is a root of

$$\Omega_1(x) = \begin{vmatrix} c_{22}-x, & c_{23}, & \ldots, & c_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ c_{m2}, & c_{m3}, & \ldots, & c_{mm}-x \end{vmatrix} = 0,$$

of multiplicity less by one than its multiplicity for $\Omega(x) = 0$.

Similarly, we have relations of the form

$$\phi_1(z+\omega') = \theta'\phi_1(z),$$
$$f_2(z+\omega') = d_{21}\,\phi_1(z) + d_{22}\,f_2(z) + \ldots + d_{2m}\,f_m(z),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$f_m(z+\omega') = d_{m1}\phi_1(z) + d_{m2}f_2(z) + \ldots + d_{mm}f_m(z):$$

the multiplier θ' being a root of

$$\Omega_1'(x) = \begin{vmatrix} d_{22}-x, & d_{23}, & \ldots, & d_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ d_{m2}, & d_{m3}, & \ldots, & d_{mm}-x \end{vmatrix} = 0,$$

of multiplicity less by one than its multiplicity for $\Omega'(x) = 0$.

The coefficients c_{ij} and d_{ij} must satisfy conditions in order to have

$$f_\kappa\{(z+\omega')+\omega\} = f_\kappa\{(z+\omega)+\omega'\},$$

for all values of κ: these conditions are

$$d_{r1}c_{1s} + d_{r2}c_{2s} + \ldots + d_{rm}c_{ms} = c_{r1}d_{1s} + c_{r2}d_{2s} + \ldots + c_{rm}d_{ms}$$
$$= k_{rs},$$

say, with the limitations

$$c_{11} = \theta,\ \ d_{11} = \theta';\quad c_{1s} = 0,\ \ d_{1s} = 0,\ \text{ if } s > 1.$$

Owing to the fact that θ is a root of $\Omega_1(x) = 0$, quantities $\kappa_3, \ldots, \kappa_m$ exist such that

$$\theta\kappa_s = c_{2s} + c_{3s}\kappa_3 + \ldots + c_{ms}\kappa_m, \qquad (s = 3, \ldots, m),$$
$$\theta = c_{22} + c_{32}\kappa_3 + \ldots + c_{m2}\kappa_m.$$

Let

$$\vartheta = d_{22} + d_{32}\kappa_3 + \ldots + d_{m2}\kappa_m,$$
$$\sigma_r = d_{2r} + d_{3r}\kappa_3 + \ldots + d_{mr}\kappa_m, \qquad (r = 3, \ldots, m);$$

then

$$\begin{aligned}
\theta\sigma_r = {} & d_{2r}\,(c_{22} + c_{32}\,\kappa_3 + \ldots + c_{m2}\,\kappa_m) \\
& + d_{3r}\,(c_{23} + c_{33}\,\kappa_3 + \ldots + c_{m3}\,\kappa_m) \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& + d_{mr}\,(c_{2m} + c_{3m}\kappa_3 + \ldots + c_{mm}\kappa_m) \\
= {} & \quad d_{22}\,c_{2r} + d_{23}\,c_{3r} + \ldots + d_{2m}\,c_{mr} \\
& + \kappa_3\,(d_{32}\,c_{2r} + d_{33}\,c_{3r} + \ldots + d_{3m}\,c_{mr}) \\
& + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& + \kappa_m\,(d_{m2}c_{2r} + d_{m3}c_{3r} + \ldots + d_{mm}c_{mr}) \\
= {} & c_{2r}\vartheta + c_{3r}\sigma_3 + c_{4r}\sigma_4 + \ldots + c_{mr}\sigma_m;
\end{aligned}$$

and therefore

$$\theta \frac{\sigma_r}{\vartheta} = c_{2r} + c_{3r}\frac{\sigma_3}{\vartheta} + c_{4r}\frac{\sigma_4}{\vartheta} + \ldots + c_{mr}\frac{\sigma_m}{\vartheta},$$

holding for all values of r. Comparing with the earlier equations in c, we have

$$\sigma_r = \vartheta\kappa_r,$$

for all values of r; and thus the second set of equations is

$$\vartheta = d_{22} + d_{32}\kappa_3 + \ldots + d_{m2}\kappa_m,$$

$$\vartheta\kappa_r = d_{2r} + d_{3r}\kappa_3 + \ldots + d_{mr}\kappa_m, \qquad (r = 3, \ldots, m).$$

Eliminating the quantities κ, we have

$$\Omega_1'(\vartheta) = 0,$$

so that θ' is the value of ϑ.

Now consider the integral

$$\chi_2(z) = f_2(z) + \kappa_3 f_3(z) + \ldots + \kappa_m f_m(z).$$

We have

$$\begin{aligned}\chi_2(z+\omega) &= f_2(z+\omega) + \kappa_3 f_3(z+\omega) + \ldots + \kappa_m f_m(z+\omega)\\ &= (c_{21} + \kappa_3 c_{31} + \ldots + \kappa_m c_{m1})\,\phi_1(z) + \theta\chi_2(z)\\ &= a_2\phi_1(z) + \theta\chi_2(z),\end{aligned}$$

say, and

$$\begin{aligned}\chi_2(z+\omega') &= f_2(z+\omega') + \kappa_3 f_3(z+\omega') + \ldots + \kappa_m f_m(z+\omega')\\ &= (d_{21} + \kappa_3 d_{31} + \ldots + \kappa_m d_{m1})\,\phi_1(z) + \theta'\chi_2(z)\\ &= b_2\phi_1(z) + \theta'\chi_2(z),\end{aligned}$$

say. When a_2 and b_2 vanish, χ_2 is doubly-periodic of the second kind: but in the general case, a_2 and b_2 are distinct from zero. The property

$$\chi_2\{(z+\omega)+\omega'\} = \chi_2\{(z+\omega')+\omega\}$$

leads to no relation between a_2 and b_2.

If the multiplicity of θ as a root of $\Omega(\theta) = 0$ and that of θ' as a root of $\Omega'(\theta') = 0$ be greater than 2, so that θ and θ' are multiple roots of $\Omega_1(x) = 0$, $\Omega_1'(x) = 0$, we proceed as above. The newly obtained integral χ_2 is used to modify the fundamental system by replacing f_2, say, so that the system consists of

$$\phi_1,\ \chi_2,\ f_3,\ \ldots,\ f_m.$$

Then, in the same way as above, it is proved that an integral χ_3 exists such that

$$\chi_3(z+\omega) = a_3\phi_1 + c_3\chi_2 + \theta\chi_3,$$
$$\chi_3(z+\omega') = b_3\phi_1 + d_3\chi_2 + \theta'\chi_3.$$

Since

$$\chi_3\{(z+\omega)+\omega'\} = \chi_3\{(z+\omega')+\omega\},$$

we find, on substitution,

$$b_2 c_3 = a_2 d_3,$$

so that we may take

$$c_3 = \lambda a_2, \quad d_3 = \lambda b_2,$$

where λ is any parameter. This parameter may clearly be absorbed into χ_3 by taking $\chi_3 \div \lambda$, and also into a_3 and b_3 by division. Thus our integrals ϕ_1, χ_2, χ_3 are such that

$$\phi_1(z+\omega) = \theta\phi_1,$$
$$\chi_2(z+\omega) = a_2\phi_1 + \theta\chi_2,$$
$$\chi_3(z+\omega) = a_3\phi_1 + a_2\chi_2 + \theta\chi_3,$$
$$\phi_1(z+\omega') = \theta'\phi_1,$$
$$\chi_2(z+\omega') = b_2\phi_1 + \theta'\chi_2,$$
$$\chi_3(z+\omega') = b_3\phi_1 + b_2\chi_2 + \theta'\chi_3.$$

And so on, until a number of integrals is obtained equal to the lesser of the orders of multiplicity of θ and θ'. Thus the next integral is χ_4 (say), where

$$\chi_4(z+\omega) = a_4\phi_1 + (a_3 + \lambda a_2)\chi_2 + a_2\chi_3 + \theta\chi_4,$$
$$\chi_4(z+\omega') = b_4\phi_1 + (b_3 + \lambda b_2)\chi_2 + b_2\chi_3 + \theta'\chi_4.$$

146. From these descriptive forms, we can proceed one stage towards the construction of an analytical form of the integrals. For this purpose, we introduce (as in § 144) the functions

$$u(z) = \pm\frac{\omega}{2\pi i}\left\{\frac{\eta}{\omega}z - \zeta(z)\right\}, \quad v(z) = \pm\frac{\omega'}{2\pi i}\left\{\zeta(z) - \frac{\eta'}{\omega'}z\right\},$$

where the doubtful sign is the same as that of the real part of $\omega' \div i\omega$; we have

$$\left.\begin{array}{l} u(z+\omega) = u(z) \\ u(z+\omega') = u(z)+1 \end{array}\right\}, \quad \left.\begin{array}{l} v(z+\omega) = v(z)+1 \\ v(z+\omega') = v(z) \end{array}\right\}.$$

Then the various integrals can be expressed as non-homogeneous polynomials in $u(z)$ and $v(z)$, the coefficients of which are doubly-

periodic functions of the second kind, with θ and θ' for multipliers. In particular, the integrals have the form

$$\begin{aligned}
&\phi_1(z) = F_1(z),\\
&\chi_2(z) = F_2(z) + I_1 F_1(z),\\
&\chi_3(z) = F_3(z) + I_1 F_2(z) + \tfrac{1}{2} I_2 F_1(z),\\
&\chi_4(z) = F_4(z) + I_1 F_3(z) + \tfrac{1}{2} J_2 F_2(z) + \tfrac{1}{6} I_3 F_1(z),
\end{aligned}$$

and so on. The functions F are doubly-periodic of the second kind with factors θ and θ'; I_1 is a polynomial of the first degree in $u(z)$ and $v(z)$; I_2 and J_2 are polynomials of the second degree in the same quantities, having I_1^2 as the aggregate of their terms of the second degree; I_3 is a polynomial of the third degree in the same quantities, having I_1^3 as the aggregate of its terms of the third degree; and so on.

To prove this, we note in the first place that $\phi_1(z)$ is a doubly-periodic function of the second kind with the multipliers θ and θ'. As for $\chi_2(z)$, we have

$$\frac{\chi_2(z+\omega)}{\phi_1(z+\omega)} = \frac{\chi_2(z)}{\phi_1(z)} + \frac{a_2}{\theta},$$

$$\frac{\chi_2(z+\omega')}{\phi_1(z+\omega')} = \frac{\chi_2(z)}{\phi_1(z)} + \frac{b_2}{\theta'}.$$

If therefore we take the function $p_{21}(z)$, where

$$p_{21}(z) = \frac{a_2}{\theta} v(z) + \frac{b_2}{\theta'} u(z),$$

we have

$$\begin{aligned}
\frac{\chi_2(z+\omega)}{\phi_1(z+\omega)} - p_{21}(z+\omega) &= \frac{\chi_2(z)}{\phi_1(z)} - p_{21}(z)\\
&= \frac{\chi_2(z+\omega')}{\phi_1(z+\omega')} - p_{21}(z+\omega');
\end{aligned}$$

and therefore

$$\frac{\chi_2(z)}{\phi_1(z)} - p_{21}(z)$$

is a doubly-periodic function of the first kind. Let $F_2(z)$ denote the product of this function and $\phi_1(z)$; then $F_2(z)$ is doubly-periodic of the second kind with multipliers θ and θ', and we have

$$\chi_2(z) = F_2(z) + p_{21}(z)\,\phi_1(z).$$

Similarly, if

$$p_{31}(z) = \left\{\tfrac{1}{2}\left(\frac{a_2}{\theta}\right)^2 - \frac{a_3}{\theta}\right\} v(z) + \left\{\tfrac{1}{2}\left(\frac{b_2}{\theta'}\right)^2 - \frac{b_3}{\theta'}\right\} u(z),$$

we find

$$\frac{\chi_3(z)}{\phi_1(z)} - p_{21}(z)\frac{\chi_2(z)}{\phi_1(z)} + \tfrac{1}{2}p_{21}{}^2(z) + p_{31}(z)$$

to be a doubly-periodic function of the first kind. Let $F_3(z)$ denote the product of this function and $\phi_1(z)$; then $F_3(z)$ is doubly-periodic of the second kind with multipliers θ and θ', and we have

$$\chi_3(z) = F_3(z) + p_{21}(z)F_2(z) + \{\tfrac{1}{2}p_{21}{}^2(z) - p_{31}(z)\}\,\phi_1(z).$$

Similarly, after reduction,

$$\begin{aligned}\chi_4(z) = F_4(z) + p_{21}(z)F_3(z) &+ \{\tfrac{1}{2}p_{21}{}^2(z) - p_{41}(z)\}\,F_2(z)\\ &+ \{\tfrac{1}{6}p_{21}{}^3(z) - p_{42}(z)\}\,\phi_1(z),\end{aligned}$$

where $F_4(z)$ is a doubly-periodic function of the second kind with multipliers θ and θ', $p_{41}(z)$ is a polynomial in u and v of the first degree, and $p_{42}(z)$ is a polynomial (not homogeneous) in u and v of the second degree.

And so on, in general: the theorem is thus established.

Construction of Integrals that are Uniform.

147. Further progress in the effective determination of the analytical forms of the integrals on the basis of the foregoing properties is not possible in the general case. When particular classes of limitations are imposed upon the coefficients in the original differential equation, such progress might be possible: but it frequently happens that some more special method leads more directly to the solution.

The simplest case is that in which the equation possesses a uniform integral, or in which the equation has several uniform integrals: but, of course, the preceding investigations in §§ 141—146 apply to all equations of the type considered, whether they have uniform integrals or not. When all the integrals are uniform (and this can be determined independently by considering their forms in the vicinity of the singularities), then the

doubly-periodic functions of the second kind arising in the preceding investigation are uniform functions of z; and a general method of constructing such functions is known*. Instead, however, of using the preceding results, it sometimes is more convenient and more direct to infer the irreducible singularities of the integrals from the differential equation itself. These are used to construct an appropriate uniform doubly-periodic function of the second kind; the remaining quantities needed for the precise determination of the integral are then inferred by substituting the expression in the differential equation.

Ex. 1. Consider the equation†

$$\frac{d^3w}{dz^3} - 3\{2\wp(z) + a\}\frac{dw}{dz} + \beta w = 0,$$

with the usual notation for the Weierstrassian elliptic functions; a and β are constants.

The only irreducible singularity that an integral can have is $z=0$. The indicial equation for $z=0$ is

$$n(n-1)(n-2) - 6n = 0,$$

the roots of which are -1, 0, 4; and the expansions that respectively correspond to the roots are easily proved to be

$$w_1 = \frac{1}{z} + \tfrac{1}{2}az + \tfrac{1}{12}\beta z^2 + \left(\tfrac{1}{40}g_2 - \tfrac{1}{8}a^2\right)z^3 + \ldots,$$
$$w_2 = 1 + \tfrac{1}{12}\beta z^3 + \tfrac{1}{40}a\beta z^5 + \ldots,$$
$$w_3 = z^4 + \tfrac{1}{7}az^6 + \ldots.$$

Thus no logarithms are involved; every integral is a uniform function of z, being of the form $Aw_1 + Bw_2 + Cw_3$; and at least one integral of the equation is thus a uniform doubly-periodic function of the second kind. We proceed to its construction.

This doubly-periodic function of the second kind cannot be devoid of poles, if it is to involve the first of the above integrals in its expression. (If it were devoid of poles, it would also‡ be devoid of zeros in the finite part of the plane: and then (*l.c.*) it could only be an exponential of the form $e^{\kappa z}$, which is manifestly not a solution of our equation.) It has one irreducible pole; it therefore has one irreducible zero in the finite part of the plane. Let the latter be denoted by $-a$, which at present is unknown.

We now consider§ the elementary function

$$\frac{\sigma(z+a)}{\sigma(z)\,\sigma(a)}e^{\lambda z},$$

* *T. F.*, §§ 137—139.

† It is a modified form of an equation given by Picard, *Crelle*, t. xc, p. 290.

‡ *T. F.*, § 139. § *T. F.*, *l.c.*

which has $z=0$ for an irreducible simple pole, and $-a$ for an irreducible simple zero; its expansion begins with z^{-1}, and the function must therefore agree with the integral above obtained in the vicinity of $z=0$. (The constants λ and a determine, or are determined by, the multipliers of the periodic function; but at present these are unknown, and so λ and a must be determined in another manner.) To expand the above function in powers of z, we have

$$\begin{aligned}\frac{\sigma(z+a)}{\sigma(a)} &= 1+z\frac{\sigma'(a)}{\sigma(a)}+\frac{z^2}{2!}\frac{\sigma''(a)}{\sigma(a)}+\frac{z^3}{3!}\frac{\sigma'''(a)}{\sigma(a)}\\ &= 1+z\zeta+\tfrac{1}{2}z^2(\zeta^2-\wp)+\tfrac{1}{6}z^3(\zeta^3-3\zeta\wp-\wp')+\dots,\end{aligned}$$

the Weierstrassian functions on the right-hand side being functions of a. Also

$$\sigma(z)=ze^{-\frac{1}{240}g_2z^4-\dots};$$

and therefore

$$\begin{aligned}w &= \frac{\sigma(z+a)}{\sigma(z)\,\sigma(a)}e^{\lambda z}\\ &= \frac{1}{z}+(\lambda+\zeta)+\tfrac{1}{2}z\{(\lambda+\zeta)^2-\wp\}+\tfrac{1}{6}z^2\{(\lambda+\zeta)^3-3(\lambda+\zeta)\wp-\wp'\}\\ &\quad +\frac{z^3}{24}\{(\lambda+\zeta)^4-6(\lambda+\zeta)^2\wp-4(\lambda+\zeta)\wp'-3\wp^2+\tfrac{3}{5}g_2\}+\dots\end{aligned}$$

This is to satisfy the differential equation, so that it must be of the form

$$Aw_1+Bw_2+Cw_3.$$

Clearly $A=1$, $B=\lambda+\zeta$, for this purpose: the value of C would be needed for the complete expression, but we merely require λ and a at present. Comparing the coefficients, we thus have

$$\begin{aligned}A&=1,\\ B&=\lambda+\zeta,\\ A\alpha&=(\lambda+\zeta)^2-\wp,\\ \tfrac{1}{2}A\beta&=(\lambda+\zeta)^3-3(\lambda+\zeta)\wp-\wp',\end{aligned}$$

so that λ and a are determined by the equations

$$\left.\begin{aligned}(\lambda+\zeta)^2-\wp&=\alpha\\ (\lambda+\zeta)^3-3(\lambda+\zeta)\wp-\wp'&=\tfrac{1}{2}\beta\end{aligned}\right\}.$$

We have

$$\lambda+\zeta=\frac{\wp'+\frac{1}{2}\beta}{\alpha-2\wp},$$

where a is determined by the relation

$$\beta\wp'+(3\alpha^2-g_2)\wp+\tfrac{1}{4}\beta^2-\alpha^3-g_3=0.$$

The function on the left-hand side is a doubly-periodic function of the first kind: it has a single irreducible pole, which is at $z=0$ and is of multiplicity three. Hence it has three irreducible zeros, say a_1, a_2, a_3; and their sum is congruent to 0, so that we may take

$$a_1+a_2+a_3=0.$$

In general, a_1, a_2, a_3 are unequal, because a and β are general constants; the discussion of the critical conditions, that lead to equalities between a_1, a_2, a_3, and of the consequent modifications in the complete primitive, is left as an exercise. Let

$$\lambda_r = -\zeta(a_r) + \frac{\wp'(a_r) + \frac{1}{2}\beta}{a - 2\wp(a_r)}, \qquad (r=1, 2, 3);$$

then

$$W_r = \frac{\sigma(z+a_r)}{\sigma(z)\,\sigma(a_r)} e^{\lambda_r z}$$

is an integral of the equation for each of the three values of r. The primitive of the equation is

$$w = \sum_{r=1}^{3} A_r \frac{\sigma(z+a_r)}{\sigma(z)\,\sigma(a_r)} e^{\lambda_r z},$$

where A_1, A_2, A_3 are arbitrary constants.

Ex. 2. Obtain the relations which express the integrals w_1, w_2, w_3 of the equation in the preceding example in terms of W_1, W_2, W_3; and determine the multipliers of the integrals.

Ex. 3. Obtain the primitive of the equation

$$(\wp' + \wp^2)\, w'' - (\wp\wp' + \wp'' - \wp^3)\, w' + (\wp'^2 - \wp^2\wp' - \wp\wp'')\, w = 0,$$

in the form

$$w = Ae^{\zeta(z)} + B\wp(z).$$

Ex. 4. Verify that the primitive of the equation

$$\frac{d^2y}{dx^2} + k^2 \frac{\operatorname{sn} x \operatorname{cn} x}{\operatorname{dn} x} \frac{dy}{dx} + n^2 y \operatorname{dn}^2 x = 0$$

is

$$y = A\cos(n \operatorname{am} x) + B\sin(n \operatorname{am} x).$$

(Gyldén.)

Ex. 5. Prove that, if I be an odd function and J be an even function, both doubly-periodic in the same periods, the integrals of the equation

$$\frac{d^2w}{dz^2} + I\frac{dw}{dz} + Jw = 0$$

can be expressed in terms of $\wp(z)$.

Hence (or otherwise) integrate the equation

$$w'' + \frac{\wp'^2 - \wp\wp''}{\wp\wp'} w' - \frac{1}{4}\frac{\wp'^2}{\wp^2} w = 0.$$

Ex. 6. Determine the relations among the constants (if any) in the equation

$$w''' + (a - 3\wp)\, w' + (\gamma + \beta\wp - \tfrac{3}{2}\wp')\, w = 0,$$

in order that every integral of the equation should be uniform; and assuming the relations satisfied, shew that the equation has three integrals of the form

$$\frac{\sigma(z+a)}{\sigma(z)\,\sigma(a)} e^{\lambda z}.$$

Ex. 7. Shew that the equation

$$y''' + (a - 3k^2 \operatorname{sn}^2 x)\, y' + (\beta + \gamma k^2 \operatorname{sn}^2 x - 3k^2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x)\, y = 0$$

has an integral of the form

$$y = \frac{H(x+\omega)}{\Theta(x)} e^{x\{\lambda - Z(\omega)\}},$$

provided

$$3a + \gamma^2 = 3(1+k^2);$$

and that it then has three integrals of that form. Obtain these integrals. (Mittag-Leffler.)

Ex. 8. Obtain the integral of the equation

$$\frac{d^3y}{dx^3} + (h - 6k^2 \operatorname{sn}^2 x)\frac{dy}{dx} + h_1 y = 0$$

in the form

$$y = \frac{H(x+\omega)}{\Theta(x)} e^{x\{\lambda - Z(\omega)\}},$$

where the constants λ and ω are given by the equations

$$h - (1+k^2) + 3(\lambda^2 - k^2 \operatorname{sn}^2 \omega) = 0,$$
$$2\lambda^3 - 6\lambda k^2 \operatorname{sn}^2 \omega + 2\lambda(1+k^2) - 4k^2 \operatorname{sn} \omega \operatorname{cn} \omega \operatorname{dn} \omega - h_1 = 0.$$

Verify that, in general, three distinct integrals are thus obtained. (Picard.)

Ex. 9. Prove that the equation

$$\frac{d^4y}{dx^4} + (a - 12k^2 \operatorname{sn}^2 x)\frac{d^2y}{dx^2} + \beta\frac{dy}{dx} + (\gamma + \delta k^2 \operatorname{sn}^2 x)\, y = 0$$

has an integral of the form

$$y = \frac{H(x+\omega)}{\Theta(x)} e^{x\{\lambda - Z(\omega)\}},$$

provided

$$2a + \delta = 8(1+k^2);$$

and that, if this relation be satisfied, it has four such integrals. Obtain them. (Mittag-Leffler.)

Ex. 10. Verify that the equation

$$(\operatorname{sn}^2 x - \operatorname{sn}^2 a)\frac{d^2y}{dx^2} - 2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x \frac{dy}{dx} + 2y\{1 - 2(1+k^2)\operatorname{sn}^2 a + 3k^2 \operatorname{sn}^4 a\} = 1$$

has an integral of the form

$$y = \frac{H(x+\omega)}{\Theta(x)} e^{x\{\lambda - Z(\omega)\}},$$

provided

$$\lambda = \frac{\operatorname{sn} \omega \operatorname{cn} \omega \operatorname{dn} \omega}{\operatorname{sn}^2 a - \operatorname{sn}^2 \omega}, \quad \operatorname{sn}^2 \omega = \frac{\operatorname{sn}^4 a\,(2k^2 \operatorname{sn}^2 a - 1 - k^2)}{3k^2 \operatorname{sn}^4 a - 2(1+k^2)\operatorname{sn}^2 a + 1};$$

and obtain the primitive.

Hence integrate the equation

$$\frac{d^2u}{dx^2} = u(6k^2 \operatorname{sn}^2 x + h),$$

where h is a constant. (Hermite.)

Ex. 11. Discuss the equation

$$\frac{d^4w}{dz^4}+\{a-12\wp(z)\}\frac{d^2w}{dz^2}+\beta\frac{dw}{dz}+\{\gamma+\delta\wp(z)\}w=0,$$

for those cases when every integral is a uniform function of z.

Ex. 12. Shew that there are three sets of values of the constants a and b, for which the equation

$$3\frac{d^2y}{dx^2}=\{\tfrac{1}{10}\wp''''(x)+7\wp''(x)+a\wp(x)+b\}y$$

admits as an integral a uniform doubly-periodic function; and obtain the integral. (Math. Tripos, Part II, 1897.)

Ex. 13. Prove that the equation

$$y''''-2n(n+1)y''\wp(z)-2n(n+1)y'\wp'(z)$$
$$+y\left\{\frac{n(n+1)(n+3)(n-2)}{6}\wp''(u)+a\right\}=0,$$

where a is an arbitrary constant and n is a positive integer, has a uniform function of z for its complete primitive. (Halphen.)

Ex. 14. Construct the equation which has

$$w=\{a_1+a_2\wp(z)+a_3\wp'(z)+a_4\wp''(z)\}f(z)$$

for its complete primitive and, for a properly determined value of $f(z)$, is devoid of the term in $\dfrac{d^3w}{dz^3}$. Likewise construct the equation which has

$$w=\{a_1+a_2\,\mathrm{sn}\,z+a_3\,\mathrm{cn}\,z+a_4\,\mathrm{dn}\,z\}f(z)$$

for its complete primitive, with the corresponding determination of $f(z)$ to remove the term in $\dfrac{d^3w}{dz^3}$. In each case, the quantities a_1, a_2, a_3, a_4 are to be regarded as arbitrary constants. (Halphen.)

Ex. 15. Prove that the primitive of the equation

$$w'''-\tfrac{4}{3}n^2w'\wp(z)-\tfrac{2}{27}n(n+3)(4n-3)w\wp'(z)=0$$

is a uniform function of z, when n is an integer multiple of 3; and discuss the primitive, when the integer n is prime to 3. (Halphen.)

Lamé's Equation.

148. One of the most important instances, in which a differential equation with uniform doubly-periodic coefficients has a uniform doubly-periodic function of the second kind for its integral, is Lamé's equation or, rather, the more general form of

Lamé's equation as discussed in the investigations of Hermite, Halphen, and others. The form used by Hermite* is

$$\frac{1}{w}\frac{d^2w}{dz^2} = n(n+1)k^2\operatorname{sn}^2 z + B,$$

where n is a positive integer and B is a general constant; the form used by Halphen† is

$$\frac{1}{w}\frac{d^2w}{dz^2} = n(n+1)\wp(z) + B,$$

with the same significance for n and B. We shall use the latter form of equation: it is selected for convenience and for its slightly greater generality owing to the functional independence of the periods. The mode of discussion is the same for the two forms.

As we are concerned with the application of the general theory‡, rather than with the special properties of the functions defined by Lamé's equation, only an outline of the solution of the equation will be given here. The detailed developments, and references to further memoirs, will be found in the authorities just quoted.

It may be not without interest to indicate how this form of equation arises from the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

characteristic of the potential in free space. When orthogonal curvilinear coordinates α, β, γ, as defined by three orthogonal surfaces

$$\alpha(x, y, z) = \alpha, \quad \beta(x, y, z) = \beta, \quad \gamma(x, y, z) = \gamma,$$

are used, then the equation becomes

$$\frac{\partial}{\partial\alpha}\left(\frac{A}{BC}\frac{\partial V}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{B}{CA}\frac{\partial V}{\partial\beta}\right) + \frac{\partial}{\partial\gamma}\left(\frac{C}{AB}\frac{\partial V}{\partial\gamma}\right) = 0,$$

* "Sur quelques applications des fonctions elliptiques," a separate reprint (1885) from the *Comptes Rendus*.

† *Traité des fonctions elliptiques*, t. II, ch. XII.

‡ That is, the theory of the uniform doubly-periodic functions of the second kind which are integrals of the differential equation. It has been proved (§ 54) that, by an appropriate transformation, the equation can be changed so as to be of Fuchsian type.

where

$$A^2 = \left(\frac{\partial\alpha}{\partial x}\right)^2 + \left(\frac{\partial\alpha}{\partial y}\right)^2 + \left(\frac{\partial\alpha}{\partial z}\right)^2,$$

$$B^2 = \left(\frac{\partial\beta}{\partial x}\right)^2 + \left(\frac{\partial\beta}{\partial y}\right)^2 + \left(\frac{\partial\beta}{\partial z}\right)^2,$$

$$C^2 = \left(\frac{\partial\gamma}{\partial x}\right)^2 + \left(\frac{\partial\gamma}{\partial y}\right)^2 + \left(\frac{\partial\gamma}{\partial z}\right)^2.$$

Choose, as the orthogonal surfaces, the three quadrics which ar[e] confocal with a given ellipsoid; and let λ, μ, ν be the roots of th[e] equation

$$\frac{x^2}{a^2+\theta} + \frac{y^2}{b^2+\theta} + \frac{z^2}{c^2+\theta} = 1,$$

a cubic in θ. Then* we take

$$a^2+\lambda = \wp(\alpha) - e_1,\quad b^2+\lambda = \wp(\alpha) - e_2,\quad c^2+\lambda = \wp(\alpha) - e_3,$$

$$a^2+\mu = \wp(\beta) - e_1,\quad b^2+\mu = \wp(\beta) - e_2,\quad c^2+\mu = \wp(\beta) - e_3,$$

$$a^2+\nu = \wp(\gamma) - e_1,\quad b^2+\nu = \wp(\gamma) - e_2,\quad c^2+\nu = \wp(\gamma) - e_3.$$

Now

$$\wp'^2(\alpha)\, A^2 = \left(\frac{\partial\lambda}{\partial x}\right)^2 + \left(\frac{\partial\lambda}{\partial y}\right)^2 + \left(\frac{\partial\lambda}{\partial z}\right)^2 = 4p_\lambda^2,$$

where

$$\frac{1}{p_\lambda^2} = \frac{x^2}{(a^2+\lambda)^2} + \frac{y^2}{(b^2+\lambda)^2} + \frac{z^2}{(c^2+\lambda)^2}$$

$$= \frac{(\lambda-\mu)(\lambda-\nu)}{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}$$

$$= \frac{4(\lambda-\mu)(\lambda-\nu)}{\wp'^2(\alpha)};$$

hence

$$A^2 = \frac{1}{(\lambda-\mu)(\lambda-\nu)}.$$

Similarly

$$B^2 = \frac{1}{(\mu-\lambda)(\mu-\nu)},\quad C^2 = \frac{1}{(\nu-\lambda)(\nu-\mu)};$$

so that the equation for the potential becomes

$$\frac{\partial}{\partial\alpha}\left\{(\mu-\nu)\frac{\partial V}{\partial\alpha}\right\} + \frac{\partial}{\partial\beta}\left\{(\nu-\lambda)\frac{\partial V}{\partial\beta}\right\} + \frac{\partial}{\partial\gamma}\left\{(\lambda-\mu)\frac{\partial V}{\partial\gamma}\right\} = 0,$$

* Greenhill, *Proc. Lond. Math. Soc.*, t. XVIII (1887), p. 275.

or, what is the same thing,

$$\{\wp(\beta)-\wp(\gamma)\}\frac{\partial^2 V}{\partial\alpha^2}+\{\wp(\gamma)-\wp(\alpha)\}\frac{\partial^2 V}{\partial\beta^2}+\{\wp(\alpha)-\wp(\beta)\}\frac{\partial^2 V}{\partial\gamma^2}=0.$$

For the purposes contemplated in the transformation, the quantity V is the product of a function of α, a function of β, and a function of γ, or is an aggregate of such products; and it is a uniform function of its variables. Hence, writing

$$V=f(\alpha)\,g(\beta)\,h(\gamma),$$

where f, g, h denote uniform functions of their arguments, we have

$$\{\wp(\beta)-\wp(\gamma)\}\frac{1}{f}\frac{d^2f}{d\alpha^2}+\{\wp(\gamma)-\wp(\alpha)\}\frac{1}{g}\frac{d^2g}{d\beta^2}$$
$$+\{\wp(\alpha)-\wp(\beta)\}\frac{1}{h}\frac{d^2h}{d\gamma^2}=0.$$

Thus

$$\frac{1}{w}\frac{d^2w}{dz^2}=A\wp(z)+B,$$

where $w=f$ when $z=\alpha$, $w=g$ when $z=\beta$, $w=h$ when $z=\gamma$, and A, B are constants independent of α, β, γ: they must be such as will, if possible, make w a uniform function of its argument. The only possible singularities of w are $z=0$ and points congruent with $z=0$; hence, after the earlier investigations, we consider the irreducible point $z=0$. The form of the equation shews that it will be an infinity of w; and thus it must be a pole, say of order n, where n is a positive integer. Thus we have, in the vicinity of the pole,

$$w=\frac{a_0}{z^n}+\frac{a_1}{z^{n-1}}+\ldots\quad=\frac{1}{z^n}R(z),$$

$$\frac{d^2w}{dz^2}=\frac{n(n+1)a_0}{z^{n+2}}+\ldots=\frac{n(n+1)}{z^{n+2}}R_1(z),$$

where $R(z)$ and $R_1(z)$ are regular functions of z, such that

$$\frac{R_1(z)}{R(z)}=1+\text{powers of } z.$$

Hence, in the vicinity of $z=0$, we have

$$\frac{1}{w}\frac{d^2w}{dz^2}=\frac{n(n+1)}{z^2}(1+\text{powers of } z),$$

and therefore

$$A = n(n+1),$$

a limitation upon the form of the constant A. But there is no limitation upon B, necessary for the existence of integrals of the type indicated; and therefore the differential equation may be taken in the form as stated.

To obtain Hermite's form, we write

$$\wp(z) - e_3 = \frac{e_1 - e_3}{\operatorname{sn}^2 y}, \quad y = z(e_1 - e_3)^{\frac{1}{2}}, \quad k^2 = \frac{e_2 - e_3}{e_1 - e_3},$$

as usual, and then take

$$y = x + iK';$$

the equation becomes

$$\frac{1}{w}\frac{d^2 w}{dx^2} = n(n+1) k^2 \operatorname{sn}^2 x + B'',$$

where B'' is a constant.

149. The method of solution of the equation is based upon the knowledge that there is at least one integral in the form of a doubly-periodic function of the second kind: the limitations, that have been imposed upon the equation, secure that this function is uniform. Moreover, the integral has only one irreducible pole, viz. at $z = 0$, and the pole is of order n.

There are two modes of using these results in order to construct the integral.

By one of them, we use* the further property that a uniform doubly-periodic function of the second kind has as many irreducible zeros as it has irreducible poles, account being taken of the orders of the points in each category. Accordingly, in the present instance, the integral has n irreducible zeros: let them be $-a_1, -a_2, \ldots, -a_n$. Consider the uniform function

$$w = \frac{\sigma(z + a_1)\, \sigma(z + a_2) \ldots \sigma(z + a_n)}{\sigma^n(z)} e^{\rho z},$$

which is doubly-periodic of the second kind; its (single) irreducible pole is of order n and is at $z = 0$; and it possesses the

* *T. F.*, §§ 139, 141.

necessary n (unknown) irreducible zeros, so that it is of a suitable form. We have

$$\frac{1}{w}\frac{dw}{dz} = \rho - n\zeta(z) + \sum_{r=1}^{n} \zeta(z + a_r).$$

In order to simplify the right-hand side, it is convenient to take

$$\rho = \lambda - \sum_{r=1}^{n} \zeta(a_r);$$

and so

$$\frac{1}{w}\frac{dw}{dz} = \lambda + \sum_{r=1}^{n} \zeta(z + a_r) - \sum_{r=1}^{n} \zeta(a_r) - n\zeta(z).$$

Hence

$$\frac{1}{w}\frac{d^2w}{dz^2} - \frac{1}{w^2}\left(\frac{dw}{dz}\right)^2 = n\wp(z) - \sum_{r=1}^{n} \wp(z + a_r).$$

But

$$\frac{1}{w}\frac{dw}{dz} = \lambda + \tfrac{1}{2}\sum_{r=1}^{n} \frac{\wp'(a_r) - \wp'(z)}{\wp(a_r) - \wp(z)},$$

and therefore

$$\begin{aligned}\frac{1}{w^2}\left(\frac{dw}{dz}\right)^2 = \tfrac{1}{4}\sum_{r=1}^{n} &\left\{\frac{\wp'(a_r) - \wp'(z)}{\wp(a_r) - \wp(z)}\right\}^2 \\ &+ \tfrac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n} \frac{\wp'(a_r) - \wp'(z)}{\wp(a_r) - \wp(z)} \cdot \frac{\wp'(a_s) - \wp'(z)}{\wp(a_s) - \wp(z)} \\ &+ \lambda \sum_{r=1}^{n} \frac{\wp'(a_r) - \wp'(z)}{\wp(a_r) - \wp(z)} + \lambda^2.\end{aligned}$$

The first term on the right-hand side is equal to

$$\sum_{r=1}^{n} \{\wp(z + a_r) + \wp(a_r) + \wp(z)\}.$$

To modify the second term, where the summation is for pairs of unequal values of r and s, we have

$$\begin{aligned}&\frac{\wp'(a_r) - \wp'(z)}{\wp(a_r) - \wp(z)} \cdot \frac{\wp'(a_s) - \wp'(z)}{\wp(a_s) - \wp(z)} \\ &= \frac{4\wp^3(z) - g_2\wp(z) - g_3 - \wp'(a_r)\,\wp'(a_s) - \wp'(z)\,\{\wp'(a_r) + \wp'(a_s)\}}{\{\wp(a_r) - \wp(z)\}\,\{\wp(a_s) - \wp(z)\}} \\ &= 4\,\{\wp(z) + \wp(a_r) + \wp(a_s)\} + L_{rs}\left\{\frac{\wp'(a_r) - \wp'(z)}{\wp(z) - \wp(a_r)} - \frac{\wp'(a_s) - \wp'(z)}{\wp(z) - \wp(a_s)}\right\},\end{aligned}$$

after easy reductions, where

$$L_{rs} = \frac{\wp'(a_r) + \wp'(a_s)}{\wp(a_r) - \wp(a_s)};$$

and thus the second term in the expression for $\frac{1}{w^2}\left(\frac{dw}{dz}\right)^2$ becomes

$$n(n-1)\wp(z) + 2(n-1)\sum_{r=1}^{n}\wp(a_r) + \tfrac{1}{2}\sum_{r=1}^{n}\frac{\wp'(a_r)-\wp'(z)}{\wp(z)-\wp(a_r)}\sum_{s=1}^{n}{}' L_{rs},$$

where the summation is for all values of s from 1 to n except only $s=r$. Then

$$\frac{1}{w}\frac{d^2w}{dz^2} = n(n+1)\wp(z) + (2n-1)\sum_{r=1}^{n}\wp(a_r) + \lambda^2 + \tfrac{1}{2}\sum_{r=1}^{n}\frac{\wp'(a_r)-\wp'(z)}{\wp(z)-\wp(a_r)}\left\{\sum_{s=1}^{n}{}' L_{rs} - 2\lambda\right\}.$$

Comparing this result with the differential equation under consideration, we naturally take

$$\sum_{s=1}^{n}{}' L_{rs} = 2\lambda,$$

for all values of r, that is,

$$\left.\begin{array}{l}
\dfrac{\wp'(a_1)+\wp'(a_2)}{\wp(a_1)-\wp(a_2)} + \dfrac{\wp'(a_1)+\wp'(a_3)}{\wp(a_1)-\wp(a_3)} + \ldots = 2\lambda \\[2ex]
\dfrac{\wp'(a_2)+\wp'(a_1)}{\wp(a_2)-\wp(a_1)} + \dfrac{\wp'(a_2)+\wp'(a_3)}{\wp(a_2)-\wp(a_3)} + \ldots = 2\lambda \\[1ex]
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\[1ex]
\dfrac{\wp'(a_n)+\wp'(a_1)}{\wp(a_n)-\wp(a_1)} + \dfrac{\wp'(a_n)+\wp'(a_2)}{\wp(a_n)-\wp(a_2)} + \ldots = 2\lambda
\end{array}\right\},$$

and
$$(2n-1)\sum_{r=1}^{n}\wp(a_r) + \lambda^2 = B.$$

Adding the n former equations together, we have

$$0 = 2n\lambda,$$

so that λ vanishes. Hence *if the n quantities $a_1, a_2, \ldots, a_n$ are determined by the equations*

$$\begin{array}{l}
\dfrac{\wp'(a_1)+\wp'(a_2)}{\wp(a_1)-\wp(a_2)} + \dfrac{\wp'(a_1)+\wp'(a_3)}{\wp(a_1)-\wp(a_3)} + \ldots = 0, \\[2ex]
\dfrac{\wp'(a_2)+\wp'(a_1)}{\wp(a_2)-\wp(a_1)} + \dfrac{\wp'(a_2)+\wp'(a_3)}{\wp(a_2)-\wp(a_3)} + \ldots = 0, \\[1ex]
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{array}$$

(which are equivalent to only $n-1$ independent equations, because the sum of the n left-hand sides is zero) and by

$$(2n-1)\sum_{r=1}^{n}\wp(a_r) = B,$$

then

$$F(z) = \frac{\sigma(z+a_1)\,\sigma(z+a_2)\ldots\sigma(z+a_n)}{\sigma^n(z)}\, e^{-z\sum\limits_{r=1}^{n}\zeta(a_r)}$$

is an integral of Lamé's equation

$$\frac{1}{w}\frac{d^2w}{dz^2} = n(n+1)\,\wp(z) + B.$$

The equation remains unchanged when $-z$ is written for z; hence $F(-z)$ is also an integral. Save in the case when the constants a are such that $F(z)$ and $F(-z)$ are effectively the same function, we have two independent integrals of the equation, which therefore is completely solved.

150. Another method of arranging the necessary analysis is as follows. Consider the equation

$$\frac{1}{w}\frac{d^2w}{dz^2} = F(z),$$

where $F(z)$ is a doubly-periodic function; by Picard's theorem (§ 142), an integral (say w_1) is known to exist in the form of a doubly-periodic function of the second kind. If then we write

$$v = \frac{1}{w_1}\frac{dw_1}{dz},$$

the quantity v is a doubly-periodic function of the first kind; and it satisfies the equation

$$\frac{dv}{dz} + v^2 = F(z).$$

The irreducible poles of w_1, in their proper order, are known from the singularities of the original equation; let them be n in number, account being taken of multiplicity. Then each of them is a pole of v, of the first order; and the sum of their residues for v is $-n$. The number* of irreducible zeros of w_1 is also n, account being taken of multiplicity; each of them is a pole of v, of the first order, and the sum of their residues for v is $+n$.

We therefore construct a uniform doubly-periodic function of the first kind, having these poles, all simple, viz. the known poles arising through the singularities of F, and the unknown poles

* *T. F.*, § 139.

arising through the zeros of w_1, taking care to have $-n$ and $+n$ for the respective sums of the residues. The general expression for such a function is known*: when substituted as a trial function in the above equation, comparison of the results leads to a determination of the constants.

As an illustration, consider the equation

$$\frac{1}{w}\frac{d^2w}{dz^2} = 2\wp(z) + B.$$

The irreducible pole of $\wp(z)$, viz., $z=0$, is the only irreducible pole of w_1, and it is of the first degree. Accordingly, it is a simple pole of v, with a residue -1. Further, there is (by the preceding argument) only one other pole of v: it is simple, and has a residue $+1$. As v is a doubly-periodic function of the first kind, an appropriate expression is

$$\begin{aligned} v &= \zeta(z-c) - \zeta(z) + k \\ &= \zeta(z-c) - \zeta(z) + \zeta(c) + b, \end{aligned}$$

say; and b, c have to be determined by substituting in the equation

$$\frac{dv}{dz} + v^2 = 2\wp(z) + B.$$

Now

$$\begin{aligned} \frac{dv}{dz} &= -\wp(z-c) + \wp(z) \\ &= 2\wp(z) + \wp(c) - \tfrac{1}{4}\left\{\frac{\wp'(z)+\wp'(c)}{\wp(z)-\wp(c)}\right\}^2; \end{aligned}$$

and by the addition-theorem for the ζ-function, we have

$$v = b + \tfrac{1}{2}\frac{\wp'(z)+\wp'(c)}{\wp(z)-\wp(c)}.$$

Substituting, we have

$$2\wp(z) + \wp(c) + b^2 + b\frac{\wp'(z)+\wp'(c)}{\wp(z)-\wp(c)} = 2\wp(z) + B,$$

which must be satisfied identically. Accordingly,

$$b = 0, \quad \wp(c) = B;$$

and thus, with a known value of c,

$$v = \zeta(z-c) - \zeta(z) + \zeta(c),$$

* *T. F.*, § 138.

so that

$$w_1 = A \frac{\sigma(z-c)}{\sigma(z)} e^{z\zeta(c)}.$$

There are two values of c, equal and opposite: the construction of the primitive is immediate.

Ex. 1. Shew that two independent integrals of the equation

$$\frac{1}{w}\frac{d^2w}{dz^2} = 2\wp(z) + B,$$

in the case when $B = e_1$, are given by

$$\{\wp(z) - e_1\}^{\frac{1}{2}}, \quad \{\wp(z) - e_1\}^{\frac{1}{2}} \{\zeta(z+\omega) + e_1 z\};$$

and obtain the integrals in the cases, when $B = e_2$, and $B = e_3$, respectively.

Ex. 2. Obtain the primitive of the equation

$$\frac{1}{y}\frac{d^2y}{dx^2} = 2k^2 \operatorname{sn}^2 x - a,$$

(where a is constant), in the form

$$y = A \frac{H(x+\alpha)}{\Theta(x)} e^{-zZ(\alpha)} + B \frac{H(x-\alpha)}{\Theta(x)} e^{zZ(\alpha)},$$

where

$$\operatorname{dn}^2 \alpha = a - k^2.$$

Discuss the solution in the three particular cases

$$a = 1 + k^2, \ 1, \ k^2. \qquad \text{(Hermite.)}$$

Ex. 3. Shew that

$$w = \frac{\sigma(z+a_1)\,\sigma(z+a_2)\,\sigma(z+a_3)}{\sigma^3(z)} e^{-z\{\zeta(a_1)+\zeta(a_2)+\zeta(a_3)\}}$$

satisfies the equation

$$\frac{1}{w}\frac{d^2w}{dz^2} = 12\wp(z) + 15b,$$

if $\wp(a_1)$, $\wp(a_2)$, $\wp(a_3)$ are the roots of the cubic equation

$$4\theta^3 - 12b\theta^2 + (24b^2 - g_2)\,\theta - 60b^3 + 4g_2 b - g_3 = 0;$$

and deduce the primitive. (Halphen.)

Ex. 4. Shew that the primitive of the equation

$$\frac{d^2w}{dz^2} = \{n(n+1)\,\wp(z) + B\}\,w$$

can be expressed in finite form for appropriate values of the constant B in the following cases:—

I. When n is an even integer, $= 2m$, then either

$$w = a_m \wp^m(z) + a_{m-1} \wp^{m-1}(z) + \ldots + a_0,$$

or

$$w = [\{\wp(z) - e_\lambda\}\{\wp(z) - e_\mu\}]^{\frac{1}{2}} \{c_{m-1} \wp^{m-1}(z) + \ldots + c_0\},$$

where e_λ, e_μ are any two of the three constants e_1, e_2, e_3:

II. When n is an odd integer, $=2m-1$, then either

$$w=\{\wp(z)-e_\lambda\}^{\frac{1}{2}}\{c_{m-1}\wp^{m-1}(z)+\ldots+c_0\},$$

or

$$w=\wp'(z)\{b_{m-2}\wp^{m-2}(z)+\ldots+b_0\}$$

where e_λ is any one of the three constants e_1, e_2, e_3.

Determine the number of solutions of the specified kind in each of the cases indicated. (Crawford.)

Ex. 5. Shew that an integral of the equation

$$\frac{1}{w}\frac{d^2w}{dz^2}=n(n+1)k^2\,\mathrm{sn}^2 z+h+\frac{m(m+1)}{\mathrm{sn}^2 z},$$

where h is a constant and m, n are integers, can be expressed in the form

$$w=\frac{\theta_1(z-z_1)\,\theta_1(z-z_2)\ldots\theta_1(z-z_{m+n})}{\{\theta_1(z)\}^m\{\theta(z)\}^m}e^{z\sum\limits_{r=1}^{m+n}Z(u_r)}$$

in the usual notation of the theta-functions, $z_1, z_2, \ldots, z_{m+n}$ being appropriate constants.

Obtain the primitive. (M. Elliott.)

Ex. 6. Obtain the primitive of the equation

$$\frac{d^2w}{dz^2}=\frac{w}{4\wp^2(z)}\{4\wp^4(z)-2g_2\wp(z)-3g_3\}$$

in the form

$$w=\{\wp'(z)\}^{\frac{1}{2}}\{Ae^{\zeta(z)}+Be^{-\zeta(z)}\}.$$

Ex. 7. Shew that there are two values of k_0, for which the equation

$$\frac{1}{w}\frac{d^2w}{dz^2}=k_0+k_1\wp(z)-2m\wp'(z)+\tfrac{1}{6}m^2\wp''(z),$$

where m is a constant, possesses an integral of the form

$$w=e^{-az+m\zeta(z)}\frac{\sigma(z-b)}{\sigma(z)};$$

and, for each such value, obtain the primitive. (Benoit.)

Ex. 8. Shew that there are $n+1$ values of k_0, for which the equation

$$\frac{1}{w}\frac{d^2w}{dz^2}=k_0+k_1\wp(z)-m(n+1)\wp'(z)+\tfrac{1}{6}m^2\wp''(z),$$

where m is a constant and n a positive integer, possesses an integral of the form

$$w=\frac{\prod\limits_{r=1}^{n}\sigma(z-b_r)}{\sigma^n(z)}e^{-az+m\zeta(z)}.$$

Prove also that, if the right-hand side of the differential equation be increased by $\Psi(z)$, where Ψ is a doubly-periodic function of the first kind having all its poles simple, a corresponding theorem holds as regards the integral, if k_0 be properly determined. (Benoit.)

Ex. 9. Integrate the equation

$$\frac{1}{w}\frac{d^2w}{dz^2}=\frac{\mu(\mu+1)}{\mathrm{sn}^2 z}+\frac{\mu'(\mu'+1)\,\mathrm{dn}^2 x}{\mathrm{cn}^2 x}+\frac{\mu''(\mu''+1)\,k^2\,\mathrm{cn}^2 x}{\mathrm{dn}^2 x}+n(n+1)\,k^2\,\mathrm{sn}^2 x+h,$$

where μ, μ', μ'', n are positive integers. (Darboux.)

151. The other mode of utilising the known properties of the integral, when it is a uniform doubly-periodic function of the second kind, is to obtain the actual expansion of the integral in the vicinity of its irreducible pole and thence to construct its functional expression in terms of the elementary function

$$\frac{\sigma(z+a)}{\sigma(z)}e^{\lambda z},$$

where a and λ are initially unknown constants. Some indication of the process is given in Ex. 1, § 147; but a slightly different form will be adopted for the present purpose. We take the elementary function in the form

$$G(z)=\frac{\sigma(z+a)}{\sigma(z)\,\sigma(a)}e^{z\{\rho-\zeta(a)\}},$$

where ρ and a are now to be regarded as the constants to be determined. The expansion of this function in the vicinity of its irreducible pole at $z=0$ is

$$G(z)=\frac{1}{z}+\rho+\tfrac{1}{2}\{\rho^2-\wp(a)\}\,z+\tfrac{1}{6}\{\rho^3-3\rho\wp(a)-\wp'(a)\}\,z^2$$
$$+\tfrac{1}{24}\{\rho^4-6\rho^2\wp(a)-4\rho\wp'(a)-3\wp^2(a)+\tfrac{3}{5}g_2\}\,z^3+\dots.$$

If, in the same vicinity, an integral of the differential equation exists in the form

$$w=(-1)^{n-1}\frac{(n-1)!}{z^n}a_n+(-1)^{n-2}\frac{(n-2)!}{z^{n-1}}a_{n-1}+\dots$$
$$\dots+\frac{a_1}{z}+a_0+\text{positive powers},$$

then we may take

$$w=a_n\frac{d^{n-1}G}{dz^{n-1}}+a_{n-1}\frac{d^{n-2}G}{dz^{n-2}}+\dots+a_1G,$$

where a comparison of expansions serves to determine the constants a and ρ. The integral thus is known.

An illustration will render the details clearer. In the case when $n = 2$, the equation is

$$\frac{1}{w}\frac{d^2w}{dz^2} = 6\wp(z) + B.$$

Let

$$w = \frac{1}{z^2} + \frac{a_0}{z} + a_1 + a_2 z + a_3 z^2 + \dots$$

be substituted in the equation; we find

$$a_0 = 0, \quad a_2 = 0, \quad a_4 = 0, \dots$$

$$a_1 = -\tfrac{1}{6}B, \quad a_3 = \tfrac{1}{24}B^2 - \tfrac{3}{40}g_2, \dots$$

so that

$$w = \frac{1}{z^2} - \tfrac{1}{6}B + (\tfrac{1}{24}B^2 - \tfrac{3}{40}g_2)\, z^2 + \dots .$$

Manifestly, the form to take is

$$w = -\frac{dG}{dz};$$

and then comparing the two expansions, we have

$$-\tfrac{1}{2}\{\rho^2 - \wp(a)\} = -\tfrac{1}{6}B,$$

$$\rho^3 - 3\rho\wp(a) - \wp'(a) = 0.$$

These equations give

$$\wp(a) = \frac{B^3 + 27g_3}{9B^2 - 27g_2},$$

$$\rho = \frac{3\wp'(a)}{B - 6\wp(a)}.$$

The former in general leads to two irreducible values of a; the latter uniquely determines ρ for each of these values of a. Denoting the two values of a by a and $-a$, and writing

$$G_1 = \frac{\sigma(z+a)}{\sigma(z)\,\sigma(a)}\, e^{z\left\{\frac{3\wp'(a)}{B - 6\wp(a)} - \zeta(a)\right\}},$$

$$G_2 = \frac{\sigma(z-a)}{\sigma(z)\,\sigma(a)}\, e^{-z\left\{\frac{3\wp'(a)}{B - 6\wp(a)} - \zeta(a)\right\}},$$

the primitive of the differential equation is

$$w = L\frac{dG_1}{dz} + M\frac{dG_2}{dz}.$$

Ex. 1. Discuss the integral of the equation

$$\frac{1}{w}\frac{d^2w}{dz^2}=6\wp(z)+B,$$

when a, as obtained in the preceding solution, has the values 0, ω, ω', ω'', respectively.

Ex. 2. Prove that an integral of the equation

$$\frac{1}{w}\frac{d^2w}{dz^2}=12\wp(z)+15b$$

is given by

$$w=\frac{d^2G}{dz^2}-3bG,$$

where

$$G(z)=\frac{\sigma(z+a)}{\sigma(z)\,\sigma(a)}\,e^{z\{\rho-\zeta(a)\}},$$

the constants ρ and a being given by the equations

$$12b\wp(a)+20b^2+\tfrac{1}{3}g_2=\left(\frac{20b^3-g_2b+g_3}{4b^2-\tfrac{1}{3}g_2}\right)^3=\Delta^3,$$

$$\rho=\frac{\wp'(a)}{\Delta-2\wp(a)-4b}.$$

Deduce the primitive.

CHAPTER X.

Equations having Algebraic Coefficients.

152. The differential equations, considered in the preceding chapters, have had uniform functions of the independent variable for their coefficients. We now proceed to consider (but only briefly) some equations without this limitation: one of the most important classes is constituted by those which have algebraic functions of the variable as their coefficients. For this purpose, let y denote an algebraic function of the independent variable x, defined by the equation

$$\psi(x, y) = 0,$$

where ψ is a polynomial in x and y, and the equation is of genus p. With this algebraic equation we associate the proper Riemann surface of connectivity $2p+1$.

We assume that the linear differential equation has uniform functions of x and y for its coefficients, so that each of these is a uniform function of position on the surface: and we write the equation in the form

$$\frac{d^m u}{dx^m} + \alpha_1(x, y)\frac{d^{m-1}u}{dx^{m-1}} + \alpha_2(x, y)\frac{d^{m-2}u}{dx^{m-2}} + \ldots + \alpha_m(x, y)\, u = 0.$$

Let

$$w = ue^{\frac{1}{m}\int \alpha_1(x, y)\, dx},$$

the exponential in the factor of u on the right-hand side being an Abelian integral; then the equation for w is

$$\frac{d^m w}{dx^m} + P_2(x, y)\frac{d^{m-2}w}{dx^{m-2}} + P_3(x, y)\frac{d^{m-3}w}{dx^{m-3}} + \ldots + P_m(x, y)\, w = 0,$$

devoid of the derivative of order $m-1$; all the coefficients $P_2, \ldots, P_m$ are algebraic functions of x, and are uniform functions of x and y. This is the form of equation which will be discussed.

Let (x_0, y_0) denote any position on the surface, which is not a singular point on the surface and in the vicinity of which each of the coefficients P is regular. Then an integral exists, which is regular everywhere over a domain in the surface, and is uniquely determined by the assignment of arbitrary values to w and to its first $m-1$ derivatives at (x_0, y_0). In fact, all the results relating to the synectic integrals of an equation with uniform coefficients hold for the present equation in the domain of (x_0, y_0).

Next, let account be taken of the singularities of the equation $\psi = 0$ and of the associated surface. As these affect all the coefficients of all differential equations of the class considered, and thus afford no relative discrimination among the functions defined by those equations, we shall assume them simplified as much as possible before proceeding to consider the properties of the functions. Accordingly, we shall suppose that, if the equation $\psi = 0$ (or the Riemann surface associated with it) possesses a complicated singularity, it is resolved* into its simplest form by means of birational transformations, so that we may write

$$x - e = g(\xi, \eta), \quad y - f = h(\xi, \eta),$$

where g and h are uniform functions which, in connection with $\psi = 0$, admit of uniform expressions for ξ and η in terms of $x - e$ and $y - f$, and are such that $\xi = 0$, $\eta = 0$ is an ordinary position on the transformed Riemann surface. The positions on the surface, that have to be considered in connection with the differential equation, are now ordinary positions: and therefore, in dealing with the theory of the equation, no generality is lost if we assume that the singularities of the equation $\psi = 0$ and of the Riemann surface are ordinary positions for the integrals. (Of course, in any particular example, it may happen that a multiple point on the curve $\psi = 0$, or a branch-point of the associated surface, is definitely a singularity of the equation. In order to discuss the nature of the integrals in the vicinity of such a point, we take†

$$x - e = \zeta^q, \quad y - f = \zeta^p S(\zeta),$$

* *T. F.*, § 252.

† *T. F.*, § 97.

where p and q are integers, and S is a holomorphic function of its argument that does not vanish when $\zeta = 0$; and then we investigate the character of the integrals in the vicinity of $\zeta = 0$.)

Lastly, let (a, b) denote a position on the Riemann surface (being a pair of values given by the differential equation) such that the coefficients of the equation are not regular in the immediate vicinity of (a, b); after the preceding explanations, we may assume that $y - b$ is a holomorphic function of $x - a$ in the immediate vicinity of the position. The character of the integrals in that region is determined, after substitution of $y - b$ in terms of $x - a$, in association with an indicial equation; and the general processes of the theory, in the case of differential equations with uniform coefficients, are applicable to the integrals in the vicinity of (a, b). As in that earlier theory, we have a fundamental system of integrals existing at any ordinary position on the surface, the system being composed of m linearly independent members. Continuation of these integrals is possible: and by taking all admissible paths from one ordinary position to any other ordinary position (care being taken to avoid the actual singularities), and assuming an arbitrary set of initial values at the first point, we shall obtain all possible integrals at the second point. Similarly, by taking all possible closed paths on the Riemann surface, which begin at an ordinary point (x_0, y_0) and return to it, we obtain new integrals at the end of the path: and each of these integrals is linearly expressible in terms of the members of the initial fundamental system.

A Fundamental System of Integrals, and the Fundamental Equation.

153. Let $w_1, w_2, \ldots, w_m$ denote a fundamental system at an ordinary position (x_0, y_0); and let the variable of position describe a closed path on the surface returning to (x_0, y_0), this closed path being chosen so as to include the singularity (a, b) but no other singularity of the differential equation. Suppose that the effect upon the fundamental system, caused by this variation of the variable of position, is to change it into

$w_1', w_2', \ldots, w_m'$: then, as in the case of uniform coefficients, the latter set also constitute a fundamental system, and the two systems are related by the equations

$$w_\mu' = \sum_{\lambda=1}^{m} a_{\lambda\mu} w_\lambda, \qquad (\mu = 1, \ldots, m),$$

where the determinant of the coefficients a is different from zero. This determinant is (as in § 14) equal to unity. For let Δ denote the determinant of the fundamental system

$$\begin{vmatrix} \dfrac{d^{m-1}w_1}{dx^{m-1}}, & \dfrac{d^{m-1}w_2}{dx^{m-1}}, & \ldots, & \dfrac{d^{m-1}w_m}{dx^{m-1}} \\ \dfrac{d^{m-2}w_1}{dx^{m-2}}, & \dfrac{d^{m-2}w_2}{dx^{m-2}}, & \ldots, & \dfrac{d^{m-2}w_m}{dx^{m-2}} \\ \cdots & \cdots & \cdots & \cdots \\ w_1 \quad , & w_2 \quad , & \ldots, & w_m \end{vmatrix},$$

and let Δ' denote the same determinant in relation to the fundamental system w'; then, if A denote the determinant of the coefficients $a_{\lambda\mu}$, we have

$$\Delta' = A\Delta.$$

Now, because the term involving the $(m-1)$th derivative of w is absent from the differential equation, we have, as in § 14,

$$\Delta = C,$$

where C is a constant. Let the function Δ, which is equal to C in the vicinity of (x_0, y_0), be traced along the closed path which the variable of position describes on its return to (x_0, y_0); it is steadily constant, and its final value is Δ', so that

$$\Delta' = C;$$

and therefore we have

$$A = 1.$$

Further, as in the cases when the coefficients are uniform functions of the independent variable, it is possible to choose a linear combination v of the members of the fundamental system such that, if v' denote the value of v obtained by making the variable of position describe the aforesaid closed path, we have

$$v' = \theta v.$$

The multiplier θ is a root of the equation

$$A(\theta) = \begin{vmatrix} a_{11} - \theta, & a_{12}, & \ldots, & a_{1m} \\ a_{21}, & a_{22} - \theta, & \ldots, & a_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1}, & a_{m2}, & \ldots, & a_{mm} - \theta \end{vmatrix}$$

$$= 1 + I_1\theta + I_2\theta^2 + \ldots + I_{m-1}\theta^{m-1} + (-1)^m\theta^m = 0.$$

This equation is independent of the choice of the fundamental system, so that its coefficients may be regarded as *invariants* of the linear substitution, which the fundamental system undergoes in the description of the closed path round (a, b).

154. If some, or if all, of the integrals in the vicinity of (a, b) are regular in the sense of § 29, then an indicial equation for the singularity exists; and if ρ be a root of this equation for an integral with a multiplier θ, then

$$\theta = e^{2\pi i\rho}.$$

If no one of the integrals is regular, there is no valid indicial equation. In the first case, the general character of an integral is determined by the value of ρ: and the explicit form is obtained by substituting an expression of the appropriate character so as to determine the coefficients. In the second case, various methods* for obtaining the value of θ have been suggested, by Fuchs†, Hamburger‡, and Poincaré§; the most general is the method of infinite determinants, due to Hill and von Koch, and expounded in Chapter VIII.

Without entering upon details, it may be said briefly that many of the properties of linear differential equations having algebraic coefficients can be treated by processes that, except as to greater complexity in the mere analysis, are the same as for equations with uniform coefficients. It therefore seems unnecessary to discuss them at any length, as they would lead to what is substantially a repetition of a discussion already effected for less complicated equations.

* See § 127.

† *Crelle*, t. LXXV (1873), pp. 177—223.

‡ *Crelle*, t. LXXXIII (1877), pp. 185—210.

§ *Acta Math.*, t. IV (1884), pp. 208 et seq.

A systematic discussion of equations having algebraic coefficients and development of many of their characteristic properties will be found in a series of memoirs by Thomé*.

Ex. 1. Consider the equation

$$\frac{d^2w}{dx^2}+\frac{\alpha}{(ax+by+c)^2}w=0,$$

where the variable y is defined by the relation

$$x^2+y^2=1,$$

and α, a, b, c are constants.

The position at infinity is a singularity of the differential equation in each of the two sheets of the Riemann surface. The integrals are regular in that vicinity in one sheet, and the exponents to which they belong are the roots of

$$\sigma(\sigma+1)+\frac{\alpha}{(a+bi)^2}=0,$$

provided $a+bi$ is not zero; but, if $a+bi=0$, the integrals are irregular at infinity in that sheet. Similarly, they are regular in the vicinity of infinity in the other sheet, and the exponents to which they belong are the roots of

$$\sigma(\sigma+1)+\frac{\alpha}{(a-bi)^2}=0,$$

provided $a-bi$ is not zero; but, if $a-bi=0$, the integrals are irregular at infinity in that sheet.

The other singularities of the equation are given by

$$\left.\begin{array}{r}ax+by+c=0\\x^2+y^2-1=0\end{array}\right\}.$$

When these are distinct from one another, let them be denoted by $x=\cos\theta$, $y=\sin\theta$; $x=\cos\phi$, $y=\sin\phi$. The integrals are regular in the vicinity of each position; and the respective indicial equations are

$$\rho(\rho-1)+\frac{\alpha}{(a-b\cot\theta)^2}=0,$$

$$\rho(\rho-1)+\frac{\alpha}{(a-b\cot\phi)^2}=0.$$

When the two singularities coincide, let the common position be denoted by $x=\cos\psi$, $y=\sin\psi$; and then

$$a-b\cot\psi=0.$$

In the vicinity, we have

$$x=\cos\psi+\xi,$$
$$y=\sin\psi-\xi\cot\psi-\tfrac{1}{2}\frac{\xi^2}{\sin^3\psi}+\tfrac{1}{2}\frac{\xi^3\cos\psi}{\sin^5\psi}-\ldots,$$

* *Crelle*, t. cxv (1895), pp. 33—52, 119—149; *ib.*, t. cxix (1898), pp. 131—147; *ib.*, t. cxxi (1900), pp. 1—39; *ib.*, t. cxxii (1900), pp. 1—29.

so that the equation is

$$\frac{d^2w}{d\xi^2}+\frac{4a\sin^6\psi}{b^2\xi^4}\left\{1+\frac{\xi\cos\psi}{\sin^2\psi}+\dots\right\}^{-2}w=0.$$

The integrals are not regular; but the equation may have one normal integral, and can even have two normal integrals, of the type

$$e^{\frac{2i\sin^3\psi}{b\xi}a^{\frac{1}{2}}}\xi^\sigma f(\xi),$$

where f is a polynomial in ξ. The forms, and the conditions necessary to significance, can be obtained as in §§ 85—87.

Ex. 2. Discuss in the same way the singularities of the same differential equation, when the irrational quantity y is given by the respective relations

$$\text{(i)}\quad x^3+y^3=1,$$
$$\text{(ii)}\quad y^2=4x^3-g_2x-g_3.$$

Ex. 3. Let u_1 and u_2 denote a fundamental system of the equation in Ex. 1, for $y=(1-x^2)^{\frac{1}{2}}$; and let v_1 and v_2 denote a fundamental system of the same equation for $y=-(1-x^2)^{\frac{1}{2}}$. Shew that the linear equation of the fourth order, which has u_1, u_2, v_1, v_2 as its integrals, has rational functions of x for its coefficients; and obtain them.

Ex. 4. The equation

$$\frac{dw}{dx}=w\phi(x, y),$$

where

$$\psi(x, y)=0,$$

has its primitive in the form

$$w=Ke^{\int\phi(x, y)\,dx}.$$

It is natural to inquire whether an equation

$$\frac{d^2w}{dx^2}=w\theta(x, y)$$

can have an integral of the type

$$w=e^{\int\varpi(x, y)\,dx},$$

where $\varpi(x, y)$ is a rational function of x and y. A general method for such an inquiry has been given by Appell*, though it is not carried to a complete issue as regards detail: it will be sufficiently illustrated by means of the equation

$$\frac{d^2w}{dx^2}=\frac{ax+\beta y}{y(ax+by)^2}w,$$

where $x^2+y^2=1$, it being required to find under what conditions, if any, the equation can have an integral of the form

$$w=e^{\int\varpi(x, y)\,dx},$$

* *Ann. de l'Éc. Norm. Sup.*, 2^{me} Sér., t. XII (1883), pp. 8—46.

where $\varpi(x, y)$ is a rational function of x and y. Since

$$\varpi = \frac{1}{w}\frac{dw}{dx},$$

we have

$$\frac{d\varpi}{dx} + \varpi^2 = \frac{\alpha x + \beta y}{y(ax+by)^2}.$$

We assume that each of the quantities $\alpha \pm \beta i$, $a \pm bi$, is different from zero. By adopting the method in the preceding Ex. 1, the integrals of the equation in w are easily seen to be regular in the vicinity of $x = \infty$, so that they have the form

$$w = x^\lambda R\left(\frac{1}{x}\right),$$

where $R\left(\frac{1}{x}\right)$ is a holomorphic function for large values of x, not vanishing when $x = \infty$; and thus

$$\varpi = \frac{\lambda}{x} + \frac{\mu}{x^2} + \ldots,$$

in the vicinity of $x = \infty$. Substituting in the equation for ϖ, we have

$$\lambda^2 - \lambda = \frac{\alpha + \beta i}{i(a+bi)^2}.$$

Now the infinities of w are included among the points

(i) $x = \infty$, which has just been considered; there are two possible values of λ in each sheet:

(ii) $y = 0$, with $x = 1$, $x = -1$, which are the branch-points of the surface:

(iii) $ax + by = 0$, in each sheet.

Moreover, the zeros of w are unknown from the differential equation: but they must be considered, because each of them gives a pole of ϖ. Let such an one be

(iv) $x = f$,

the number of such points being unknown. All these points, whether infinities of w or zeros of w, can be singularities of ϖ.

As regards the branch-points (ii), we may take

$$y = \eta, \quad x = 1 - \tfrac{1}{2}\eta^2 + \ldots,$$

in the vicinity of 1, 0, where η is small; and then

$$-\frac{1}{\eta}\frac{d\varpi}{d\eta} + \varpi^2 = \frac{\alpha}{a^2\eta},$$

so far as the governing term in ϖ is concerned. If this be

$$\varpi = \frac{A}{\eta^n},$$

where $n > 0$, then

$$n+2=2n, \quad A^2+nA=0.$$

Thus $n=2$; and we can have $A=-2$, or $A=0$, as possible values.

Similarly for the vicinity of -1, 0.

Next, at the two points (iii), where $ax+by=0$, we have

$$\frac{x}{b}=\frac{y}{-a}=\frac{1}{(a^2+b^2)^{\frac{1}{2}}},$$

say

$$x=\sin\psi, \quad y=\cos\psi, \quad a\tan\psi=-b.$$

Then, in the vicinity, we take

$$x=\sin\psi+\xi, \quad y=\cos\psi-\xi\tan\psi+\ldots,$$

so that

$$ax+by=\xi(a-b\tan\psi)+\ldots$$

$$=\frac{a^2+b^2}{a}\xi+\ldots.$$

Thus the equation is

$$\frac{d\varpi}{d\xi}+\varpi^2=-\frac{(ab-\beta a)\,a}{(a^2+b^2)^2}\frac{1}{\xi^2},$$

so far as the governing term in ϖ is concerned. If this governing term be

$$\varpi=\frac{\sigma}{\xi}=\frac{\sigma(a^2+b^2)}{a}\frac{1}{ax+by},$$

we have

$$\sigma^2-\sigma=-\frac{a(ab-\beta a)}{(a^2+b^2)^2}.$$

Thus there are two possible values of σ at each of the two points.

Lastly, as regards a point such as $x=f$ in the set (iv), it is easy to see that, if the governing term in ϖ be

$$\frac{B}{(x-f)^n},$$

then

$$2n=n+1, \quad B^2=nB;$$

that is, $n=1$, and either $B=1$, $B=0$, are possible values. This holds for every such point $x=f$ and in each sheet.

Our required function $\varpi(x, y)$, if it exists, is to be a rational function of x and y, and we have obtained all the singularities that, in any circumstances, it might possess. We accordingly must take some combination of the possible infinities, which are

$x=\infty$, with any of the values of λ,

$x=\pm 1$, $y=0$, with either $A=-2$, or $A=0$,

$ax+by=0$, with any of the values of σ,

$x=f$, with $B=1$, or $B=0$.

A possible form is clearly

$$\varpi = \frac{C}{ax+by},$$

where C is a constant. We have (if this be admissible)

$$\frac{C}{a+bi} = \lambda,$$

from the first of the possible infinities: we take $A=0$ from the second: then

$$C = \sigma \frac{a^2+b^2}{a},$$

from the third: and we take $B=0$ from the fourth. Hence we must have

$$\lambda = \sigma \frac{a-bi}{a},$$

for some possible values of λ and of σ: that is,

$$1 \pm \left\{1 - \frac{4ai - 4\beta}{(a+bi)^2}\right\}^{\frac{1}{2}} = \frac{a-bi}{a}\left[1 \pm \left\{1 - \frac{4a}{(a^2+b^2)^2}(ab - \beta a)\right\}^{\frac{1}{2}}\right],$$

the signs being at our disposal. This leads to a single value of β, viz.

$$\beta = \frac{a^2}{b^2} - a\frac{a}{b};$$

and the condition is satisfied by taking the negative sign on both sides. We then have

$$C = \frac{a}{b};$$

so that, with the above value of β, an integral of the equation

$$\frac{d^2w}{dx^2} = w \frac{ax+\beta y}{y(ax+by)^2}$$

is given by

$$w = e^{\int \frac{a\,dx}{b\,(ax+by)}}.$$

Actual evaluation of the integral in the exponential can easily be effected.

Of course, it would have been possible to discuss the particular equation by taking

$$x = \frac{2t}{1+t^2}, \quad y = \frac{1-t^2}{1+t^2},$$

with t as the new independent variable; for the algebraic relation is of genus zero, and therefore* the variables can be expressed as rational functions of a new parameter. The new form of equation would then have uniform coefficients. But the foregoing method, that has been adopted, is possible for an equation $\psi(x, y) = 0$ of any genus.

* *T. F.*, § 247.

Association with Automorphic Functions.

155. It is manifest that some of the complexity in the analysis associated with the construction of integrals, either in general or in the vicinity of particular points, would be removed, if the equation could be changed so that, in its new form, its coefficients are uniform functions of the independent variable. This change would be secured, if both the variables x and y in the relation

$$\psi(x, y)=0$$

were expressed as uniform functions of a new variable z.

Now it is known* that, when the genus of this relation is zero, both x and y can be expressed as rational functions of a new variable z, which itself is a rational function of x and y: moreover, the expressions contain (explicitly or implicitly) three arbitrary parameters, which may be used to simplify the form of the resulting equation. Again†, when the genus of the relation is unity, both x and y can be expressed as uniform doubly-periodic functions of a new variable z, while $\wp(z)$ and $\wp'(z)$ are rational functions of x and y; moreover, the expressions contain (explicitly or implicitly) one arbitrary parameter, which again may be used to simplify the form of the resulting equation. And, in each case, definite processes are known by which the formal expressions of x and y, in terms of the new variable, can actually be obtained.

When the genus of the algebraical relation

$$\psi(x, y)=0$$

is greater than unity, a corresponding transformation is possible by means of automorphic functions: not merely so, but such a transformation can be effected in an unlimited number of ways. Further, it is possible to choose transformations that simplify the properties of the integrals of the differential equations to which they are applied. But, down to the present time, the instances in which the complete formal expressions of x and y have been obtained, and the application to the differential equations has been made, are comparatively rare. The results that have been

* *T. F.*, § 247.

† *T. F.*, § 248.

established are of the nature of existence-theorems. It is true that indications for the construction of formal expressions are given; but the detailed analysis required to carry out the indications is of so elaborate a character that it may fairly be said to be incomplete. The subject presents great, if difficult, opportunities for research in its present stage.

A brief account, based mainly on the work* of Poincaré, is all that will be given here. References to the investigations of Klein and others in the region of automorphic functions will be found elsewhere†.

The main properties of infinite discontinuous groups and of functions, which are automorphic for the substitutions of the groups, will be regarded as known. It is convenient to associate with any group a region of variation of the variable which is a fundamental region; and for the sake of simplicity in the following explanations, it will be assumed that this region is such that, when the substitutions are applied to it in turn, the whole plane is covered once, and once only. Further, also for the sake of simplicity, it will be assumed that the axis of real quantities in the plane is conserved by the substitutions of the group. There are corresponding investigations, which establish the results when these assumptions are not made; but, as already indicated, the results are mainly of the nature of existence-theorems and cannot be regarded as possessing any final form, so that the kind of consideration adduced will be sufficiently illustrated by dealing with the simplest cases. In order to deal with the most general cases, it is necessary to utilise the theory of automorphic functions in all its generality; yet the subject still is merely in a stage of growth, being far from its complete development‡.

156. It is known§ that, if x and y be two uniform functions of a variable z, which are automorphic for an infinite

* This work is best expounded in his five valuable memoirs in *Acta Mathematica*, t. I (1882), pp. 1—62, 193—294, *ib.*, t. III (1883), pp. 49—92, *ib.*, t. IV (1884), pp. 201—312, *ib.*, t. V (1884), pp. 209—278.

† *T. F.*, chapters XXI, XXII.

‡ The most consecutive account of the subject is to be found in Fricke und Klein's *Vorlesungen ü. d. Theorie d. automorphen Functionen* (Leipzig, Teubner; vol. I, 1897; vol. II, part I, 1901).

§ *T. F.*, § 309.

discontinuous group of substitutions effected on z, then some algebraic relation

$$\psi(x, y) = 0$$

subsists between them. Conversely, if this algebraic equation be given, it is desirable to express the variables x and y as uniform automorphic functions of a new variable z. For this purpose, we note that for general values of x, the variable y is a uniform analytic function* of x; but there are special values of x, being the branch-points, at and near which y ceases to be uniform. Now suppose that x can be expressed as a uniform automorphic function of z, say

$$x = f(z),$$

the fundamental polygon being such that the branch-point values of x correspond to its corners (or to some of them), which include all the essential singularities of the uniform function $f(z)$. Then, when substitution is made in the above relation, it becomes an equation defining y as a function of z; so long as z varies within the polygonal region, y does not approach those values where it ceases to be uniform, for they are given only by the corners of the polygon. Hence y becomes a uniform function† of z; and as x is automorphic for the group of the polygon, it is at once seen that y also is automorphic for that group.

Further, suppose that at the same time there is given a linear differential equation of any order, in which the coefficients are rational functions of x and y. In addition to the branch-points which may be singularities of the equation, it may have a limited number of other singularities. Let such a singularity be $x = a$, $y = b$, where of course $\psi(a, b) = 0$: for the moment, the question of the regularity of the integrals in the vicinity is not raised. If the polygon is constructed, so that $x = a$ corresponds to one of its corners which is an essential singularity of the group, then that corner is an essential singularity of $f(z)$. Hence, when the differential equation is transformed so that z becomes the independent variable, the original singularities no longer occur so long as z is restricted to variation within the fundamental polygon: they can occur only for the special values at the corresponding corners. If, further, the function $f(z)$ is such that no special

* *T. F.*, § 97.

† Another method of obtaining this result is indicated in § 160.

singularities for values of z are introduced for values of x that are ordinary points of the equation, which will be the case if $f'(z)$ does not vanish within the polygon, then all the values of z within the polygon are ordinary points of the equation, and all the integrals are synectic everywhere within the polygon. The singularities have been transferred to the boundary of the z-region; and thus the variables x and y, as well as all the integrals of the given linear differential equation which has rational functions of x and y for its coefficients, can be expressed as uniform functions of z within the region of its variation.

Automorphic Functions and Conformal Representation.

157. The relation between the variable z and the function $x=f(z)$ can be considered in two different ways, the analytical expression of the significance being the same for the two ways.

In the first place, the relation can be regarded as one of conformal representation. Assuming for the sake of simplicity that all the singular values of x are real, consider the problem* of representing the upper half of the x-plane bounded by the axis of real quantities conformally upon a polygon in the z-plane, bounded by circular arcs and having m sides: this conformal representation is known to be possible. If its expression be

$$x=f(z),$$

then $f'(z)$ must not become zero or infinite anywhere within the polygon, that is, for any finite values of x; for otherwise, the magnification would be zero or infinite there, a result that is excluded save at possible singularities on the boundary.

It is manifest that the representation remains substantially the same, if the z-plane be subjected to any homographic transformation

$$z=\frac{a'\zeta+b'}{c'\zeta+d'},$$

where $a'd'-b'c'=1$; for this will merely change the polygon bounded by circular arcs into another polygon similarly bounded.

* *T. F.*, § 271.

Hence, in constructing the function for the conformal representation, account must be taken of this possibility; and therefore, as

$$\{z, x\} = \{\zeta, x\},$$

where $\{z, x\}$ is the Schwarzian derivative, we construct this function $\{z, x\}$. We have*

$$\{z, x\} = \tfrac{1}{2} \Sigma \frac{1-\alpha^2}{(x-a)^2} + \Sigma \frac{A_0}{x-a} = 2I(x),$$

say, where the summation on the right-hand side extends over all the singular values a of x; the internal angle of the z-polygon at the corner homologous with a is $\alpha\pi$, and the coefficients A_0 are real quantities. If ∞ is an ordinary value of x, so that no angular point of the polygon is its homologue, then

$$\begin{aligned} 0 &= \Sigma A_0, \\ 0 &= \Sigma a\, A_0 + \tfrac{1}{2}\Sigma(1-\alpha^2), \\ 0 &= \Sigma a^2 A_0 + \Sigma a(1-\alpha^2). \end{aligned}$$

If ∞ is a singular value of x, which has an angular point of the polygon as its homologue, with the internal angle equal to $\kappa\pi$, then

$$\begin{aligned} \Sigma A_0 &= 0, \\ \Sigma a A_0 &= -\tfrac{1}{2}\Sigma(1-\alpha^2) + \tfrac{1}{2}(1-\kappa^2), \end{aligned}$$

the summations being over all the finite singular values of x.

The number of constants is sufficient for the representation. In the case when ∞ is not the homologue of an angular point of the polygon, we have m constants a, m constants α, and m constants A_0, subjected to three relations as above; as all these constants are real, there are $3m-3$ independent constants. But, if

$$x = \frac{a''X + b''}{c''X + d''},$$

where $a''d'' - b''c'' = 1$ and the constants a'', b'', c'', d'' are real, then the upper half of the x-plane is transformed into itself; hence the m constants a are effectively only $m-3$ in number, and thus the constants in $I(x)$ are equivalent to $3m-6$ independent constants, which can be used to make a solution determ-

* The whole investigation is due to Schwarz: see *T. F.*, § 271.

inate. On the other hand, to determine the polygon, $3m$ constants are needed, viz. two coordinates for each of the m corners and a radius for each arc: but these are subject to a reduction by 6, for the representation is determinate subject to a transformation

$$z = \frac{a'\zeta + b'}{c'\zeta + d'},$$

where $a'd' - b'c' = 1$, and the constants a', b', c', d' are complex, so that there are six real parameters undetermined. The number of available constants is therefore sufficient for the number of conditions that must be satisfied.

In the case when ∞ is the homologue of an angular point, we have $m-1$ constants a, m constants α, and m constants A_0, subjected to two relations as above; as all these constants are real, they are equivalent to $3m-3$ independent constants. The remainder of the argument is the same as before; and we infer that the number of constants is sufficient to satisfy the number of conditions for the conformal representation.

It need hardly be pointed out that, thus far, the polygon bounded by circular arcs is any polygon whatever; it has been taken arbitrarily, and it does not necessarily satisfy the conditions of being a fundamental region suited for the construction of automorphic functions.

158. That polygons can be drawn in the z-plane, suited to the construction of automorphic functions in connection with a given algebraic relation $\psi(x, y) = 0$, may be seen as follows. For simplicity, let the polygon be of the first family*, and let it have $2n$ edges arranged in n conjugate pairs; and suppose that q is the number of cycles of its corners, each cycle being closed. The genus p of the group is given by

$$2p = n + 1 - q.$$

When the surface included by the polygon is deformed and stretched in such a manner that conjugate edges are made to coincide by the coincidence of homologous points, then for each cycle in the polygon there is a single position on the closed

* *T. F.*, §§ 292, 293.

surface obtained by the deformation. This closed surface corresponds* to the Riemann surface for the equation

$$\psi(x, y) = 0,$$

which also is of genus p; and thus there are q positions on the surface, each associated with one of the q cycles. Each such position requires a couple of real parameters to define it; and thus we have $2q$ real parameters. Equations, which are birationally transformable into one another, are not regarded as independent: and therefore the effective number of constants in $\psi(x, y) = 0$ to be taken into account is $3p - 3$, being the number† of class-moduli which are invariantive under birational transformation. Each of these is complex, so that the number of real parameters thus arising is $6p - 6$. We therefore have to provide for $6p - 6 + 2q$ real parameters, by means of the polygon.

In order that the polygon may be properly associated with a Fuchsian group, it must satisfy certain conditions. Its sides must be arcs of circles, the centres of which lie in the axis of real quantities. As it has $2n$ sides, we therefore require $2n$ centres on that axis and $2n$ radii, making $4n$ real quantities in all; but three of the centres may be taken arbitrarily, for the polygon now under consideration is substantially unaffected by a transformation

$$\left(z, \frac{az + b}{cz + d}\right),$$

where a, b, c, d are real; so that the total number of real quantities necessary is effectively $4n - 3$. They are, however, not sufficient of themselves to specify an appropriate polygon: for conjugate sides must be congruent, a property that imposes one condition for each pair of edges, and therefore n conditions in all: and the sum of the angles in a cycle must be a submultiple of 2π, so that q conditions in all are thus imposed. Hence the total number of real quantities necessary is

$$\begin{aligned} &= 4n - 3 - n - q \\ &= 3n - 3 - q \\ &= 6p - 6 + 2q, \end{aligned}$$

in effect, the same as the number of real parameters given.

* *T. F.*, § 310.

† *T. F.*, § 246.

Automorphic Functions and Linear Equations of the Second Order: Fuchsian Equations.

159. In the second place, the variable z, and the automorphic functions x and y, can be associated with a linear differential equation of the second order. Let

$$v_1 = \left(\frac{dx}{dz}\right)^{\frac{1}{2}}, \quad v_2 = z\left(\frac{dx}{dz}\right)^{\frac{1}{2}},$$

so that

$$z = \frac{v_2}{v_1};$$

then it is easy to verify that

$$\frac{1}{v_1}\frac{d^2v_1}{dx^2} = \frac{1}{v_2}\frac{d^2v_2}{dx^2} = \tfrac{1}{2}\frac{\{x, z\}}{x'^2},$$

where $\{x, z\}$ is the Schwarzian derivative of x with regard to z, and $x' = dx/dz$. It is a known property* that, if x is an automorphic function of z, then the function

$$\frac{\{x, z\}}{x'^2}$$

is automorphic for the same group; hence it can be expressed rationally in terms of x and y, where

$$\psi(x, y) = 0.$$

Denoting its value by $-2I$, where I is a rational function of x and y, which may be a rational function of x alone, we have v_1 and v_2 as linearly independent integrals of the equation

$$\frac{d^2v}{dx^2} + Iv = 0;$$

the quantity z is the quotient of the two integrals.

The analytical relation is effectively the same as before; for if

$$\{z, x\} = 2I,$$

we know† that z is the quotient of two integrals of

$$\frac{1}{v}\frac{d^2v}{dx^2} + I = 0.$$

* *T. F.*, § 311.

† *Treatise on Differential Equations*, § 61.

Moreover,

$$\{x, z\} = -x'^2 \{z, x\};$$

so that the results agree in form. The difference is that, regarding the relation as a problem of conformal representation, we have been able to calculate the value of I in greater detail than in the alternative mode of regarding the relation: but the considerations adduced in connection with the differential equation have been of only the most general character, and have not permitted any discussion of the form of I.

When an equation of the form

$$\frac{d^2v}{dx^2} + Iv = 0$$

is given, where I is a rational function of x, or a rational function of two variables x and y, connected by an algebraic equation

$$\psi(x, y) = 0,$$

it may happen that x and y are uniform functions of z, the quotient of two integrals of the differential equation. But these uniform functions are not necessarily, nor even generally, automorphic for a group of substitutions of z. Judging from the result of the consideration of the question as a problem of conformal representation, we should be led to expect that the constants, which survive in I after the conditions for uniformity are satisfied, might be determinable so that the uniform functions of z are automorphic. When this determination is effected, the equation is called* *Fuchsian* by Poincaré, if the group be Fuchsian.

160. We proceed to consider more particularly the properties of the equation

$$\frac{d^2v}{dx^2} + Iv = 0,$$

in relation to the quotient of its integrals. Let $x = a$, $y = b$ be a singularity of the equation, where $\psi(a, b) = 0$; and let

$$\text{Limit } [(x-a)^2 I]_{x=a} = \rho,$$

so that the indicial equation for a is

$$n(n-1) + \rho = 0.$$

* *Acta Math.*, t. IV, p. 223.

Let n_1 and n_2 be its roots, when they are unequal; then two integrals of the equation are of the form

$$v_1 = (x-a)^{n_1} + \dots,$$
$$v_2 = (x-a)^{n_2} + \dots;$$

and so

$$z = \frac{v_2}{v_1} = (x-a)^{n_2-n_1} + \dots .$$

If $\alpha\pi$ be the internal angle of the z-polygon at the angular point homologous with a, we must have

$$n_2 - n_1 = \alpha,$$

and therefore

$$1 - 4\rho = \alpha^2,$$

that is,

$$\rho = \tfrac{1}{4}(1-\alpha^2),$$

so that

$$I = \tfrac{1}{4}\frac{1-\alpha^2}{(x-a)^2} + \dots,$$

the remaining terms being of index higher than -2.

This is valid, if α is not zero. When α is zero, so that $n_1 = n_2$ and therefore $\rho = \frac{1}{4}$, the integrals of the equation are

$$v_1 = (x-a)^{n_1} + \dots,$$
$$v_2 = (x-a)^{n_1}[\{1+\dots\}\log(x-a) + \text{powers of } x-a],$$

and so, in the immediate vicinity of a, we have

$$z = \frac{v_2}{v_1} = \log(x-a) + \text{powers};$$

and then

$$I = \frac{\frac{1}{4}}{(x-a)^2} + \dots,$$

the remaining terms again being of index higher than -2.

The quantity α, in terms of which the leading fraction in I is expressed, depends upon the character of the singularity at (a, b). If the latter denote a singular combination of values for the equation

$$\psi(x, y) = 0,$$

then it is known* that the variables x and y can be expressed in the form

$$x - a = \zeta^q, \quad y - b = \zeta^p S(\zeta),$$

* *T. F.*, § 97.

where $S(\zeta)$ is a regular function of ζ, which does not vanish when $\zeta = 0$, and the expressions are valid in the immediate vicinity of the position. Let r be the least common multiple of p and q, and write

$$\alpha = \frac{1}{r}, \quad \zeta^{\frac{q}{r}} = \zeta^{\frac{1}{p'}} = z + \dots;$$

then in that vicinity, we have

$$(x-a)^\alpha = z + \dots,$$
$$y - b = z^{pp'} S(z^p),$$

so that both x and y are uniform functions of z in the vicinity.

The commonest instance occurs, when (a, b) is a simple branch-point; we then have

$$p, q = 1, 2,$$

so that $\alpha = \frac{1}{2}$.

If (a, b) be a singularity of some given differential equation of any order, say

$$\frac{d^m w}{dx^m} = \sum_{n=0}^{m-1} \phi_n(x, y) \frac{d^n w}{dx^n},$$

where $\psi(x, y) = 0$, three cases arise.

Firstly, let all the integrals be free from logarithms, and let all the exponents to which the members of a fundamental system of integrals (supposed regular) belong be commensurable: then they are integer multiples of a quantity k^{-1}, and we take

$$\alpha = \frac{1}{k}, \quad (x-a)^\alpha = z + \dots.$$

In that case, any integral is of the form

$$w = (x-a)^\mu R(x-a)$$
$$= (x-a)^{\frac{M}{k}} R(x-a)$$
$$= z^M R(z),$$

so that the integrals of the equation, as well as the variables x and y, become uniform functions of z in the vicinity of $z = 0$.

Secondly, let the integrals (still supposed regular) of the fundamental system belong to exponents some of which at least are not commensurable quantities. We take

$$z = \log(x-a) + \text{powers};$$

and then an integral of the form

$$(x-a)^{\mu} R(x-a),$$

becomes

$$e^{\mu z} R_1(z),$$

i.e., a uniform function of z, valid for large values of $|z|$: and this uniformity is maintained whether μ is commensurable or not.

Thirdly, let $x=a$ be an essential singularity of one or more of the integrals, supposed irregular there. As in the last case, we take

$$z = \log(x-a) + \text{powers};$$

the integral may or may not become uniform for large values of $|z|$.

In the last two cases, if the expression for x in terms of z, say

$$x = f(z),$$

be automorphic, then $z=\infty$ is an essential singularity of the function $f(z)$; and then, when z varies within the polygonal region, x does not approach the value a for which the integrals of the equation cease to be regular. Within the region, the integrals are uniform. It is to be noted that the relation, adopted in the second case and the third case, would be effective in the first case also, so far as securing uniformity; but the converse does not hold. The relation which, as seen above, corresponds to the vicinity of an angular point of the polygon where the sides touch, is the most generally applicable of all: the form of relation, corresponding to the first case, is applicable only under the somewhat restricted conditions of that case.

161. These conditions and limitations affect the quantity I in the equation

$$\frac{d^2v}{dx^2} + Iv = 0,$$

for they determine the leading coefficient in its expansion near any of its poles; but, in general, they do not determine I completely. On the other hand, we so far have only secured the uniformity in character of the functional expression of x in terms of z: the automorphic property of the functional expression has not been secured. The latter is effected by the proper assignment of the remaining parameters in I.

As a special instance, take the case in which the genus of the group and of the permanent equation is zero; so that, if the polygon has $2n$ edges, the number of cycles q is given by

$$q = n + 1.$$

Taking the angular points in order as $A_1, A_2, \ldots, A_{2n}$, and making the sides

$$\left.\begin{matrix} A_1 A_2 \\ A_1 A_{2n} \end{matrix}\right\}, \quad \left.\begin{matrix} A_2\ A_3 \\ A_{2n} A_{2n-1} \end{matrix}\right\}, \ldots \Big\}, \quad \left.\begin{matrix} A_{n-1} A_n \\ A_{n+3} A_{n+2} \end{matrix}\right\}, \quad \left.\begin{matrix} A_n\ \ A_{n+1} \\ A_{n+2} A_{n+1} \end{matrix}\right\},$$

to be conjugate pairs, the necessary $n+1$ cycles are

$$A_1;\ A_2,\ A_{2n};\ A_3,\ A_{2n-1};\ \ldots;\ A_n,\ A_{n+2};\ A_{n+1}.$$

To define the polygon of $2n$ circular arcs, which have their centres on the axis of real quantities, we require the $4n$ coordinates of the angular points; but these effectively are only $4n - 3$ quantities, because the z-plane is determinate, subject only to a transformation

$$\left(z, \frac{az+b}{cz+d}\right),$$

where a, b, c, d are real. In each cycle, the sum of the angles is a submultiple of 2π: so that $n + 1$ conditions are thus imposed. Again, the edges in a conjugate pair must be congruent; so that n further conditions are thus imposed. Accordingly, there remain $2n - 4$ real independent constants to determine the polygon.

The polygon thus determined defines a Fuchsian function; as the genus is zero, every function can be expressed rationally in terms of x, so that the equation for v (leading to z, as the quotient of two integrals) is

$$\frac{d^2v}{dx^2} + Iv = 0,$$

where I is a rational function of x. Corresponding to the $n+1$ cycles, there are $n+1$ values of x; let these be

$$a_1,\ a_2,\ \ldots,\ a_n,\ \infty.$$

Let $\alpha_r \pi$ be the sum of the internal angles of the z-polygon corresponding to a_r, so that α_r is the reciprocal of an integer; and take α_{n+1} to be the quantity α for ∞. Then in the vicinity of a_r, we have

$$I = \frac{1}{4} \frac{1 - \alpha_r^2}{(x - a_r)^2} + \ldots,$$

for each of the values of r. Thus, if we write

$$A = (x - a_1)(x - a_2) \dots (x - a_n),$$

and remember that I is a rational function of x, we have

$$I = \frac{G(x)}{A^2},$$

where

$$G(a_r) = \left[\frac{dA}{dx}\right]^2_{x=a_r},$$

for $r = 1, \dots, n$. In order to satisfy the condition for $x = \infty$, $G(x)$ must be of order $2n-2$, and

$$G(x) = \tfrac{1}{4}(1 - \alpha^2_{n+1})\, x^{2n-2} + \dots .$$

The number of coefficients in $G(x)$ is $2n-1$; but the coefficient of the highest power is known, and there are n relations among the rest, owing to the conditions at $a_1, \dots, a_n$; hence there remain $n-2$ coefficients independent of one another. Each of these is complex in general, so that they are effectively equivalent to $2n-4$ real constants. Assuming that the quantities $a_1, \dots, a_n$ are known, it is to be expected that the $2n-4$ conditions for the polygon determine these $2n-4$ real constants.

In the simplest case, we have $n = 2$; and we may take $a_1 = 0$, $a_2 = 1$, so that

$$I = \tfrac{1}{4}\frac{1 - \alpha_1^2}{x^2} + \tfrac{1}{4}\frac{1 - \alpha_2^2}{(x-1)^2} + \frac{\rho}{x} + \frac{\sigma}{x-1}.$$

The conditions for $x = \infty$ give

$$\rho + \sigma = 0,$$

$$\tfrac{1}{4}(1 - \alpha_1^2) + \tfrac{1}{4}(1 - \alpha_2^2) + \sigma = \tfrac{1}{4}(1 - \alpha_3^2),$$

where $\alpha_1, \alpha_2, \alpha_3$ are the reciprocals of integers; the quantity I then is the invariant of the hypergeometric series.

162. As another illustration, which may be treated somewhat differently, consider the equation

$$y^2 = x(1-x)(1-cx),$$

where c is a real constant less than unity; and write

$$ac = 1,$$

so that a is a real constant greater than unity. Here, the points $x=0, 1, a, \infty$ are each of them singular; and the value of α is $\frac{1}{2}$ for each of them. Consequently,

$$I=\frac{\frac{1}{4}(1-\frac{1}{4})}{x^2}+\frac{\frac{1}{4}(1-\frac{1}{4})}{(x-1)^2}+\frac{\frac{1}{4}(1-\frac{1}{4})}{(x-a)^2}+\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x-a};$$

and the conditions for $x=\infty$ give

$$A+B+C=0,$$

$$\tfrac{9}{16}+B+Ca=\tfrac{3}{16}.$$

One constant in I is left undetermined by these conditions; thus

$$I=\frac{\frac{3}{16}}{x^2}+\frac{\frac{3}{16}}{(x-1)^2}+\frac{\frac{3}{16}}{(x-a)^2}-\frac{\frac{3}{8}x+\lambda}{x(x-1)(x-a)},$$

say, where λ is the undetermined constant. It is possible to determine λ, so that x is a Fuchsian function of z, where z is the quotient of two solutions of the equation

$$\frac{d^2v}{dx^2}+Iv=0.$$

As regards this Fuchsian function, its polygon may be obtained simply as follows. We take four points A, B, C, D in the z-plane to be the homologues of $0, 1, a, \infty$; owing to the value of α, which is $\frac{1}{2}$ in each case, the internal angles of the polygon must each be $\frac{1}{2}\pi$. We make the edges AB, CD conjugate, and likewise the edges BC, DA; and then there is a single cycle, $ADCB$, the sum of the angles in which is 2π. With the former notation, we thus have $q=1$, $n=2$; so that

$$2p=2+1-1=2,$$

and therefore $p=1$, as should be the case. Further, the sum of the angles of a curvilinear triangle, entirely on one side of the real axis, is less than π, when the centres of the circular arcs lie on the real axis: so that, if our polygon be curvilinear, the sum of its angles would be less than 2π (for it could be made up of two triangles), whereas the sum is actually 2π. Hence the polygon can only be a rectangle, and the Fuchsian functions are doubly-periodic. We therefore take

$$x=\text{sn}^2 z,\quad y=\text{sn}\, z\ \text{cn}\, z\ \text{dn}\, z,$$

as is manifestly permissible; and then

$$z' = \frac{dz}{dx} = \frac{1}{2y} = \frac{1}{2c^{\frac{1}{2}} x^{\frac{1}{2}} (x-1)^{\frac{1}{2}} (x-a)^{\frac{1}{2}}},$$

which leads to

$$\begin{aligned}\{z, x\} &= \tfrac{3}{8}\left[\frac{1}{x^2} + \frac{1}{(x-1)^2} + \frac{1}{(x-a)^2}\right] - \tfrac{1}{4}\,\frac{3x-a-1}{x(x-1)(x-a)} \\ &= 2I,\end{aligned}$$

so that we have

$$\lambda = -\tfrac{1}{8}(a+1).$$

This value of λ renders x (and so y) a Fuchsian function of the quotient of two solutions of the equation

$$\frac{d^2v}{dx^2} + Iv = 0.$$

As regards the integrals of this equation, the indicial equation of $x=0$ is

$$\rho(\rho-1) + \tfrac{3}{16} = 0,$$

so that $\rho = \frac{1}{4}$, $\rho = \frac{3}{4}$. Denoting by v_1 and v_2 the integrals that belong to $\frac{1}{4}$ and $\frac{3}{4}$ respectively, we have

$$v_1 = x^{\frac{1}{4}}\left\{1 + 2\frac{\lambda}{a}x + \dots\right\},$$

$$v_2 = x^{\frac{3}{4}}\left\{1 + \tfrac{2}{3}\frac{\lambda}{a}x + \dots\right\}.$$

Hence

$$\zeta = \frac{v_2}{v_1} = x^{\frac{1}{2}}\,\frac{1 + \frac{2}{3}\frac{\lambda}{a}x + \dots}{1 + 2\frac{\lambda}{a}x + \dots},$$

and therefore

$$\begin{aligned}x &= \zeta^2 + \tfrac{8}{3}\frac{\lambda}{a}\zeta^4 + \dots \\ &= \zeta^2 - \tfrac{1}{3}(1+c)\zeta^4 + \dots \\ &= \operatorname{sn}^2\zeta,\end{aligned}$$

after the earlier analysis.

Similarly, in the vicinity of $x=1$, we find integrals

$$V_1 = (x-1)^{\frac{1}{4}}\left\{1 + 2\frac{\lambda + \frac{3}{8}}{1-a}(x-1) + \dots\right\},$$

$$V_2 = (x-1)^{\frac{3}{4}}\left\{1 + \tfrac{2}{3}\frac{\lambda + \frac{3}{8}}{1-a}(x-1) + \dots\right\};$$

and then, taking

$$\zeta_1 = \frac{V_2}{V_1},$$

we find

$$\begin{aligned} x - 1 &= \zeta_1{}^2 + \tfrac{8}{3}\,\frac{\lambda + \frac{3}{8}}{1 - a}\,\zeta_1{}^4 + \dots \\ &= \zeta_1{}^2 + \tfrac{1}{3}\,\frac{a - 2}{a - 1}\,\zeta_1{}^4 + \dots \\ &= \zeta_1{}^2 + \tfrac{1}{8}\,\frac{1 - 2c}{1 - c}\,\zeta_1{}^4 + \dots. \end{aligned}$$

Now

$$\begin{aligned} x - 1 &= -\operatorname{cn}^2 \zeta \\ &= -(1 - c)\,\frac{\operatorname{sn}^2(\zeta - K)}{\operatorname{dn}^2(\zeta - K)} \\ &= -(1 - c)(\zeta - K)^2 + \tfrac{1}{3}(1 - 2c)(1 - c)(\zeta - K)^4 - \dots, \end{aligned}$$

so that

$$\zeta_1 = (c - 1)^{\frac{1}{2}}(\zeta - K).$$

Hence

$$\frac{V_2}{V_1} = (c - 1)^{\frac{1}{2}}\left(\frac{v_2}{v_1} - K\right),$$

so that, as

$$\begin{aligned} V_2 &= A v_2 + B v_1, \\ V_1 &= C v_2 + D v_1, \end{aligned}$$

where $AD - BC = 1$, because

$$V_2 \frac{dV_1}{dx} - V_1 \frac{dV_2}{dx} = \text{constant},$$

we have

$$\left.\begin{aligned} V_2 &= (c - 1)^{\frac{1}{4}}(v_2 - K v_1) \\ V_1 &= (c - 1)^{-\frac{1}{4}} v_1 \end{aligned}\right\}.$$

Again, in the vicinity of $x = a$, we find integrals

$$U_1 = (x - a)^{\frac{1}{4}}\left\{1 + 2\,\frac{\lambda + \frac{3}{8}a}{a(a - 1)}(x - a) + \dots\right\},$$

$$U_2 = (x - a)^{\frac{3}{4}}\left\{1 + \tfrac{2}{3}\,\frac{\lambda + \frac{3}{8}a}{a(a - 1)}(x - a) + \dots\right\};$$

and then, taking

$$\zeta_2 = \frac{U_2}{U_1},$$

we find

$$x - a = \zeta_2^2 + \tfrac{8}{3} \frac{\lambda + \frac{3}{8}a}{a(a-1)} \zeta_2^4 + \dots$$

$$= \zeta_2^2 + \tfrac{1}{3} \frac{(2-c)c}{1-c} \zeta_2^4 + \dots.$$

Also

$$\begin{aligned} x - a &= \operatorname{sn}^2 \zeta - \frac{1}{c} \\ &= -\frac{1}{c} \operatorname{dn}^2 \zeta \\ &= \frac{1-c}{c} \frac{\operatorname{sn}^2(\zeta - K - iK')}{\operatorname{cn}^2(\zeta - K - iK')} \\ &= \frac{1-c}{c}(\zeta - K - iK')^2 + \tfrac{1}{3} \frac{(1-c)(2-c)}{c} (\zeta - K - iK')^4 + \dots, \end{aligned}$$

so that

$$\zeta_2 = \left(\frac{1-c}{c}\right)^{\frac{1}{2}} (\zeta - K - iK').$$

Proceeding as before, this leads to the relations

$$\left.\begin{aligned} U_2 &= \left(\frac{1-c}{c}\right)^{\frac{1}{2}} \{v_2 - (K + iK') v_1\} \\ U_1 &= \left(\frac{1-c}{c}\right)^{-\frac{1}{2}} v_1 \end{aligned}\right\}.$$

Lastly, for large values of x, we have

$$W_2 = x^{\frac{1}{2}} \left\{1 - \tfrac{1}{12}(1+a)\frac{1}{x} + \dots\right\},$$

$$W_1 = x^{\frac{1}{2}} \left\{1 - \tfrac{1}{4}(1+a)\frac{1}{x} + \dots\right\};$$

and then, taking

$$\zeta_3 = \frac{W_2}{W_1},$$

we find

$$\frac{1}{x} = \zeta_3^2 - \tfrac{1}{3}(1+a)\zeta_3^4 + \dots.$$

Now

$$\begin{aligned} \frac{1}{x} &= \frac{1}{\operatorname{sn}^2 \zeta} \\ &= c \operatorname{sn}^2(\zeta - iK') \\ &= c(\zeta - iK')^2 - \tfrac{1}{3} c^2 (\zeta - iK')^4 (1+a) + \dots, \end{aligned}$$

so that

$$\zeta_3 = c^{\frac{1}{2}}(\zeta - iK').$$

Proceeding as before, this leads to the relations

$$\left.\begin{aligned} W_2 &= c^{\frac{1}{4}}(v_2 - iK'v_1) \\ W_1 &= c^{-\frac{1}{4}} v_1 \end{aligned}\right\}.$$

The relations, in fact, have enabled us to construct the expressions for each fundamental system in terms of the first and, therefore by inference, in terms of every other.

Ex. 1. Discuss in the same way the Fuchsian differential equation

$$\frac{1}{v}\frac{d^2v}{dx^2} + \tfrac{3}{16}\left\{\frac{1}{(x-e_1)^2} + \frac{1}{(x-e_2)^2} + \frac{1}{(x-e_3)^2}\right\} - \frac{\frac{3}{8}x}{(x-e_1)(x-e_2)(x-e_3)} = 0,$$

connected with the equation

$$y^2 = 4x^3 - g_2 x - g_3.$$

Ex. 2. Shew that, if

$$x = \wp(\log z),$$

where $\wp$ denotes Weierstrass's elliptic function,

$$\{z, x\} = \tfrac{3}{8}\left[\frac{1}{(x-e_1)^2} + \frac{1}{(x-e_2)^2} + \frac{1}{(x-e_3)^2}\right] - \tfrac{1}{8}\frac{1}{(x-e_1)(x-e_2)(x-e_3)};$$

and discuss the significance of the integral relation in regard to its pseudo-automorphic character for the equation

$$y^2 = 4x^3 - g_2 x - g_3.$$

Ex. 3. A fundamental polygon in the z-plane is composed of two semicircles, one upon a diameter in the real axis for values of z corresponding to values of x equal to 0 and 1, the other upon a similar diameter for values of x equal to 1 and a, (where $a > 1$), and of two straight lines drawn, through points corresponding to 0 and a, perpendicular to the axis of real quantities. Prove that the subsidiary equation of the second order, for the construction of x as an automorphic function of the quotient of two of its integrals, is

$$\frac{1}{v}\frac{d^2v}{dx^2} = -\tfrac{1}{4}\left[\frac{1}{x^2} + \frac{1}{(x-1)^2} + \frac{1}{(x-a)^2}\right] + \frac{\frac{1}{2}x + \mu}{x(x-1)(x-a)},$$

where the constant μ is to be properly determined.

Automorphic Functions used to make the Integrals of any Equation Uniform.

163. If, for any given equation, there is only one singularity, it can be made to lie at the origin.

In order to obtain a variable z, in terms of which the integrals of the given equation can be expressed uniformly, we construct an

equation of the second order which has $x = 0$ for a singularity, of such a form that the indicial equation for $x = 0$ has equal roots (§ 160). This auxiliary equation may have other singularities, but otherwise it may be kept as simple as possible. Such an equation is

$$\frac{d^2v}{dx^2} = \frac{\lambda}{x^2} v;$$

the indicial equation for $x = 0$ is

$$\theta(\theta - 1) = \lambda,$$

so that $\lambda = -\frac{1}{4}$ if it has equal roots. Thus the equation is

$$\frac{d^2v}{dx^2} + \frac{1}{4}\frac{v}{x^2} = 0.$$

Two integrals are given by

$$v_1 = x^{\frac{1}{2}}, \quad v_2 = x^{\frac{1}{2}} \log x;$$

thus

$$z = \frac{v_2}{v_1} = \log x,$$

which is the new independent variable.

An equation of the kind indicated is (§ 45, Ex. 6)

$$\frac{d^2u}{dx^2} + \left(\frac{3}{x} + \frac{1}{2x^2}\right)\frac{du}{dx} + \left(\frac{1}{x^3} - \frac{1}{2x^4}\right)u = 0:$$

when the variable is changed from x to z, where $x = e^z$, the equation becomes

$$\frac{d^2u}{dz^2} + (2 + \tfrac{1}{2}e^{-z})\frac{du}{dz} + (e^{-z} - \tfrac{1}{2}e^{-2z})u = 0.$$

The integrals are synectic for all finite values of z.

164. When a given differential equation has two singularities, a homographic transformation can be applied so as to fix them at $x = 0$, $x = 1$.

To obtain a variable z in terms of which the integrals of the given equation can be expressed uniformly, we construct an equation of the second order, having 0 and 1 as its singularities and such that the respective indicial equations have repeated roots. An appropriate equation is

$$\frac{d^2v}{dx^2} = \frac{\alpha + \beta x}{x^2(x-1)^2} v.$$

The indicial equation for $x=0$ is

$$\rho(\rho-1)=\alpha,$$

so that $\alpha=-\frac{1}{4}$; the indicial equation for $x=1$ is

$$\rho(\rho-1)=\alpha+\beta,$$

so that $\alpha+\beta=-\frac{1}{4}$, and therefore $\beta=0$, so that the equation is

$$\frac{d^2v}{dx^2}+\frac{\frac{1}{4}}{x^2(x-1)^2}v=0.$$

One integral is easily found to be

$$v_1=x^{\frac{1}{2}}(x-1)^{\frac{1}{2}};$$

and then z, the quotient of another integral by v_1, is given by

$$\frac{dz}{dx}=\frac{C}{v_1{}^2}=\frac{-1}{x(x-1)},$$

on particularising the constant C, which may be arbitrary. Thus

$$x=\frac{e^z}{e^z-1}$$

gives a new variable z, such that the integrals of the given differential equation are uniform functions of z.

Thus let the equation be

$$\frac{d^2y}{dx^2}+\frac{2x+a}{x(x-1)}\frac{dy}{dx}+\frac{b}{x^2(x-1)^2}y=0,$$

which has $x=0$ and $x=1$ for real singularities: it is easy to verify that $x=\infty$ is not a singularity but only an ordinary point for every integral. When the equation is transformed so that z is the independent variable, it becomes

$$\frac{d^2y}{dz^2}-(a+1)\frac{dy}{dz}+by=0,$$

the integrals of which clearly are uniform functions of z.

165. When a given differential equation has three singularities, a homographic transformation can be used so as to fix them at $x=0, 1, \infty$.

We may proceed in two ways. It may be possible to choose, as the fundamental region in the z-plane, a triangle, having circular arcs for its sides, and having $\lambda\pi$, $\mu\pi$, $\nu\pi$ for its internal angles at points which are the homologues of $0, \infty, 1$ respectively:

λ, μ, ν being the reciprocals of integers. Then the subsidiary equation may be taken in the form

$$\frac{1}{v}\frac{d^2v}{dx^2}+\frac{\frac{1}{4}(1-\lambda^2)}{x^2}+\frac{\frac{1}{4}(1-\nu^2)}{(x-1)^2}+\frac{1}{4}\frac{\lambda^2-\mu^2+\nu^2-1}{x(x-1)}=0,$$

which is the normal form of the equation of the hypergeometric series with parameters α, β, γ, where

$$\lambda^2=(1-\gamma)^2, \quad \mu^2=(\alpha-\beta)^2, \quad \nu^2=(\gamma-\alpha-\beta)^2.$$

The variable z may be taken as the quotient of two solutions of the subsidiary equation; and so

$$z=x^{1-\gamma}\frac{F(\alpha+1-\gamma,\ \beta+1-\gamma,\ 2-\gamma,\ x)}{F(\alpha,\ \beta,\ \gamma,\ x)}.$$

It is known* that x, thus defined, is a uniform automorphic function of z.

This transformation will render uniform the integrals of a differential equation, which has no singularities except at 0, 1, ∞, provided the integrals are regular in the vicinity of those singularities and belong to indices which are integer multiples of λ, ν, μ respectively. If these conditions are not satisfied, in particular, if the singularities are essential for the integrals, then we proceed by an alternative method.

We take a subsidiary equation having 0, 1, ∞ for singularities, such that the indicial equation for each of them has equal roots. Let it be

$$\frac{d^2v}{dx^2}=\frac{\alpha'+\beta'x+\gamma'x^2}{x^2(x-1)^2}v,$$

where α', β', γ' are to be chosen so that the indicial equation for each of the singularities has equal roots. These equations are

$$\rho(\rho-1)=\alpha', \quad \sigma(\sigma-1)=\alpha'+\beta'+\gamma', \quad \tau(\tau+1)=\gamma',$$

so that

$$\alpha'=-\tfrac{1}{4}, \quad \beta'=\tfrac{1}{4}, \quad \gamma'=-\tfrac{1}{4},$$

and thus the equation is

$$\frac{d^2v}{dx^2}+\tfrac{1}{4}v\frac{1-x+x^2}{x^2(x-1)^2}=0.$$

* *T. F.*, § 275.

The coefficient of v is the invariant of a hypergeometric equation, of which the parameters are

$$\alpha = \beta = \tfrac{1}{2}, \quad \gamma = 1;$$

so that z, the quotient of two integrals v, is also the quotient of two integrals of the equation

$$x(1-x)\frac{d^2w}{dx^2} + (1-2x)\frac{dw}{dx} - \tfrac{1}{4}w = 0.$$

This is the equation of the quarter-periods in elliptic functions: so that

$$z = \frac{K(x)}{K'(x)}.$$

This relation effectively defines x as a modular function* of z: the fundamental region is a curvilinear triangle. The function exists over the whole z-plane: the axis of real quantities is a line of essential singularity.

Any differential equation, having $x = 0, 1, \infty$ for all its singularities no matter what their character may be, can be transformed by the preceding relation so that z is the independent variable: its integrals are then expressible as functions of z which are uniform over the whole of the z-plane, their essential singularities lying on the axis of real quantities.

Ex. A differential equation has only three singularities at $x = a, b, c$, such that the roots of the indicial equations of those points are integer multiples of α, β, γ respectively, where α, β, γ are reciprocals of integers. Shew that a variable, in terms of which the integrals can be expressed as uniform functions, is given by taking the quotient of two Riemann P-functions with the appropriate singularities and indices.

Automorphic Functions applied to General Linear Equations of any Order.

166. At the beginning of the preceding explanations and discussions, it was assumed (§ 157) that all the singular values of x are real. The assumption was then made for the sake of simplicity: it can be proved† to be unnecessary.

* *T. F.*, § 303.

† Poincaré, *Acta Math.*, t. IV, pp. 246—250.

Firstly, let the singularities be constituted by $a_1, a_2, \ldots, a_m$, all of which are real, and by c, which will be supposed complex. With these we shall associate c_0, the conjugate of c; and we write

$$\phi(x) = (x - c)(x - c_0),$$

a quadratic polynomial with real coefficients. Then all the quantities

$$0, \ \phi(a_1), \ \phi(a_2), \ \ldots, \ \phi(a_m), \ \phi(\tfrac{1}{2}c + \tfrac{1}{2}c_0)$$

are real. Construct a fundamental region in the z-plane, such that the foregoing $m+2$ quantities are the homologues of the corners; and let

$$X = F(z)$$

be the relation that gives the conformal representation of the region upon half the X-plane, so that $F(z)$ is a Fuchsian function of z.

Consider the variable x, as defined by the equation

$$\phi(x) = F(z).$$

So long as z remains within the fundamental region, x is a uniform function of z; it could cease to be so, only if

$$\phi'(x) = 0,$$

that is, if $x = \frac{1}{2}c + \frac{1}{2}c_0$, and then we should have

$$F(z) = \phi(\tfrac{1}{2}c + \tfrac{1}{2}c_0),$$

which is not possible for values of z within the region. Also, $\dfrac{dx}{dz}$ is not zero for any value of z within the region; for then we should have

$$F'(z) = 0,$$

which would make a zero magnification between the X-plane and the z-region: this we know to be impossible for internal z-points. This uniform function x, whose derivative does not vanish within the polygon, cannot acquire either of the values c or c_0 within the polygon, for then we should have

$$F(z) = 0,$$

which is possible only at a corner. Nor can it acquire any of the values $a_1, a_2, \ldots, a_m$ for points within the z-polygon: for at any such value, we have

$$F(z) = \phi(a),$$

which again is possible only at a corner.

Now since $X = F(z)$ is a relation that conformally represents the half X-plane upon a z-polygon bounded by circular arcs (this polygon being otherwise apt for the construction of automorphic functions), we have (§ 157)

$$\{z, X\} = 2\psi(X),$$

where $\psi(X)$ is a rational function of X. But for any variables X and x, we have

$$\{z, x\} = \{z, X\}\left(\frac{dX}{dx}\right)^2 + \{X, x\};$$

and therefore, in the present case,

$$\begin{aligned}\{z, x\} &= 2(2x - c - c_0)^2 \psi(x^2 - cx - c_0 x + cc_0) - \frac{6}{(2x - c - c_0)^2} \\ &= 2\Psi(x),\end{aligned}$$

say, where $\Psi(x)$ is a rational function of x. Hence z is the quotient of two integrals of the equation

$$\frac{d^2v}{dx^2} + v\Psi(x) = 0.$$

Now x is known to be a uniform function of z; it is therefore a Fuchsian function of z. And we have proved that, for values of z within the polygon, x cannot acquire any of the real values $a_1, a_2, \ldots, a_m$ or either of the complex values c, c_0, and, further, that $\dfrac{dx}{dz}$ does not vanish.

Secondly, to extend this result to the case, when x is not to acquire any one of any number of complex values for z-points within the polygon, we adopt an inductive proof; we assume the result to hold when there are $q-1$ pairs of conjugate complex values, and shall then prove it to hold when there are q pairs. It has been proved to hold, (i), when there are no complex values and, (ii), when there is a pair of conjugate complex values: it thus will be proved to hold generally.

Suppose, then, that the given x-singularities are made up of a number m of real values $a_1, a_2, \ldots, a_m$, and of a number of complex values. Let the latter be increased in number by associating with each complex value its conjugate complex, whenever that conjugate does not occur in the aggregate; and let the increased aggregate be denoted by

$$c_1, c_1'; \; c_2, c_2'; \; \ldots; \; c_q, c_q';$$

arranged in conjugate pairs. Write

$$\phi(x) = \prod_{r=1}^{q} (x - c_r)(x - c_r'),$$

which is a polynomial of degree $2q$ with real coefficients. The equation

$$\frac{d\phi(x)}{dx} = 0,$$

of degree $2q - 1$ with real coefficients, certainly possesses one real root; its other roots, when not real, can be arranged in conjugate pairs, the number of pairs not being greater than $q - 1$. Let its roots be denoted by

$$b_1, \; b_2, \; \ldots, \; b_{2q-1},$$

an aggregate which contains not more than $q - 1$ conjugate pairs.

In the series of quantities

$$0; \; \phi(a_1), \; \ldots, \; \phi(a_m); \; \phi(b_1), \; \ldots, \; \phi(b_{2q-1});$$

there are certainly $m + 2$ real quantities; and there are not more than $q - 1$ conjugate pairs of complex quantities. According to our hypothesis, a Fuchsian function $G(z)$ can be constructed, such that the foregoing $m + 2 + 2(q - 1)$ quantities are the homologues of the corners of an appropriate fundamental region, and $G'(z)$ does not vanish within the region. Then, proceeding on the same lines as in the simpler case, we consider a variable x, defined by the relation

$$\phi(x) = G(z).$$

So long as z remains within the fundamental region, x is a uniform function of z; it could cease to be so, only if

$$\phi'(x) = 0,$$

that is, if $x = b_1, b_2, \ldots,$ or b_{2q-1}, and then we should have

$$G(z) = \phi(b_1), \; \phi(b_2), \; \ldots, \; \text{ or } \; \phi(b_{2q-1}),$$

which is not possible for values of z within the region. Also, $\frac{dx}{dz}$ does not vanish for values of z within the region; for otherwise we should have

$$G'(z) = 0$$

for such values, and this is known not to be the case. Further, x, being a uniform function of z whose derivative does not vanish for values within the polygon, cannot acquire any of the values c_r or c_r', for $r = 1, \ldots, q$, within the polygon; if it could, we should have $\phi(x) = 0$ there, and then

$$F(z) = 0,$$

which is possible only at a corner. Nor can it acquire any of the values $a_1, \ldots, a_m$ for values of z within the polygon: if it could, we should have

$$F(z) = \phi(a_1), \quad \phi(a_2), \quad \ldots, \quad \text{or} \quad \phi(a_m),$$

which again is possible only at a corner.

Now since $Y, = G(z)$, is an automorphic function, it follows* that

$$\left(\frac{dY}{dz}\right)^{-2} \{Y, z\},$$

which is equal to $-\{z, Y\}$, also is an automorphic function. Consider the upper half of the Y-plane. So far as the equation $Y = G(z)$ is concerned, certain points on the upper side of the axis of real quantities are exceptional, not more than $q-1$ in number; these can be considered as excluded, and cuts drawn from them to singular points on the real axis. We then can regard this simply-connected and resolved half-plane as conformally represented upon the polygon by the equation $Y = G(z)$; hence†

$$\{z, Y\} = 2\theta(Y),$$

where $\theta(Y)$ is a rational function of Y. But

$$\phi(x) = Y,$$

where $\phi(x)$ is a polynomial; hence

$$\begin{aligned}\{z, x\} &= \{z, Y\}\left(\frac{dY}{dx}\right)^2 + \{Y, x\}\\ &= 2\theta[\phi(x)]\left[\frac{d\phi(x)}{dx}\right]^2 + \{\phi(x), x\}\\ &= 2\Theta(x),\end{aligned}$$

* *T. F.*, § 311.

† *T. F.*, § 271.

say, where $\Theta(x)$ is a rational function of x. Hence z is the quotient of two integrals of the equation

$$\frac{d^2w}{dx^2} + w\Theta(x) = 0.$$

Now x is known to be a uniform function of z. It is therefore a Fuchsian function of z, which acquires the particular assigned values only at the corners of the fundamental region and nowhere within the region; its derivative does not vanish anywhere within the region.

The statement is thus established.

167. The preceding explanations, outlines of proofs, and analysis, will give an indication of the kind of result to be obtained, and the kind of application to differential equations to be made. It will be recognised that such proofs as have been adduced are not entirely complete: thus, when a number of real constants is to be determined by the same number of equations, whether algebraical or transcendental, it would be necessary to shew that the constants, if determined in the precise number, are real. As, however, it was stated at the beginning of these sections that only an introductory sketch of the theory would be given, there will be no attempt to complete the preceding proofs: we shall be content with referring the student, for the long and complicated processes needed to establish even the existence of certain results without evaluating their exact form, to the classical memoirs by Poincaré, and to the treatise by Fricke and Klein, which have already been quoted*. It may be convenient to recount the most important and central results of Poincaré's investigations, which have any application to the theory of linear differential equations.

Let

$$\frac{d^qw}{dx^q} = \sum_{\kappa=0}^{q-1} \phi_\kappa(x, y) \frac{d^\kappa w}{dx^\kappa}$$

be a linear equation of order q, having rational functions of x and y for its coefficients, where y is defined in terms of x by the algebraic equation

$$\psi(x, y) = 0;$$

* A memoir by E. T. Whittaker, "On the connexion of algebraic functions with automorphic functions," *Phil. Trans.* (1899), pp. 1—32, may also be consulted.

this equation in w will be called the *main* equation. Let

$$\frac{d^2v}{dx^2} = v\theta(x, y)$$

be another equation, in which $\theta(x, y)$ is a rational function of x and y; it will be called the *subsidiary* equation, and its elements are entirely at our disposal.

Let $x = a_\mu$, $y = b_\mu$, be a singularity of the main equation. If all the integrals are regular at this singularity, if they are free from logarithms, and if they belong to exponents, which are commensurable quantities (no two being equal), let k^{-1} (where k is an integer) be a quantity such that the exponents are integer multiples of k^{-1}. We make $x = a_\mu$, $y = b_\mu$, a singularity of the subsidiary equation. In the case of the indicated hypothesis as to the integrals of the main equation, we make the difference of the two roots of the indicial equation of the subsidiary equation equal to k^{-1}. In every other case, we make those two roots equal. This is to be effected for each of the singularities of the main equation.

Thus the subsidiary equation is made to possess all the singularities of the main equation. It may have other singularities also; for each of them, the difference of the two roots of the corresponding indicial equation is made either zero or the reciprocal of an integer, at our own choice. By these conditions, the coefficient $\theta(x, y)$ will be partly determinate: but a number of parameters will remain undetermined.

The effect of these conditions is, by the analysis of § 160, to make x and y uniform functions of z, where z is the quotient of two linearly independent integrals of the subsidiary equation; and no further conditions for this purpose need be imposed upon the parameters, which may therefore be used to secure other properties of the uniform functions. The various forms of θ, corresponding to the various determinations of the parameters, determine a corresponding number of differential equations; all of these are said to belong to the same *type*, which thus is characterised by the singularities and their indicial equations.

Poincaré has proved a number of propositions connected with the results that can be obtained by the appropriate assignment

of values to these parameters. Of these, the most important are :—

I. It is possible to assign a unique set of values in such a way as to secure that x and y are Fuchsian functions of z, existing only within a fundamental circle.

II. It is possible to assign sets of values, unlimited in number, in such a way in each case as to secure that x and y are Kleinian functions of z, existing over only part of the z-plane.

III. It is possible to assign a unique set of values in such a way as to secure that x and y are Fuchsian functions or Kleinian functions of z, existing over the whole of the z-plane.

There are limiting cases when the Fuchsian function becomes doubly-periodic, or simply-periodic, or rational.

Poincaré's Theorem that any Linear Equation can be integrated by means of Fuchsian and Zetafuchsian Functions.

168. Consider now the integrals of the main differential equation, when they are expressed in terms of the variable z. We shall assume that x and y have been determined as Fuchsian functions of z, existing only within the fundamental circle.

Near an ordinary point x_0, y_0, any integral w is a holomorphic function of $x - x_0$; near such a point, x is a holomorphic function of $z - z_0$; so that w, when expressed as a function of z, is a holomorphic function of z.

In the vicinity of a singularity (a, b), there are two cases to consider. If all the exponents to which the integrals belong are commensurable quantities, so that they are integer multiples of some proper fraction k^{-1}, where k is an integer, and if the integrals are free from logarithms, then every integral is of the form

$$w = (x - a)^{\frac{\mu}{k}} S(x - a),$$

where S is a holomorphic function of $x-a$. As in § 160, we have

$$z-c=(x-a)^{\frac{1}{k}}\,T(x-a),$$

so that

$$x-a=(z-c)^k\,R(z-c),$$

where T and R are holomorphic functions. Hence

$$w=(z-c)^\mu\,G(z-c),$$

where the function G is holomorphic in the vicinity of c. Thus w is a uniform function of z; if μ is positive, then c is an ordinary point; if μ is negative, it is a pole.

In all other cases, whether the integrals involve logarithms, or the exponents to which they belong are not all commensurable, or the singularity is one where some of the integrals, or even all the integrals, are irregular, the roots of the indicial equation for the subsidiary equation are equal. In consequence, the two circular arcs of any polygon touch, and thus the angular point is on the fundamental circle. As we consider the values of z within the fundamental circle, the character of the integral, when expressed as a function of z, does not arise for the point of the kind under consideration.

It thus appears that, when z is restricted to lie within the fundamental circle of the Fuchsian functions which are the representative expressions of x and y, any integral of the main equation is a uniform function of z. When this uniform function has poles, it can be represented in the form

$$\frac{G(z)}{G_1(z)},$$

where the zeros of $G_1(z)$ are the poles of the integral in unchanged multiplicity, and both $G(z)$ and $G_1(z)$ are holomorphic functions of z within the fundamental circle. When the uniform function representing the integral has no poles, it can be expressed in the form

$$H(z),$$

where the function $H(z)$ is holomorphic everywhere within the fundamental circle.

Hence we have Poincaré's theorem* that *the integrals of a linear differential equation with algebraic coefficients can be expressed as uniform functions of an appropriately chosen variable.*

* *Acta Math.*, t. IV, p. 311.

169. The characteristic property of these uniform functions can be obtained as follows. Taking the equation in the form

$$\frac{d^q w}{dx^q} = \sum_{\kappa=0}^{q-2} \phi_\kappa(x, y) \frac{d^\kappa w}{dx^\kappa}, \qquad \psi(x, y) = 0,$$

where it is supposed that the term (if any) which involved $\dfrac{d^{q-1}w}{dx^{q-1}}$ has been removed from the equation by the usual substitution (§ 152), we denote by $\theta_1, \theta_2, \ldots, \theta_q$ a fundamental system of integrals in the vicinity of any singularity (a_μ, b_μ). Let a closed path on the Riemann surface, associated with the permanent equation, be described round the singularity; then, when the path is completed, the members of the fundamental system have acquired values $\theta_1', \theta_2', \ldots, \theta_q'$, such that

$$\theta_n' = a_{1,n}^{(\mu)} \theta_1 + a_{2,n}^{(\mu)} \theta_2 + \ldots + a_{q,n}^{(\mu)} \theta_q, \qquad (n = 1, 2, \ldots, q),$$

where the coefficients $a^{(\mu)}$ are constants such that their determinant is unity, because the derivative of order next to the highest is absent from the differential equation.

Now x and y are Fuchsian functions of z, existing only within the fundamental circle in the z-plane; hence, when the path on the Riemann surface, which cannot be made evanescent, is completed, x and y return to their initial values, and z has described some path which is not evanescent. It follows, from the nature of the functions, that the end of the z-path is a point in another polygon, homologous with the initial position, so that the final position of z is of the form

$$\frac{\alpha_\mu z + \beta_\mu}{\gamma_\mu z + \delta_\mu}.$$

The integrals $\theta_1, \theta_2, \ldots, \theta_q$ are uniform functions of z; let them be denoted by $\phi_1(z), \phi_2(z), \ldots, \phi_q(z)$. Moreover, θ_n' is the value of θ_n at the conclusion of the path; thus

$$\theta_n' = \phi_n\left(\frac{\alpha_\mu z + \beta_\mu}{\gamma_\mu z + \delta_\mu}\right),$$

so that the integrals in the fundamental system consist of a set of uniform functions of z, which are characterised by the property

$$\phi_n\left(\frac{\alpha_\mu z + \beta_\mu}{\gamma_\mu z + \delta_\mu}\right) = a_{1,n}^{(\mu)} \phi_1(z) + a_{2,n}^{(\mu)} \phi_2(z) + \ldots + a_{q,n}^{(\mu)} \phi_q(z),$$
$$(n = 1, \ldots, q).$$

Corresponding to the substitution of the Fuchsian group, we have a linear substitution S_μ in the quantities $\phi_1, \phi_2, \ldots, \phi_q$: the aggregate of these linear substitutions S_μ forms a group, which is isomorphic with the Fuchsian group.

Functions of this pseudo-automorphic character are called* *Zetafuchsian* by Poincaré: and thus we can say that *linear differential equations can be integrated by means of Fuchsian and Zetafuchsian functions which are uniform.* It is, however, necessary to obtain explicit expressions for the functions ϕ, in order that the equation may be regarded as integrated. This is effected (*l.c.*) by Poincaré as follows.

Let

$$\left\| A^{(\mu)}_{m,n} \right\|$$

represent the substitution inverse to S_μ, so that the quantities $A^{(\mu)}_{m,n}$ are the minors of the determinant of S_μ. Take any q arbitrary rational functions of z, say $H_1(z), H_2(z), \ldots, H_q(z)$; and by means of them, in association with the Fuchsian group, construct p infinite series, defined by the equations

$$\xi_\mu(z) = \sum_i \sum_{\nu=1}^{q} A^{(i)}_{\mu,\nu} H_\nu\left(\frac{\alpha_i z + \beta_i}{\gamma_i z + \delta_i}\right) \frac{1}{(\gamma_i z + \delta_i)^{2m}},$$

for the q values $1, \ldots, q$ of μ; the quantity m is a positive integer; and the summation with regard to i is over all the substitutions of the Fuchsian group. This integer m is at our disposal: by choosing it sufficiently large, and by limiting the rational functions H, so that no one of the quantities

$$H_\nu\left(\frac{\alpha_i z + \beta_i}{\gamma_i z + \delta_i}\right)$$

is infinite on the fundamental circle, all the series can be made absolutely converging: but we do not stay to establish this result†. Assuming this convergence, and writing

$$s_i(z) = \frac{\alpha_i z + \beta_i}{\gamma_i z + \delta_i}, \quad s_i(s_k(z)) = s_\rho(z),$$

* *Acta Math.*, t. v, p. 227.

† It can be established on the same lines as the convergence of Poincaré's Thetafuchsian series: *T. F.*, §§ 304, 305.

so that, for any value of k and all the values of i, we get all the values of ρ for the group, we have

$$\xi_\mu\left(\frac{\alpha_k z+\beta_k}{\gamma_k z+\delta_k}\right)=\sum_\rho \sum_{\nu=1}^{q} A_{\mu,\nu}^{(i)} H_\nu\left(\frac{\alpha_\rho z+\beta_\rho}{\gamma_\rho z+\delta_\rho}\right)\left(\frac{\gamma_k z+\delta_k}{\gamma_\rho z+\delta_\rho}\right)^{2m}.$$

But

$$\sum_{n=1}^{q} a_{\mu,n}^{(k)} \xi_n(z)=\sum_\rho \sum_{\nu=1}^{q} \sum_{n=1}^{q} a_{\mu,n}^{(k)} A_{n,\nu}^{(\rho)} H_\nu\left(\frac{\alpha_\rho z+\beta_\rho}{\gamma_\rho z+\delta_\rho}\right)\frac{1}{(\gamma_\rho z+\delta_\rho)^{2m}}.$$

Owing to the properties of the isomorphic groups, we have

$$S_i S_k = S_\rho,$$

and therefore

$$S_k S_\rho^{-1} = S_i^{-1},$$

that is,

$$\sum_{n=1}^{q} a_{\mu,n}^{(k)} A_{n,\nu}^{(\rho)} = A_{\mu,\nu}^{(i)};$$

and therefore

$$\xi_\mu\left(\frac{\alpha_k z+\beta_k}{\gamma_k z+\delta_k}\right)=(\gamma_k z+\delta_k)^{2m}\sum_{n=1}^{q} a_{\mu,n}^{(k)} \xi_n(z).$$

Now let $\Theta(z)$ represent a Thetafuchsian series*, with the parametric integer m, and possessing the foregoing Fuchsian group: then, for each substitution of the group, we have

$$\Theta\left(\frac{\alpha_k z+\beta_k}{\gamma_k z+\delta_k}\right)=(\gamma_k z+\delta_k)^{2m}\,\Theta(z).$$

We introduce functions $Z_1, Z_2, \ldots, Z_q$, defined by the relations

$$Z_\mu(z)=\frac{\xi_\mu(z)}{\Theta(z)}, \qquad (\mu=1, \ldots, q).$$

They satisfy the conditions

$$Z_n\left(\frac{\alpha_k z+\beta_k}{\gamma_k z+\delta_k}\right)=a_{1,n}^{(k)} Z_1(z)+a_{2,n}^{(k)} Z_2(z)+\ldots+a_{q,n}^{(k)} Z_q(z);$$

and therefore we may take

$$\phi_\mu(z)=Z_\mu(z),$$

or the q functions Z, which are Zetafuchsian functions, constitute a system of integrals of the differential equation.

170. As regards the Zetafuchsian functions thus constructed, it will be noted that the rational functions $H_1, \ldots, H_q$, which

* *T. F.*, § 305.

enter into their construction, are arbitrary; so that an infinite number of Zetafuchsian functions can be formed, admitting a Fuchsian group G and the linear group (say $\bar{G}$) isomorphic with G.

Further, the Thetafuchsian series $\Theta(z)$ with the parametric integer m is any whatever; but, as

$$x = f(z) = f\left(\frac{\alpha_k z + \beta_k}{\gamma_k z + \delta_k}\right),$$

we have

$$\frac{dx}{dz} = f'(z) = f'\left(\frac{\alpha_k z + \beta_k}{\gamma_k z + \delta_k}\right)\frac{1}{(\gamma_k z + \delta_k)^2},$$

so that we may take

$$\Theta(z) = \left(\frac{dx}{dz}\right)^m P(x, y),$$

where $P(x, y)$ is any uniform function of x and y. The simplest case occurs when $P(x, y) = 1$.

Again, we have

$$Z_n\left(\frac{\alpha_k z + \beta_k}{\gamma_k z + \delta_k}\right) = a_{1,n}^{(k)} Z_1(z) + a_{2,n}^{(k)} Z_2(z) + \ldots + a_{q,n}^{(k)} Z_q(z);$$

and therefore

$$\frac{1}{(\gamma_k z + \delta_k)^2} Z_n'\left(\frac{\alpha_k z + \beta_k}{\gamma_k z + \delta_k}\right) = a_{1,n}^{(k)} \frac{dZ_1}{dz} + a_{2,n}^{(k)} \frac{dZ_2}{dz} + \ldots + a_{q,n}^{(k)} \frac{dZ_q}{dz},$$

so that

$$\frac{1}{f'\left(\frac{\alpha_k z + \beta_k}{\gamma_k z + \delta_k}\right)} Z_n'\left(\frac{\alpha_k z + \beta_k}{\gamma_k z + \delta_k}\right) = a_{1,n}^{(k)} \frac{\frac{dZ_1}{dz}}{\frac{dx}{dz}} + a_{2,n}^{(k)} \frac{\frac{dZ_2}{dz}}{\frac{dx}{dz}} + \ldots + a_{q,n}^{(k)} \frac{\frac{dZ_q}{dz}}{\frac{dx}{dz}},$$

that is,

$$\frac{d}{dx} Z_n\left(\frac{\alpha_k z + \beta_k}{\gamma_k z + \delta_k}\right) = a_{1,n}^{(k)} \frac{dZ_1}{dx} + a_{2,n}^{(k)} \frac{dZ_2}{dx} + \ldots + a_{q,n}^{(k)} \frac{dZ_q}{dx}.$$

Hence

$$\frac{dZ_1}{dx}, \frac{dZ_2}{dx}, \ldots, \frac{dZ_q}{dx},$$

are a Zetafuchsian system, admitting the Fuchsian group G and the isomorphic linear group $\bar{G}$.

The same property is possessed for all the derivatives of any order of the system $Z_1, Z_2, \ldots, Z_q$ with regard to x.

This property is used by Poincaré to obtain the most general expression for a Zetafuchsian system, admitting the groups G and $\bar{G}$. Let it be $T_1, T_2, \ldots, T_q$; and construct the matrix

$$\left\| \begin{array}{llll} Z_1, & \dfrac{dZ_1}{dx}, \ldots, & \dfrac{d^{q-1}Z_1}{dx^{q-1}}, & T_1 \\ Z_2, & \dfrac{dZ_2}{dx}, \ldots, & \dfrac{d^{q-1}Z_2}{dx^{q-1}}, & T_2 \\ \cdots & \cdots & \cdots & \cdots \\ Z_q, & \dfrac{dZ_q}{dx}, \ldots, & \dfrac{d^{q-1}Z_q}{dx^{q-1}}, & T_q \end{array} \right\|.$$

Denote by $(-1)^{a-1}\Delta_{a-1}$ the determinant obtained by cutting out the a^{th} column from the matrix: then, by a known property of determinants, we have

$$\Delta_0 Z_n + \Delta_1 \frac{dZ_n}{dx} + \ldots + \Delta_{q-1}\frac{d^{q-1}Z_n}{dx^{q-1}} + \Delta_q T_n = 0,$$

for all values of n. Hence

$$T_n = -\frac{\Delta_0}{\Delta_q} Z_n - \frac{\Delta_1}{\Delta_q}\frac{dZ_n}{dx} - \ldots - \frac{\Delta_{q-1}}{\Delta_q}\frac{d^{q-1}Z_n}{dx^{q-1}}.$$

When z is subjected to any transformation of the group G, the quantities in any column in the matrix are subjected to the corresponding linear transformation of the group $\bar{G}$; so that each of the $q+1$ determinants $\Delta_0, \Delta_1, \ldots, \Delta_q$ is multiplied by the determinant of the linear transformation. Hence $\Delta_r \div \Delta_q$ is unaltered, that is, it is automorphic for the substitution of the group G; and therefore, as this property is possessed for each substitution, $\Delta_r \div \Delta_q$ is automorphic for the group G. Consequently, $\Delta_r \div \Delta_q$ is a rational function of x and y, say

$$\frac{\Delta_r}{\Delta_q} = -F_r, \qquad (r = 0, 1, \ldots, q-1);$$

and therefore

$$T_n = F_0 Z_n + F_1 \frac{dZ_n}{dx} + \ldots + F_{q-1}\frac{d^{q-1}Z_n}{dx^{q-1}},$$

for $n = 1, 2, \ldots, q$. This is Poincaré's expression for the most general Zetafuchsian system, admitting the Fuchsian group G and the isomorphic linear group $\bar{G}$.

We can immediately verify that $Z_1, \ldots, Z_q$ satisfy a linear differential equation, having coefficients that are rational in x and y. For

$$\frac{d^q Z_1}{dx^q}, \quad \frac{d^q Z_2}{dx^q}, \ldots, \quad \frac{d^q Z_q}{dx^q},$$

are a Zetafuchsian system, admitting the Fuchsian group G and the isomorphic linear group $\overline{G}$; and therefore rational functions $\phi_0, \phi_1, \ldots, \phi_{q-1}$ exist, such that

$$\frac{d^q Z_n}{dx^q} = \phi_0 Z_n + \phi_1 \frac{dZ_n}{dx} + \ldots + \phi_{q-1} \frac{d^{q-1} Z_n}{dx^{q-1}},$$

holding for all values of n. Thus $Z_1, \ldots, Z_q$ are integrals of the linear differential equation

$$\frac{d^q Z}{dx^q} = \phi_0 Z + \phi_1 \frac{dZ}{dx} + \ldots + \phi_{q-1} \frac{d^{q-1} Z}{dx^{q-1}}.$$

Similarly, $T_1, \ldots, T_q$ are integrals of a linear differential equation also of order q, having rational functions of x and y for its coefficients, and characterised by the same groups G and $\overline{G}$ as characterise the equation satisfied by $Z_1, \ldots, Z_q$.

Concluding Remarks.

171. The Zetafuchsian and Thetafuchsian functions thus used occur, for the most part, in the form of series of a particular kind; as they were first devised by Poincaré, his name is frequently associated with them. The main aim in constructing them was to obtain functions which should exhibit, simply and clearly, the organic character of automorphism under the substitutions of the groups; and they are avowedly intended* to be distinct in nature from series adapted to numerical calculation, such as series in powers of z.

Unless both these properties, viz. the exhibition of the organic character of the function and its adaptability to numerical calculation, are possessed by the functions involved, it is manifest that they are not in the most useful form. It is unlikely that the best development of the general theory can be effected, until

* *Acta Math.*, t. v, p. 211.

functions have been obtained in a form that possesses both the properties indicated. In this connection, Klein* quotes a parallel instance from the theory of elliptic functions, viz. the series of the form

$$\Sigma\,\Sigma\,(m\omega + m'\omega')^{-\mu},$$

used† by Eisenstein, which exhibit the characteristic automorphic property of the modular functions, but are not adapted to numerical calculation. Their deficiency in this respect has been met by the possession of the theta-functions and the sigma-functions. The generalisation of the Jacobian theta-function and the Weierstrassian sigma-function, required for automorphic functions, has not yet been attained.

We thus return to the statement made at the beginning of the foregoing sketch of Poincaré's theory of linear differential equations with algebraic coefficients. The explicit analysis connected with the theory of automorphic functions has not yet acquired sufficiently comprehensive forms upon which to work; and therefore its application to linear differential equations, as to any other subject, can be only partial and imperfect in its present stage. The theory of automorphic functions in general presents great possibilities of research: the gradual realisation of these possibilities will be followed by corresponding developments in many regions of analysis.

* *Vorlesungen ü. lineare Differentialgleichungen d. zweiten Ordnung*, (Göttingen, 1894), p. 496. See also Fricke und Klein, *Theorie der automorphen Functionen*, t. II, p. 155.

† For references, see *T. F.*, § 56.

INDEX TO PART III.

(The numbers refer to the pages in this volume. The Table of Contents at the beginning of the volume may be consulted.)

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